N I Watkins

describes

INSTITUTES

OF

NATURAL PHILOSOPHY,

THEORETICAL AND PRACTICAL.

BY WILLIAM ENFIELD, LL. D.

WITH SOME CORRECTIONS;

CHANGE IN THE ORDER OF THE BRANCHES;

AND THE ADDITION OF

AN APPENDIX TO THE ASTRONOMICAL PART,

SELECTED FROM

MR. EWING'S PRACTICAL ASTRONOMY.

BY SAMUEL WEBBER, A. M. A. A. S.
Late President of Harvard College.

FOURTH AMERICAN EDITION, WITH IMPROVEMENTS.

Omnis philosophiæ difficultas in eo versari videtur, ut a phænomenis motuum investigemus vires naturæ, deinde ab his viribus demonstremus phænomena reliqua.—*Newton.*

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R. J. Watkins

TO THE

REV. JOSEPH PRIESTLEY,

DOCTOR OF LAWS, FELLOW OF THE ROYAL SOCIETY, &c,

IN TESTIMONY OF RESPECT

FOR A CHARACTER EMINENTLY DISTINGUISHED

BY

COMPREHENSIVE AND ENLARGED VIEWS OF SCIENCE,

ASSIDUOUS AND SUCCESSFUL RESEARCHES INTO NATURE,

AN ARDENT LOVE OF TRUTH,

INDEFATIGABLE ZEAL IN THE SERVICE OF RELIGION,

SIMPLICITY OF MANNERS,

AND

AN ACTIVE SPIRIT OF PHILANTHROPY,

THIS WORK

IS INSCRIBED

BY HIS AFFECTIONATE FRIEND,

AND OBEDIENT SERVANT,

WILLIAM ENFIELD.

PREFACE.

Nothing can be an adequate apology for obtruding upon the world a new Elementary Work, in a branch of Science already well understood, except the plea of utility. It is wholly upon this ground, that I venture to submit the following Treatise to the public inspection.

The difficulty which I met with, in providing my Classes* with a Text-book in Natural Philosophy, neither, on the one hand, materially deficient in Mathematical Demonstration, nor, on the other, too copious, or too abstruse, for the purpose of elementary instruction, first suggested the idea of this work. And the apprehension that others may have met with the same difficulty, induces me to make it public, in hopes that it may be of some use to those who wish to study, or to teach, this science systematically.

To that class of readers who are satisfied with general views, this work will be of little service. Sketches of philosophy, sufficiently comprehensive to answer their purpose, will easily be found. But the knowledge, which is gathered up in this cursory manner, must unavoidably be superficial, and will in many particulars, be confused and inaccurate. What Cicero says of philosophy in general, is particularly true of natural Philosophy: *Difficile est enim in philosophiâ pauca esse ei nota, cui non sint aut pleraque, aut omnia.*† It may be laid down as an universal maxim, that there is no easy method of obtaining excellence. The small portion of learning, or science, which is to be acquired by the help of facilitating expedients, has been justly compared to a temporary edifice built for a day.‡ It is as unreasonable to hope to acquire knowledge without undergoing the labour by which it is usually gained, as it would be to expect that an acorn will become an oak, without passing through the ordinary process of vegetation.

* In the Warrington Academy. † Tuscul. Quæst. II. 1. ‡ Knox on Liberal Education, § 9.

All the knowledge of Natural Philosophy which can be acquired by cursory reading, without the assistance of mathematical learning, must consist in an acquaintance with leading facts and general conclusions. To understand the manner in which the laws of nature have been inferred from these facts, and to be able with certainty and precision to apply these laws to the explanation of particular phenomena, necessarily requires a previous knowledge of the elements of Geometry, Trigonometry, the Conic Sections, and Algebra. A mechanic, who should set about making a machine without the requisite tools, would not act more absurdly, than a student who should attempt to understand the science of Natural Philosophy without these helps. A preceptor, who professes to teach this science in the easy and amusing method of experiment alone, is an architect without his rule, plumb-line, and compasses.

Facts are, it is true, the materials of science; and much praise is unquestionably due to those who have increased the public store, by new experiments accurately made, and faithfully related. But it is not in the mere knowledge, nor even in the discovery of facts, that philosophy consists. One who proceeds thus far, is an experimentalist; but he alone, who, by examining the nature, and observing the relation of facts, arrives at general truths, is a philosopher. A moderate share of industry may suffice for the former: patient attention, deep reflection, and acute penetration, are necessary in the latter. It is therefore no wonder, that amongst many experimentalists there should be few philosophers.

The hardy perseverance, and the vigorous exertions, which are necessary to form this character, are so contrary to that effeminacy and frivolity which distinguish the present age, that, if it were not for the provision which is made in our universities, and other seminaries, for the propagation of sound learning of every kind, there would be some reason to apprehend, that all the more abstruse and difficult branches of science would be excluded from the modern system of education, and consequently would fall into disesteem and neglect.

It is by no means the intention of this treatise to encourage the indolent spirit of the times, by opening a bye-path to the Temple of Philosophy. The known and beaten road is the safest and the best. It has been with a view of assisting the student in his progress, that I have attempted to ar-

range the leading truths of Natural Philosophy in a perspicuous method, and to demonstrate them with conciseness; adding a brief description of experiments, adapted to illustrate and confirm the propositions to which they are respectively subjoined.

Being more desirous to be useful than to appear original, I have freely selected from a variety of authors such materials as suited my design. Those who are conversant with this class of writers will perceive that, amongst many others, I have made use of the works of NEWTON, *Keil*, *Whiston*, *Gravesande*, *Cotes*, *Smith*, *Helsham*, *Rowning*, and lastly, *Rutherforth*, whose arrangement I have in part adopted.

With respect to any inaccuracies or mistakes which may have escaped my attention, I must rely on that candour, which those who are best acquainted with the extent and difficulty of this undertaking will be most inclined to exercise.

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TO THE SECOND LONDON EDITION,

BY THE EDITOR.

In laying before the public a new edition of "*The Institutes of Natural Philosophy, by the late Dr. Enfield*," the Editor feels it incumbent on him to assure the reader, that he has endeavoured, as far as was consistent with an elementary book, to avail himself of those advantages which the publication of new discoveries, and new works in science, has afforded him; and although the limits of an advertisement will not allow him to particularize all the additions that will be found interwoven with the various parts of the volume, yet it may be expected that, in this place, some notice should be taken of the most material of them; and it is presumed that the following account will be deemed sufficient for the purpose.

In the first book, the propositions on the divisibility of matter, and the attraction of cohesion, are more fully discussed, and a very useful corollary is drawn from that on the attraction of capillary tubes. To the first and third propositions of the second book, considerable additions are subjoined; and in the second chapter is inserted a new proposition, from which, in conjunction with others, are deduced many corollaries and scholiums, connected with the remaining parts of the book.* Several examples are also given in the two first sections of the fifth chapter, which will be found useful to the young student, as illustrative of the theory of falling bodies.

In the third book is given, independently of the additions noticed in the margin,† an important proposition on the specific gravities of bodies, with which are connected examples, and a table of the comparative weights of many of the most useful subtances in nature. Descriptions, accompanied

* See Prop. A, (p. 9.) and Cors. and Schols. &c. to Prop. 14, 17, 24, 26, 28, 30, 31, 36, 44, 46, 49, 52, 53, 54, 57, and 58.

† See additions to Prop. 3, 6, 12, 13, 18, A, (p. 60,) 50, and 55.

with figures, are likewise given of the Pyrometer, Air-Pump, Barometer, with its application to the measuring of altitudes, &c. Fahrenheit's Thermometer, with a Table of heat; different kinds of Hygrometers, the Steam-Engine, and the Hydrometer.

The principal additions to the book of Optics will be found connected with the propositions mentioned below;* in the course of which are introduced Mr. Delaval's Theory of Colours; brief accounts of Dr. Blair's Achromatic Lenses, and Dr. Herschel's grand Telescope.

On the subject of Astronomy, are arranged under the different articles several useful Tables, and the important discoveries of the illustrious Dr. Herschel, which have been carefully selected from the last twenty volumes of the Philosophical Transactions. The reference in the margin,† will direct the reader to those propositions to which the most material additions are subjoined.

Some valuable treatises, on Magnetism and Electricity, particularly those of Mr. Cavallo, having appeared since the original publication of this volume, it was thought necessary very considerably to enlarge this part of the work; and it is hoped that the principal discoveries in these branches of science will now be found under their respective heads.

By the suggestion of a friend, on whose judgment the public has long placed great confidence, it has been deemed proper that the first principles of chemistry should form a part of the present volume;‡ and although we have chiefly confined ourselves to the interesting discoveries of the philosophers, Black, Priestley, and Lavoisier, on Heat and the Factitious Airs, it is nevertheless presumed, that enough has been said on these subjects to render the doctrines and introductory practice to modern chemistry perfectly intelligible to any person who may be desirous of farther prosecuting the study of this amusing and useful science.

The reader ought to be apprized, that besides additions to the old plates, two new ones are now given:—one, as already noticed, accompanying descrip-

* See Prop 5, 13, 22, 42. B, (p. 120) 61, 62, 66, 68, 69, 76, D, (p. 133) 94, 122, 128, 144, and 145.

† See Def. 1, 12. Prop. 8, 16, 20, 32, 35, 39, 51, 57, 72, 78, 79, 83, 109, 116, 117, 118, 119, 120, 123, 136, 167, 168, 177, 179, A, B, (p. 323) 182, 183.

‡ [It has been deemed best to omit the Introduction to Chemistry. As an elementary treatise, it has been found defective, and as far as our information extends, it has not been generally used in those seminaries, where these Institutes are taught, the Chemical Professors generally recommending their respective favourite authors.]

tions of several pneumatic and hydraulic machines, and the other containing figures relating to subjects in magnetism, electricity, and chemistry.

It is hoped that the augmentations to the volume, although they compose about one third of the whole work, will be found such as ought, at this period, to be comprehended in an elementary book of science; and that the speculations of Dr. Herschel, towards the end of the astronomical part, will not be considered as an exception: they are at least the speculations of a great mind, and capable of exciting, in every well-disposed heart, emotions of interest and exquisite pleasure, inasmuch as they lead to the grandest and most sublime notions of the great Author of the universe.

The editor will only add, that in the additions to this work, he has uniformly aimed at conciseness; and he will consider his exertions well rewarded, if it be found, by a candid and discerning public, that he has not sacrificed perspicuity to brevity, and that he has not omitted, within his prescribed limits, any material article that might serve to render the original work, in its present enlarged state, generally acceptable and useful.

May 14, 1799.

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TO THE FIRST AMERICAN EDITION.

The principal object in undertaking this American Edition of Dr. Enfield's Institutes of Natural Philosophy was to supply our Colleges with a book, which is held in so high estimation for the use of Students in recitations to their Instructers. In some of these seminaries of learning Ferguson's Astronomy is recited, and in one or more of them after the exercises in Enfield's Philosophy, exclusive of the astronomical part. It was the opinion of the Rev. Dr. Willard, President of Harvard College, and Professor Webber, that it would be a valuable improvement in this part of Collegial Instruction to substitute Enfield's Astronomy for Ferguson's, provided some additions were made, which should be equally or more important than certain Articles in the latter, particularly those relating to the Calculation and Projection of Solar and Lunar Eclipses, including the necessary Tables. On account of some particular circumstances, measures were not taken to make the proposed additions till the printing was far advanced, and assurances given that it would be finished within a short period. Hence it was extremely difficult to execute the plan. Professor Webber, however, consented to attempt what then seemed to be practicable, and what, it was hoped, might in some good degree answer the purpose. And there appears to be a propriety in giving a particular account of the alterations that have been made.

The errors, which occurred, are corrected. And it was thought expedient to retain the Introduction to the First Principles of Chemistry, but to annex it as an Appendix.*

A change is made in the order of the branches by inserting Magnetism and Electricity between Pneumatics and Optics, instead of placing them after Astronomy; as it was thought a more natural and useful arrangement for a regular course of instruction, the propriety of closing with Astronomy being particularly obvious.

In the astronomical part, the alterations are comprised in a few particulars, the substance of which is almost entirely taken from Ferguson's Astronomy, as well as the figures to which they refer.†

An appendix of about 80 pages is subjoined to the Astronomical part, and constitutes the most distinguishing and important peculiarity of this edition. It contains the most useful Solar and Lunar

* See note p. ix.

† The particular additions are the explanations of the circles of perpetual apparition and occultation under Prop. XX.—Scholium 2 to Prop. LX.—Scholium to Prop. LXXXII.—Scholium to Prop. CXIV.—Explanation of the figures of orbits of Satellites under Scholium to Prop. CXVI. Scholium to Prop. CXLIII.—Cor. to Prop. CLXXIII.—Cor. to prop. CLXXV. A small alteration is made in the demonstration of Prop. CXIX. The Scholium to Prop. CXIX. and Scholium 3 to Prop. CXX. are omitted. The figures, together with four projections of eclipses, which are also added with the Appendix to the Astronomical part, fill two plates.

Tables, and a Table of Logistical Logarithms, together with their explanation, and twenty-two Problems, illustrating their use and application, and exemplifying the projection of Eclipses. These are selected from a work of Mr. Alexander Ewing, Teacher of Mathematics at Edinburgh, entitled "Practical Astronomy," and published in 1797. With respect to the Projection of Eclipses, there is a small alteration in making two distinct Problems for the Lunar and Solar Projections, instead of placing them under the respective Problems for calculating those Eclipses. And two notes are added to the Problem for projecting a Solar Eclipse. The former contains the necessary directions for a different mode of projection; and the latter the Rev. President Willard's method of finding the point on the sun's limb, where a Solar Eclipse begins; the knowledge of which point is of great importance to observers.

In the whole execution of the work it has been the unfeigned endeavour of the editors to merit the approbation of the public.

Boston, January, 1802.

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TO THE THIRD AMERICAN EDITION.

The following work has been, we believe, for some years, a classic in every College in New England, but considerable complaints having lately been made of its incorrectness and deficiency, it was expected that some other system would be brought forward to supply its place. None however has appeared, and in the mean time the former editions have been taken up, and copies were continually called for, which could not be furnished, so that the public necessity seemed absolutely to require another edition to be immediately put to the press. Such corrections, however, have been made as the time permitted, and they will be found much more considerable, both in number and importance, than those in any former edition. The principal alterations occur in Book II. Props. 52, 57 and 80; B. III. P. 16 and 51; B. IV. P. 4; B. VI, P. 13, 21, 22, 23, 55, 56, 61, and 147; B. VII. P. 7, 36, 39, 45, 60, 70; Prob. at the end of Chap. IV. P. 80, 115, 136; Table at the end of part I. P. 167 and 170. In the Appendix, Explanation of Tables I. II. and VI, and the examples under Problems 8 and 15. Less considerable alterations occur in every part of the work, too numerous to be particularized. These corrections have been made by the direction of James Dean, A. M. A. A. S. of Windsor, Vt. It is not pretended that the work even now is rendered perfect. The articles of Electricity and Magnetism, especially, are much behind the present state of science, and Physical Astronomy, also, is too concise and obscure; but even as it has hitherto been, it seems to be the almost unanimous opinion of instructers, that it is better adapted to the state of science in this country than any other work extant; and the publishers can venture to assure the public that the present edition is very much improved.

Boston, January, 1820.

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TO THE FOURTH AMERICAN EDITION.

In correcting this fourth edition, the publishers have availed themselves of the minute and able Remarks on the third, contained in the third volume of Silliman's American Journal of Science and Arts, page 125. Of the corrections there suggested, those were made which would not cause a considerable change in the text; and the remainder are here subjoined.

Book I. Prop. 10. "Some bodies appear to possess a power the reverse of the attraction of cohesion, called repulsion." Of the five experiments adduced in support of this proposition, the first four,—namely, the depression of mercury around iron, and in capillary tubes, the suspension of a needle by water, and the depression of the surface around a floating piece of tin-foil,—are so far from furnishing any evidence of a repulsive power in bodies, that they are mere examples of cohesion, modified by circumstances. If we suppose the force of aggregation between the particles of mercury to be more than *twice* their force of cohesion to iron and to glass,* it appears from the investigations of Clairaut, that a depression ought to be the consequence.—The suspension of a small needle on water is owing to a certain degree of viscidity in this fluid,—in consequence of which the particles of the uppermost stratum present more resistance to separation than can be overcome by the downward pressure of the needle. Those who are acquainted with the extended researches of Count Rumford on this subject, will find no more reason for ascribing the support of a needle on water to a positive repellency, than the support of a cannon ball on ice. Both are alike owing to the cohesion of the upper surface. The only difference is, that as the cohesion is many times the least in the former case, the weight of the supported body must be proportionally less, when compared with the surface it exposes.

Book II. Prop. A. This proposition, (besides that the demonstration is unsatisfactory,) is out of place; as the chapter is confined by its title to the comparison of *uniform* motions. It ought to have been deferred to ch. v.

Prop. 22. Cor. 1. is evidently erroneous. A second "Cor. 1." is inserted under this proposition, which belongs to the preceding one; for it is true only of non-elastic bodies.

Prop. 37. The demonstration is defective, from not being extended, as it easily might be, to the supposition that motion is lost in passing from one plane to another. The demonstration of the 38th is also inconclusive, because it has not been previously shown that the total loss of motion in passing through a set of planes becomes evanescent, when the planes become indefinitely numerous, and their successive inclinations indefinitely small.

Prop. 47. The demonstration shews that we may form a body by assembling particles round a *given point*, such that the body shall balance itself about this point; but it by no means shews that when the *body is given*, a point about which it will balance itself can be found;—much less that this point, as the proposition implies, is the same for all positions of the same body.

Prop. 49. The diagram employed in the proof of this proposition is drawn so inaccurately as to render it scarcely intelligible. There was the less reason for this inaccuracy, as in Rutherforth, from whom the diagram is copied, it is drawn correctly.

Prop. 51. The demonstration of this important theorem is less general than the enunciation requires, by being confined to the case in which the bodies move in the same plane. The statement with which the first corollary begins is true only under such limitations as the student can scarcely be supposed able to apply.

Prop. 56. In the great majority of instances in which the screw is employed, the resisting force is not moved up through an inclined plane, as the demonstration supposes. It would be far more simple and satisfactory to infer the law of equilibrium directly from the relative velocities of the points of application of the power and resistance.

* Perhaps it ought rather to be said,—to "the *film of moisture* which is ordinarily attached to the surface of glass." See Haüy—Traité de Physique, I. 225. Biot—I. 455.

Prop. 57. Schol. 1. "In all compound machines there will be an equilibrium, when the sum of the powers are to the weight, as the velocity of the weight is to the sum of the velocities of the powers." No interpretation can be put upon this statement which will render it true. The error arose, we presume, in some such manner as the following. It was apparent that in compound machines, (or rather in machines where several powers put several resistances in equilibrio,) the sum of the products of the powers each into its velocity, was equal to the sum of the products of the weights each into its velocity. This equation had the appearance of being capable of resolution into an analogy; and the resolution was accordingly made. But in doing it, two things were confounded which are widely different: the *sum of the products*, and the *product of the sums*.—It would have been better if the compiler had not attempted to deviate from Rutherforth. Supposing only one power, and only one resisting force which balance each other through the intervention of a series of mechanical powers,—the power will be to the weight simply as the velocity of the weight is to the velocity of the power.*

Prop. 60. Schol. The method here given of finding the initial velocity of a projectile, gives only that part of it which is in a direction perpendicular to the horizon. To obtain the whole velocity, this result ought to have been increased in the ratio of radius to the cosecant of the angle of elevation. But it would have been altogether preferable to omit noticing a method so entirely useless and even deceptive in practice, and to have substituted for it that by the ballistic pendulum.

Prop. 68. In the last edition, several palpable errors which formerly perplexed the demonstration have been corrected; but the reasoning is still far from being demonstrative. The erroneous figure of former editions is also retained. The circle GNV, instead of GML, should have had T for its centre, and GML should have been an ellipse having T for its farther focus.

Def. 16. Schol. It is improperly asserted in this scholium, that "the projectile and the centrifugal forces differ from each other as the whole from the part." These forces are dissimilar in kind, and are incapable of comparison. They stand to one another in the same relation as pressive and percussive forces. If the tangent which measures the projectile, and the subtense which measures the centrifugal force, be diminished indefinitely, as they must be before we can properly make the attempt to compare them, the latter becomes evanescent in respect to the former. The centrifugal is rather a *consequence* of the projectile force, than a *part* of it.

Prop. 70. "When bodies revolve in a circular orbit about a centre, the centripetal and centrifugal forces are equal." This proposition implies that in other orbits these forces are not equal. But the demonstration is such as would prove them no less equal, in all cases whatever.—We must confess ourselves at a loss to assign any consistent meaning to the term "centrifugal force," in relation to orbits not circular. Is this force measured by the distance by which a revolving body would be more remote from the centre, in a given small time, if the centripetal force were suspended, than it actually is while the centripetal force acts? If so, the centripetal and centrifugal forces are always equal, for the same point of an orbit, whatever be its figure. Or is it measured by the absolute increase of distance from the centre, which would take place in a given small time, if the body were abandoned to its projectile force? If so, in passing from the higher to the lower apsis of an eccentric orbit, the centrifugal force is a negative quantity.

Lemma 4. The inference concerning the equality of the arc to the chord and tangent in their vanishing state, is inconclusive when the tangent is less than the arc, as it will be in certain positions of the subtense. The demonstration may be rendered complete in the following manner:—$AB : AD+DB :: AC : AB+BC$. But $AB+BC$ is ultimately equal to AC; hence $AD+DB$ is ultimately equal to AB, and much more is the arc AB equal to the chord AB.

The first Part of Book III. which treats of Hydrostatics, presents us with several instances of explicit or implied error; particularly prop. 2, prop 13, Schol. and propositions 24, 26, 31, and 36. But we have no room to dwell upon them; and shall therefore pass directly to the second Part, which is devoted to Pneumatics.

Prop. 51. The force with which wind strikes a sail of given dimensions, was stated in former editions as varying in the duplicate ratio of the *cosine* of the angle of incidence. In the last edition, the term *sine* is substituted for *cosine*. The phrase "angle of incidence" was before used in the same sense as in optics: it is now employed in the ordinary sense of mechanics. But this correc-

* See Rutherforth's System, Vol. I. Art. 72.

tion goes but little way towards freeing the proposition from exception. If the sail be supposed confined to move in the same direction with the wind, which the demonstration seems to imply, a resolution of the force on a third independent account was necessary, which would have reduced that part of the force which is effective to the ratio of the *cube* of the sine, instead of the square.* But as the variation of the force determined from experiment differs totally both from the square and the cube, it would have been better to erase the proposition than to attempt to amend it.

Prop. 55. Schol. The mode in which the constant velocity of sound is attempted to be explained, (which, like the rest of the scholium, is copied from Rowning,) is wholly erroneous: nor do we think it easy to substitute an unexceptionable theory of the mechanism of aerial pulsations, without involving mathematical principles of a higher order than the student is supposed to be acquainted with.

Prop. 56. Schol. 3. "It is found by experiment, that air is necessary to the existence of sound, of animal life, of fire, and of explosion." This, like several other statements scattered up and down the work, in which chemical principles are alluded to, needed correction to render it accordant with the present state of our knowledge on these subjects. The experiments of Biot and Chladni shew that sound is transmitted by solid bodies as well as gaseous ones, and that it may be conveyed to the organs of hearing without the intervention of air, by forming a communication between the sounding body and the head by means of a solid conductor.—That fire and explosion require air for their existence, is true only in the most loose and popular sense of the terms. In particular, that the explosion of gunpowder cannot be effected in a vacuum, as is implied in one of the annexed experiments, is an entire mistake.†

The articles on the barometer, thermometer, hygrometer, and steam-engine, are extremely defective in point of valuable information, compared with what might have been said about them within the same limits, and in several respects are calculated to leave erroneous impressions: but we must content ourselves with giving this general caution against placing implicit confidence in them ‡

Book IV. Prop. I. Scholium. The editor of the London edition of 1799, left the question undecided, as Mr. Cavallo had done, whether any other than ferruginous bodies are capable of magnetic properties. This scholium has been retained; although the experiments of Tourte of Berlin,§ and Lampadius, place it beyond a doubt, that nickel and cobalt possess the same capacity, in this respect, with iron, except in an inferior degree. According to Lampadius, (see Thomson's Annals, 1815,) the "magnetic energies" of iron, nickel, and cobalt are as the numbers 55, 35, and 25 respectively.

Prop. 2. Schol. If it were worth the while to retain the experiments of Musschenbroek on the variation of the magnetic force at all, in place of the far more important ones of Coulomb, the numbers ought at least to have been corrected. Had the editor of the second edition taken these experiments from Musschenbroek himself, instead of taking them from Cavallo, he would have avoided the error of making the distances all ten times too large.‖ The denomination employed in the original statement (see Philos. I. 206. Ed. 1744,) is *tenths* of inches, instead of inches.

Prop. 10 To illustrate the mode of finding the declination of the needle by amplitudes of the heavenly bodies, the following example is stated: "If the magnetic amplitude is 80° eastward of the north, and the true amplitude is 82° towards the same side, then the variation of the needle is 2° west." This statement cannot be reconciled to *any* definition of the term "amplitude;" and it cannot be reconciled with the usual definition and the one given in the Astronomy, except by making all the alterations which follow: "If the magnetic amplitude be 10° N. of E. and the true amplitude is 8° towards the same side, then the variation of the needle is 2° east."

* This theoretical determination may be seen, Gregory's Mechanics, I. 539, &c.

† Robins, Hutton's Math Dictionary, &c.

‡ It may be proper just to state, for the information of those who may have access to no other rule than that given page 75, in making loose estimates of altitudes from the barometer, that the ascent corresponding to 1-10 in. fall of the mercury, instead of being one hundred and three feet, is at a mean, (that is, when the barometer itself is at 30 in. and the thermometer at 60°,) only about ninety-three feet.

§ Nicholson's Journal. Vol 26

‖ If perfect exactness in such a case were of any importance, it would be necessary to recollect, that Musschenbroek's denominations were Rhinland measure; which are greater than the English in the ratio of 1,03 to 1.

Prop. 10. Schol. 2. The late and accurate observations of Mr. Gilpin* and Col. Beaufoy on the diurnal variation of the needle in different months of the year, present wide deviations from the results here given from Mr. Canton. Both the observers just mentioned, make the extremes of the mean diurnal variation in different months, about 11′ and 4′; and both place the time of the maximum earlier in the year than was done by Mr. Canton. Col. Beaufoy (Thomson's Annals, 1819,) places it in April.

Prop. 11. Schol 1. The dip of the needle is here represented as probably "unalterable at the same place."—Whatever be the cause of the dip, this supposition is extremely improbable, while the declination is known to be variable, and to be, in common with the dip, the result of the tendency by which the needle places itself in the magnetical line. Nor do the observations made at London during the last century, warrant the inference made in this Scholium. As measured by Whiston in 1724, it was 75° 10 : and nearly accordant with this result is that of Graham, obtained in the following year. Cavendish, in 1775, found it to be 72° 30′, and Gilpin, in 1805, 70° 21′. These observations, after every allowance is made for the imperfection of the instruments employed, leave no doubt that the inclination of the needle has undergone a gradual diminution in London, during the last century. According to M. Humboldt, (see Biot—Traité de Physique, III. 136,) a similar diminution has taken place, during the same period, in France.

Book VI. Prop. 13. Schol. 1. The statements concerning the ratio of the sine of the angle of incidence to that of deviation, in passing to and from water and glass, are true only under a limitation which is not distinctly pointed out,—namely, that the angle of incidence is indefinitely small.

Schol. 2. The partial reflection of light by the second surface of transparent media, when the angle is within the limit for light to be refracted, is erroneously ascribed to "inequalities" of the surface. If this were the true cause, no distinct image of an object could be seen by light thus reflected.

Prop. 17. Exp. The effects of a single dense medium, bounded by a convex surface, on parallel, diverging, and converging rays, can never be illustrated by a convex lens, which produces *two* successive refractions,—one by a convex surface of the denser, and the other by a concave surface of the rarer medium. The lens presents the combined result of the former part of prop. 17, and the latter part of prop. 18. In particular, a convex lens can never render converging rays "less converging," as is asserted in the fourth paragraph under the Exp.

Precisely similar remarks might be repeated concerning the introduction of the concave lens to illustrate the several cases of prop. 18.—Both these experiments, if introduced at all, should have been placed after prop. 18; and the manner in which each illustrates *both* propositions should have been pointed out.†

Prop. 22. Cor. 2. The corollary is right; but the investigation which is given of it, is incorrigibly wrong. By comparison with the figure, it will be seen that it gives the position of the principal focus of a glass sphere *within* the sphere ; and that of a sphere of water, coincident with the hinder surface. The proper mode of proceeding would be, first to determine the focus of parallel rays entering a denser medium by prop. 22 ; and then to find by prop. 23, the focus of rays converging (to the point just found,) when passing out of a denser medium into a rarer, through a concave surface of the rarer.

Prop. 26. "The image will not be distinct, unless the plane surface on which it is received be placed at the distance of the *principal* focus of the lens." For "principal," read—"corresponding to the distance of the object."

Prop. 35. "Though the distance of the object from the lens be varied, the image may be preserved distinct without varying the distance of the plane surface which receives it." The distance of the plane surface from *what* ? The second mode of doing it, pointed out in the demonstration, is inconsistent with the supposition that the distance of the plane surface, either from the object, or the lens, remains unaltered. Those who will consult Rutherforth's Optics, Ch. VII. will see that this inconsistency has arisen from an attempt to blend into one, two propositions of which the conditions were different. We will add, although the remark has no relation to the last edition, that the mistake in the statement of the magnifying power of the double microscope (prop. 147.) arose

* Philos. Trans. 1816.

† Both these propositions have materially suffered in point of clearness, from employing as diagrams sections of lenses, instead of media indefinite in the direction towards which the rays proceed after refraction,—as well as from the inaccurate manner in which some of the lines are drawn.

from precisely the same source. Rutherforth investigated the two ratios on which the magnifying power depends in separate propositions,—first supposing the eye at the station of the object glass, and then at the limit of distinct vision. In uniting these two propositions into one, Enfield inadvertently retained the condition of the former.

Prop. 44. "Reflection is caused by the powers of attraction and repulsion in the reflecting bodies." This proposition is altered and abridged from the following in Rutherforth: "Bodies refract and reflect light by one and the same power, differently exercised in different circumstances." The illustration of this proposition by the original author is an excellent one, considering the state of optical knowledge at the time he wrote; but in the hands of his abridger, although all the suppositions made by Rutherforth are retained, and we are required to admit that "bodies attract those rays which are very near them, and repel those a little farther from them," yet no use is made of the attracting surface, and the most interesting part of the proposition, the reflection produced by the second surface of the medium, (in regard to which so much pains had been taken in the previous scholium to exclude other hypotheses,) is entirely omitted. The student is left to wonder why "attraction" is mentioned in the proposition as having any concern with reflection; and the identity of action in the medium by which refraction and reflection are produced, is kept out of his sight.

Prop. 46. Schol. Although perhaps nothing positively erroneous is advanced in this scholium concerning Sir Isaac Newton's theory of fits of easy transmission and reflection, we cannot but object to a naked statement of a theory, stripped of all the facts which it was formed to explain, and made at the same time in so obscure a manner as must impair the respect of the student for its illustrious author. The hypothesis of fits, however it may seem fitted to excite ridicule as exhibited in this scholium, is now justly regarded as one of the most striking displays which Newton ever made of his transcendant genius. In the hands of Biot and his companions in the career of discovery, it has acquired an importance of which Newton himself could have had no adequate conception.—Whether the principles of this now highly interesting and important department of Optics can be reduced to the level of a system of elementary instruction, is deserving of serious inquiry. A digest of the phenomena and laws of polarization, involving no difficulties which would render it inaccessible, or deprive it of its interest with those who aim at nothing more than general views of science, appears at least to be as yet a desideratum.

Prop. 58. "In all mirrors, plane or spherical, &c." This proposition, in regard to spherical mirrors, is true only of those pencils of reflected light which are indefinitely near the perpendicular.

Prop. 69. In the demonstration it is stated, that "by prop. 31, the diameter of the image, when the object is given, is inversely as the distance of the object." This is not said, in prop. 31; nor is it true, except when the object is very remote. The image formed by a lens is not in circumstances analogous to that produced on the retina of the eye; for the lens has no provision for preserving the image distinct, for different distances of the object, without varying the distance of the plane surface which receives it.

Prop. 73. "When equal objects in the same right line are seen obliquely, their apparent lengths are inversely as the squares of their distances from the eye." The limitation "in the same right line," has been very properly inserted by the editor of the present edition; but to render it correct, it wants another limitation which the proposition originally had as given by Rutherforth; that is, "When equal objects are seen *very* obliquely &c." When the object is of finite magnitude, the obliquity must be very great, in order that the proposition may hold true,—unless indeed the object itself be very small; in which case it holds true for every degree of obliquity. But under this last modification, it requires a different demonstration; and is more properly referred to the subject of apparent velocity than of apparent magnitude. As referred to the head of apparent velocity, the proposition might have been thrown into the following simple and not inelegant form: "When a body moves uniformly in a right line, its apparent velocity will be inversely as the square of the distance from the eye."

In demonstrating the 83d and 85th propositions, it is stated as the reason why the image produced by a convex or concave lens is erect, that the axis of the pencils which proceed from the extremities of the object "only cross one another at the lens." It should be, "because they only cross one another at the eye." The pencils which pass from the extreme points through a lens, do not, in fact, meet each other till they reach the eye. Figs. 8 and 9 convey no idea of the manner in which the pencils come to the eye, except in the single case in which the eye is in contact with

the lens; nor is there any other diagram in the Optics which gives the student any information on this important point.—The remark scarcely need be added, that almost all the propositions in this chapter which state the effect of lenses on apparent magnitude, have unsatisfactory demonstrations. It is taken for granted that at whatever distance from the lens the eye is placed, the pencil which enters it from the same point of the object diverges as if from the same point in space. But the fact is, that as the eye recedes from the lens, the rays which enter the pupil from the same point of the object, gradually change: the axis of the pencil, instead of coinciding with the centre of the lens, passes above or below it, according as the point of the object is above or below. Hence it is improper to assume that the pencil from A (figs. 8 and 9) diverges as if from the same point D for all distances of the eye from the lens. The assumption is erroneous, except when the object is extremely small, and it ought not to be made even in this case without proof.*

Prop. 89. If this proposition were one of the least value, it would be desirable that it should have a more satisfactory demonstration than its present one, which on several accounts is wholly inconclusive.

The statements concerning the *brightness* of the image, made in different propositions of this chapter, are not legitimately proved; for the number of rays received by the pupil from any one point of the object may be increased, and the brightness nevertheless diminished,—on account of the increase of apparent magnitude.

Props. 108, 110, and 111, assert unconditionally concerning the magnifying power of mirrors, what is true only in certain positions of the eye. If, for example, the object be nearer a concave mirror than its principal focus, and the eye be in the centre of concavity, the image, instead of "appearing larger" than the object, as is asserted in prop. 108, will appear of the same magnitude; and if the eye be brought still nearer the mirror, the image will appear the smallest.

Prop. 144. Schol. 1. "Of two refracting telescopes which magnify equally, the shorter will give a more imperfect image than the longer. For the image appearing equal in both, but being farther from the object-glass in the longer than the shorter, must be in reality larger or more magnified; whence the defect arising from the different refrangibility of the rays, will be more visible in the longer than in the shorter telescope."—The statement with which this paragraph begins is correct. The reasoning subjoined is evidently erroneous, and leads to a conclusion the reverse of what was first asserted. If two telescopes were exactly similar in all their parts, differing only in size, it is manifest that the imperfection of the image arising from unequal refrangibility, would be the same in both. But the smaller would have the disadvantage of rendering the object less bright, in the duplicate ratio of the linear dimensions. To render the brightness the same in both, the object glasses must be made equal; in which case the one of least focal distance, being a greater portion of a sphere, would produce the most imperfect image.

Schol. 2. The account of achromatic lenses in this scholium omits the essential circumstance on which the whole explanation turns. We are told that a convex lens of crown glass is to be united with a concave one of flint glass in such a manner that "the excess of refraction in the crown glass may destroy the colour caused by the flint glass" Here the student will naturally inquire, how the crown glass can possess an excess of refraction, without also possessing an excess of dispersive power? For the removal of this difficulty, no hint is given of the great fact which lies at the foundation of Dollond's improvements, viz. that the dispersive power of different media is not proportioned to their mean refractive power. Unless he has the sagacity to conjecture that this may be the case from the obscure statement above quoted, he will remain in ignorance of what has been justly regarded as the greatest discovery made in Optics since the period of Newton.

Book VII. Prop. 13. To make the demonstration from fig. 10, consistent, EPLH ought to be regarded as a circle in the heavens; it is therefore improper to place the spectator at P. The diagram should have been constructed like fig. 2, with a small concentric circle to denote the earth.

Prop. 35. Cor. "Hence it appears that the earth, at the winter solstice or Capricorn, is in its perihelion." The student will be apt to infer, from this mode of expression, that the two points mentioned have some *necessary* connexion. But so far is this from being true, that the time when the earth is in its perihelion is about ten days later than that of the winter solstice. The angular motion of the earth in the interval (for 1820) is about 9° 50.

* The remarks made in this paragraph are equally applicable to the propositions in the succeeding chapter, which relate to vision as affected by mirrors.

Prop. 35. Prob. 6. The method of finding the bearing of two places on the earth's surface, here described, is manifestly erroneous, except when the places are very near each other. This part of the problem does not appear capable of a solution on the artificial globe.

Chapter III. on Twilight, has undergone several material improvements in the last edition. The Cor. to prop. 37, is however out of place, and should have been expunged. The demonstration of prop. 39, is freed from several theoretical errors; although we think the attempt to distinguish between the sun's centre and upper limb, in an angle liable to so much uncertainty as the sun's depression at the commencement of twilight, attended with no advantage sufficient to compensate for the additional complexness it gives the demonstration.—After all, we should have been much better pleased to see the proposition entirely omitted, than any attempt made to amend it. The hypothesis, that the rays which come to the eye at the end of twilight are brought by a single reflection, is a very questionable one. The power of reflecting light possessed by the atmosphere, must depend on one or both of two causes: 1. It may reflect some of the rays which pass through it in consequence of a defect of transparency. 2. It may reflect in the same manner as light is ordinarily bent back into a denser medium. This last mode of reflection, if it ever takes place without an abrupt change of density, is evidently more likely to take place, in proportion as the variation of density is more rapid. Now whichever of these causes operates to produce twilight, it must evidently exist in a far higher degree in the lower, than in the higher regions of the atmosphere. Hence instead of a single reflection at the height of forty-two miles, two or more successive reflections may quite as probably transmit to the eye the light with which twilight closes.*—But even admitting the correctness of the assumption that twilight is produced by a single reflection, it is most obvious that no inference can be adduced concerning "the height of the atmosphere," or even the height at which it ceases to reflect light. The only legitimate conclusion is, that forty-two miles is the limit beyond which light is not reflected in sufficient quantity to affect the organs of vision. If, instead of this vague proposition, the law of atmospheric density at different altitudes had been inserted in its proper place in the Pneumatics, the subject would have been exhibited in a far more interesting and instructive form.

The subject of the moon's librations, in props. 78—82, is managed with singular infelicity. The introductory proposition should be, that "the time of the moon's rotation on its axis is equal to the mean time of its revolution round the earth,"—instead of beginning with the fact that "the moon always has nearly the same side towards the earth," and drawing the strange inference that "*if* the moon revolves about its axis, its periodical time must be equal to that of its revolution round the earth." The librations should be assigned each to its proper cause; that in latitude to the obliquity of the axis to the plane of the orbit, and that in longitude to the eccentric form of the orbit,—instead of blending the explanations of both under the loose proposition, "the librations of the moon may be explained on the supposition that the moon has a revolution on its axis." In prop. 81, the equality of the times of rotation and revolution is inferred from the librations; while it is in fact a matter of direct observation, and must be presupposed in explaining the librations themselves. In prop. 82, the *elliptical* form of the moon's orbit is inferred from the libration in longitude. We very much doubt whether the species of oval to which the moon's orbit most nearly approaches, could have been determined from direct observations on so trifling a change of phase. The proper mode of presenting this part of the subject would be, to infer the elliptical form of the orbit from the observed relation between the anomaly and the apparent diameter; and then to employ this conclusion for the explanation of the libration in longitude.

Prop. 83. Schol. In stating the results of Dr. Herschel's observations on the altitude of the lunar mountains, it is mentioned, that "one was found to be about a mile in height; but none of the others which he measured proved to be more than half that altitude." By consulting the original memoir in the Philos. Trans. and another which he has published since, it will be seen that Dr. Herschel's results differ much less from the estimates of the older Astronomers, and from the recent and accurate measurements of Schröter, than is here represented. Dr. H. makes several over a mile, and one, nearly two miles in height.

Prop. 106. "If the moon, when new, is in one of its nodes, the eclipse of the sun will be central." It should be,—"to the inhabitants of some part of the earth it will be centrally eclipsed in the zenith." To those parts of the earth at which the moon is never vertical, a central eclipse can happen only when the moon is *not* in its node, at the time of conjunction.

* See Vince's Ast. I. Art. 206.

Prop. 113, which affirms a motion "*in antecedentia*," of the satellites of the superior planets while passing from one elongation to the next through their inferior conjunction, is no less erroneous than the propositions of former editions concerning the retrogradation of the primary planets; and should, like them, have been rectified or struck out. All that can be said with truth is, that during the specified interval the motion of the secondary is retrograde *relatively* to its primary; and even this statement can scarcely be extended to the satellites of Herschel.*

Prop. 114. "The greatest elongations of a satellite on each side are equal" This proposition has several exceptions. The orbits of the third and fourth satellites of Jupiter have a very sensible eccentricity; and the same is true of the fourth (now more generally numbered as the sixth) satellite of Saturn. See Laplace: Syst. du Monde. The latter, according to Delambre, (Ast. III. 510,) has an ellipticity nearly equal to that of our moon.

Prop 123. Several of the particulars inserted in the annexed scholium from Sir I. Newton, have now become obsolete. In particular the quantities of matter in Jupiter and Saturn, instead of being to that in the sun in the ratios of 1 to 1100 and 2360, are now known to be in the ratios of 1 to 1067 and 3534.†

Prop. 135 is founded on the erroneous theory of retrogradation previously laid down; and therefore should have been corrected.

Prop. 155. The demonstration of this proposition is in part fallacious. It is said to be contrary to prop. 51. cor. of Book II, that the centre of gravity of two gravitating bodies should move; and is inferred that if one of the bodies is projected in any direction, the other must acquire (by what means we are not told) an equal motion in the opposite direction. Now this is so far from following as a necessary consequence, that the other body will not in fact acquire any such motion; and if a projectile movement be given to one of them alone, the common centre of gravity of the two will not continue at rest. Nor does this contradict the proposition referred to in the Mechanics; for the common centre will move uniformly in a right line. The proposition should have stood thus: "The sun and any planet revolve round a common centre of gravity, which remains at rest, or has a uniform rectilineal motion."

Prop. 162. This theorem, as it stood in Rowning, was preceded by an investigation of the motion of the apses produced by a force varying in a greater or less than the inverse duplicate ratio of the distance. As nothing analogous to this investigation has been retained by Enfield, the assertion that when the force varies faster than in the inverse duplicate ratio of the distance the line of the apses will move forward, and *vice versa*, made in the course of the demonstration, is wholly gratuitous.

Prop. 163. The demonstration is not only irrelevant to the proposition, but from an inadvertent change in the conditions as laid down by Rowning, a blunder is carried through it and the annexed corollary. The demonstration affirms, that if the moon is passing from the higher to the lower apsis and its gravity increases too fast, "it will approach nearer to the earth" than it would otherwise do, "and describe a portion of an orbit less eccentric, or nearer a circle." The former statement is correct; but it contradicts the latter. So in the corollary we are told that "when the gravity of the moon towards the earth decreases too fast, the eccentricity of the orbit will increase; and when her gravity towards the earth increases too fast, the eccentricity will decrease." The fact is, that in both cases alike the eccentricity will increase. It is when the gravity increases or diminishes *too slow*, that the eccentricity will decrease. Those who will give themselves the trouble of consulting the prop. as it stands in Rowning, will find no difficulty in perceiving how a hasty abridger might shift the conditions of the demonstration.

Props. 164 and 166. Why two propositions so nearly identical should find a place in this chapter we can give no account,—unless that the compiler had forgotten that he had given a theorem on the motion of the nodes from Rowning, and therefore looked for one in some other author.

* We have not attempted to rectify the periodical times of the satellites of Herschel; for with the exception of the second and fourth, their distances from their primary are wholly conjectural; nor is even their number regarded by Dr. Herschel as yet fully ascertained His last determination of the synodical revolutions of the second and fourth, given in the Philos. Trans. for 1815, is as follows:

II. 8d. 16h 56′ 5″.
IV. 13d 11h. 8′ 59″.

The inclination of their orbits to the ecliptic he finds to be 78° 58′,—much farther from perpendicularity than has been heretofore supposed.

† Méc. Céleste. Part II. Ch. 9.

So much at least is certain,—that prop. 166, and this only, among those which compose the chapter, is borrowed from Rutherforth.

Prop. 168. Schol. This method of finding the direction of gravity includes only the effect of the centrifugal force. Including the joint effect of rotation and of figure, the direction is manifestly that of a perpendicular to the tangent plane of the earth's surface, or of a normal to the elliptical curve of the meridian passing through the given place.

Prop. 173. In the concluding paragraph of the demonstration, the relative forces of the sun and moon to raise tides are erroneously stated. The real forces are directly as the masses, and inversely as the cubes of the distances.

The concluding scholium of the Astronomy consists of extracts from a paper of Dr. Herschel's in the Philos. Trans. for 1795. These extracts are so unskilfully made, and are presented in so disjointed a form, as to afford scarcely any idea of the train of argument pursued in the original. But in the original itself, high as is the estimation in which the author is justly held as an observer, we must be permitted to think that there are several statements which cannot be defended. With the view of multiplying the points of analogy between the sun and the planets, and thus increasing the presumption that the former is inhabited, he endeavours to shew that both primaries and secondaries shine in some measure by their own light. The partial illumination of the moon, for example, during a total eclipse, cannot be entirely ascribed to the light which may reach it from the earth's atmosphere;—"because, in some cases, the focus of the sun's rays refracted by the earth's atmosphere must be many thousand miles beyond the moon." Dr. Herschel assumes as the basis of this calculation, that the rays of the sun are bent by the atmosphere at only an angle of 31′. He seems to have inadvertently neglected the circumstance that the rays undergo a second equal refraction in passing out of the atmosphere. In consequence of this, the real inflection is 62′ (or rather 66′, taking 33′ as the mean horizontal refraction,) so that the focus of the sun's rays as refracted by the earth's atmosphere can never in fact be so distant as the moon. An observer stationed at the moon, even during a central eclipse of the sun, would see a luminous ring encircling the earth. The light thus thrown upon the moon's disc is amply sufficient to explain its partial illumination during a central eclipse. Were Dr. H.'s assumption concerning the amount of atmospheric refraction correct, his conclusion would not follow; for the same agency of the atmosphere which produces *twilight* to an observer stationed on the earth's surface, will produce the same effect to a second spectator, stationed any where *behind* the first, and in the same tangent plane of the earth.* Another obvious proof that Dr. H. was misled by his zeal to find points of analogy between the sun and the other bodies of the system, at least so far as the phosphorescent quality of the moon is concerned, is, that light is not given off in any sensible degree from the crescent which is unenlightened by the sun, just before and after opposition.

The attempt to remove an objection to the sun's being inhabited by supposing that "heat is produced by the sun's rays only when they act on a calorific medium," and that they are the cause of heat only "by uniting with the matter of fire which is contained in the substances that are heated," together with the arguments advanced in support of these strange positions, certainly ought, for the credit of one who has deserved so highly of astronomical science, to have been suppressed. They are too far behind the present state of Chemistry, and too little essential to the object which their author had in view, to deserve transcribing into the pages of an elementary work, which is intended to be employed in instruction.

In passing to the *Appendix*,—our limits will not allow us to notice a variety of errors which occur in the progress of the examples; nor a number of small inaccuracies unnecessarily introduced into the mode of projecting solar eclipses. The tables of epochs (which terminate with the present year) should have been extended; and might also have been advantageously corrected from those of Delambre and Burckhardt.—But the most important positive error, perhaps, which occurs in the Appendix, relates to the method of finding the arguments of the moon's latitude. In Ewing's Astronomy and all the editions of Enfield except the last, we have given, over the IIId table, "Arg. I—☽'s mean anomaly;" and over the Vth, "Arg. IV—☉'s mean anomaly." Both these arguments are wrong. Those who may have the curiosity to look into Mason's edition of Mayer's Lunar Ta-

* Hence the 93d Proposition, which ascribes the light transmitted to the moon's disc during a total eclipse to "the *reflection* of rays of light falling upon the earth's atmosphere," is doubtless in part correct.

bles, from which Ewing's were abridged, will see at a glance how these erroneous captions originated. They are in fact the 3d and 5th arguments of Mayer's tables; but Mayer's 3d table is omitted, and his 10th is made Ewing's 5th. The captions were inadvertently copied, although they belonged, in consequence of these omissions, to the wrong tables. In the last edition, the caption of the third table is altered to make it agree with the general directions for finding the arguments of Latitude given in Prob. 8th; but that of the 5th still remains erroneous, as well as the general rule under Prob. 8th. It should be, "subtract the moon's mean anomaly from the *second* argument," &c.; and the caption of table 5th should be, "Arg. II — 's mean anomaly."

The principal part of the corrections and alterations made by the editor of the last edition have our entire approbation. Particularly in regard to two highly important propositions, the one relating to the law of refraction, in the Optics, and that on the sun's parallax, in the Astronomy, he has probably done the best that the elementary character of the work admitted. There are a few instances, however, of alterations, the propriety of which appears very questionable, and which justice to the labours of former editors requires us briefly to notice.

Thus in the first proposition, "Matter *may be, and mere extension* is infinitely divisible," the clause in italics is peculiar to the last edition. We recollect having seen in Hutton's Dictionary an attempt to establish a distinction between "actual" and "potential divisibility;" but we could not understand it; nor are we any more fortunate in regard to the language just quoted. If the term "divisibility" itself means nothing more than the possibility of being divided, to say that matter *may be* infinitely divisible is a solecism. The distinction between the divisibility of matter and that of extension seems to depend on the definition of the term. If by "divisibility" be meant merely the possibility of being ideally divided by mathematical planes without any separation of parts, then the property is one which belongs to matter and to extension in precisely the same sense. But if in the phrase "divisibility of matter" be included the additional idea of discerptibility, or the possibility of being separated into parts not in contact, then the property is one which belongs *in no degree* to pure extension. In neither case does the distinction made in the proposition as quoted above appear to have any foundation. That matter *is* infinitely divisible in the first sense, is almost self-evident: whether it is so in the last, (admitting the exercise of any supposable power which does not change the nature of matter,) is a question which lies beyond the reach of the human faculties.

We notice, in the second place, that three experiments, on the approach of light bodies floating on water to each other, or to the side of the vessel, have been transferred from prop. 5, where they were originally placed to illustrate the cohesion between solids and fluids, to prop. 4, where, if they illustrate any thing, it must be the cohesive attraction between two solid bodies. It is true that these phenomena are only indirect consequences of the attraction between solids and fluids; and a scholium was very properly added by the author, (which has been omitted in all the subsequent editions,) to aid the student in tracing their connexion with the proposition. But it is most certain that they have nothing to do with the cohesive force of two solid bodies. When there is an elevation or depression of the fluid around both of two floating bodies, they will approach: when there is an elevation around one and a depression around the other, they will recede. These are mere results of capillary action; and as such, admit of an easy explanation from the general theory of Laplace.*—A popular idea of the mechanism of these phenomena will be gained from the following experiment, by which we have been much amused, and which we do not recollect to have seen noticed. Two small globules of mercury, carefully laid upon water, will swim. Let these globules be brought within one or two inches, and it is surprising to observe the rapidity with which they dart together. If one of the globules is forced to the edge of the water, (the vessel being of such materials as to be capable of being moistened,) it will recede with an activity which might seem the effect of animation. But on holding the vessel in the light, the secret of these motions will be apparent. Each globule will be seen to have a depression around it, which perceptibly extends to the distance of more than half an inch. The globules will be seen to rush together, not from any mutual attraction, but because, in doing it, each *descends down an inclined plane.* Two needles, laid on water and kept parallel to each other, will exhibit similar appearances.

In the Optics, under Prop. 18. Exps. 22, 23, 55, 56, &c. the term *focus*, as used to denote the

* See Méc. Céleste. Sup. au dix. Livre: Biot—Traité de Physique, I. 462.: Haüy I. 237.

point *as if* from which diverging rays proceed after refraction or reflection, is changed into *imaginary radiant.* The latter term is doubtless the most descriptive of the actual condition of the rays, and by some writers is uniformly employed instead of *virtual* or *negative focus.* But to introduce this distinction increases the complexness of enunciation of several important theorems which are already too complex.* It were to be wished, for the sake of these theorems, that we had some term which should merely express the point where the lines of direction of a pencil of rays meet, *before* refraction or reflection,—without including the idea of divergency or convergency; and another to denote the same thing *after* refraction or reflection. As long as this is not the case, we are not confident that any advantage is gained by changing the denomination of *focus*, when virtual, to *imaginary radiant.* But if the change is made at all, it ought at least to be carried through. This has not been done by the Editor: and the consequence is, that several propositions contain an implied error. He has inserted "imaginary radiant" after "focus" in prop. 55; but in props. 22, 23, and 56, which equally required a similar addition, and in prop. 54, which required a substitution, neither has been made. Such an addition would, it is true, have rendered the enunciation of some of these propositions exceedingly perplexed; but consistency demanded that it should be done, or that the language of former editions should be left unaltered.†

The publishers have also to acknowledge their obligations to the learned gentleman by whom the third edition was so much improved, for suggesting some additional corrections. These partly relate to points noticed in the preceding extract from Silliman's Journal. Such of them as were received in season, were adopted in the text. [See Cor. 4. Prop. xxvi. B. II. (transferred from Prop xii. B. vi.)—Problem, page 182—Problems viii. and xv. in the Appendix.] The rest of them follow here.

Add to the Scholium, Prop. 36, B. II. the following sentence. "When the body descends in a curve line, the deflection being less than any assignable angle, its cosine may be reckoned equal to radius, and the retardation as nothing."

Let Prop. 45. B. II. be omitted, and the 46th, with the two paragraphs which contain its demonstration, take its place; and let the one given below, with its demonstration and corollary, be inserted as Prop. 46. The schol. under Prop. 45, may follow the corollary given below as Schol. 1, while the two scholia under Prop. 46, may be numbered 2 and 3 respectively.

"Prop. 46. If a pendulum be made to vibrate in a cycloid, the time of a vibration, whether the arc be long or short. will be to the time in which a body will fall freely through half the length of the pendulum, as the circumference of a circle to its diameter.

Pl. 2. Fig 4. Let a body descend in a cycloid from B to X, and it will acquire the same velocity as a body falling freely from D to X (by Prop. 36), that is, a velocity which, if uniformly continued during the time of the fall from D to X, would carry it through twice that distance (Prop. 27.) or BX. (Lem. 5. cor) But a body moving along BX tends towards X with a force every where proportional to its distance therefrom (Lem. 6.); and the time of describing a line with a motion accelerated by such a force, from whatever point of the line it commence its motion, or *half the time of a*

*Such in Enfield's Optics, are props. 21, 23, 54, 56.

† It must be admitted that the language of former editions, in this respect, was not entirely consistent with itself. Defs. 8 and 18, and the Schol. to def. 13, needed modification.

vibration (by Prop. 38.) : the time of describing the same line with the velocity acquired at X continued uniformly through the line, or *the time of falling through DX, half the length of the pendulum* :: *half the circumference of a circle* : *its diameter*. And, doubling the antecedents, the whole time of a vibration : the time of falling through half the length of the pendulum :: the whole circumference of a circle : its diameter.

Cor. From this analogy the lengths of pendulums and the spaces described by falling bodies may be compared, and either of them computed from the other. For since the spaces described by falling bodies are as the squares of the times, reckoning both from the beginning of the descent (Prop. 26.), then the square of the time of one vibration of a pendulum : the square of the time in which a body would fall through half its length :: the square of the circumference of a circle : the square of its diameter :: 9.8696 : 1 :: the space described by a falling body in any given time from the beginning of its fall : half the length of a pendulum which performs a vibration in the same time. Or inversely (as the length of a pendulum and time of its vibration are more easily and more accurately ascertained by experiment than the velocity of falling bodies,) 1 : 9.8696 :: half the length of a pendulum : the space through which a body will fall while it performs one vibration."

On page 234, omit lines 26–30, and let the next paragraph begin with "The" instead of "Another."

ERRATA.

Page	Line						
83	15	from the top,	for	*to the magnet,*	read	*of the magnet.*
100	12	— bottom,	—	*zinc,*	—	*the zinc.*
108	2	— top,	—	*it,*	—	*if.*
125	15	— bottom,	—	*distance,*	—	*distances.*
166	19	— top,	—	*latitude,*	—	*altitude.*
167	16	— —	—	15″ *one minute,*	—	15′ *one minute.*
187	7	— bottom,	—	*is will,*	—	*it will.*
193	3	— —	—	1 58,	—	0 58.

at Page — 158 — 11 — from Altitude read altitude

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BOOK I.

OF MATTER.

DEFINITION I. **MATTER is an extended, solid, inactive, and moveable substance.**

SCHOLIUM. Extension and solidity are discovered to be properties of matter by the senses. Both by the sight and touch we perceive material substances to have length, breadth, and thickness, that is, to be extended: and from the resistance which they make to the touch, we acquire the idea, and infer the property of solidity. It is unnecessary here to inquire, whether solidity necessarily supposes impenetrability. Natural Philosophy, being employed in investigating the laws of nature by experiment and observation, and in explaining the phenomena of nature by these laws, has no concern with metaphysical speculations, which are generally little more than unsuccessful efforts to extend the boundaries of human knowledge beyond the reach of the human faculties.

DEF. II. **A body is any portion of matter.**

COROLLARY. All bodies have some figure; for, being portions of matter, they are finite, and therefore bounded by lines either straight or curved.

PROPOSITION I.

Matter may be, and mere extension is, infinitely divisible, or capable of being divided beyond any supposed division.

1. Any particle of matter, placed upon a plane surface, has an upper and a lower side, or a part which touches, and another which does not touch the plane, and is therefore divisible.

2. Let CO, MD, be two parallel right lines, to which let AB be drawn perpendicular. In the line MD, on one side of the perpendicular AB, take, at equal distances, the points E, F, G, H. On the other side of AB, in the line CO, take any point C, and join CE, CF, &c. Each of the lines CE, CF, &c. will cut off a portion from AB: But whatever number of lines be drawn in the same manner from C to MD produced, there will still remain a portion of AB not cut off; because no line can be drawn from the point C to the line MD, which shall coincide with CO: The line AB is therefore infinitely divisible. Plate 1. Fig. 1.

3. Let the right lines AC and GH be drawn perpendicular to the right line BF. In AC, produced at pleasure, take any points C, C, &c. from which, as centres, with the distances CA, CA, &c. describe arcs of circles KAL, NAO, &c. touching BF in the point A, and cutting HG. The farther the central point is taken from A, the greater will be the circles, and the nearer will the arcs approach to the line BF; but (El. III. Pr. 16. Cor.) the arcs, touching BF in A, cannot touch it in any other point. The line HG is therefore infinitely divisible. Plate 1. Fig. 2.

From this proposition the following theorems are derived by Dr. Keill, in his fifth lecture.

THEOREM I. Any quantity of matter, however small, and any finite space, however large, being given, (as for example, a cube circumscribed about the orb of Saturn,) it is possible for the small quantity of matter to be diffused throughout the whole space, and to fill it so that there shall be no pore or interstice in it, whose diameter shall exceed a given finite line.

COR. Hence there may be given a body, whose matter, being reduced into a space absolutely full, that space may be any given part of its former magnitude.

Theorem ii. There may be two bodies of equal bulk, whose quantities of matter, being unequal, in any proportion; yet the sum of their pores, or quantity of void space in each of the two bodies shall be to each other nearly in a ratio of equality.—Example. Suppose 1000 cubic inches of gold when reduced into a space absolutely full, to be equal to one cubic inch: then 1000 cubic inches of water, which is 19 times lighter than gold, will, when reduced, contain $\frac{1}{19}$th part of an inch of matter. Consequently the void spaces in the gold, will be to those in water as 999 to $999\frac{18}{19}$, or nearly in the ratio of equality.

In the present state of knowledge, it is impossible to determine how far the division of matter can be actually carried, or whether there be any indivisible atoms by the arrangement and combination of which all sensible bodies are formed. We are, however, furnished both by art and nature with many astonishing instances of minute division.—If a pound of silver and a grain of gold be melted together, the gold will be equally diffused through the whole silver; and if a grain of the mass, containing only the 5761th part of a grain of gold, be dissolved in aqua fortis, the gold will subside.

A grain of gold may be spread by the gold-beater into a leaf containing 50 square inches, and this leaf may be divided into 500,000 parts: and by a microscope, magnifying the diameter of an object 10 times, and its area 100 times, the 100th part of each of these, that is, the 50 millionth part of a grain of gold, will be visible.

The *natural* divisions of matter are still more wonderful. In odoriferous bodies a surprising subtilty of parts is perceived: several bodies scarcely lose any sensible part of their weight in a great length of time, and yet continually fill a very large space with odoriferous particles. Dr. Keill has computed the magnitude of a particle of assafœtida to be only the $\frac{38}{1,000,000,000,000,000,000}$th part of a cubic inch. Again, Mr. Leewenhoeck informs us that there are more animals in the milt of a cod-fish than there are men on the whole earth, and a single grain of sand is larger than four millions of these animals. Moreover, a particle of the blood of one of these animacula has been found, by calculation, to be as much smaller than a globe of the $\frac{1}{10}$th of an inch in diameter, as that globe is smaller than the whole earth. Nevertheless, if these particles be compared with the particles of light, they will be found to exceed them in bulk as much as mountains do single grains of sand.

These instances may serve to show the amazing fineness of the parts of bodies, which are nevertheless compounded. Gold, when reduced to the finest leaf, still retains those properties which arise from the modifications of its parts. Microscopic animalcula are, without doubt, organized bodies, and the globules of their blood are possessed of specific qualities. Even the rays of light are compounded of an indefinite variety of particles, which, when separated, have the power of exciting the ideas of colours.

Def. III. That force by which the parts of the same body, or of different bodies, on their contact, or near approach, are united to or tend towards each other, is called the *attraction of cohesion.*

PROP. II. The attraction of cohesion appears in solid bodies.

Exp. 1. Observe the different degrees of cohesion in different kinds of wood, suspending weights, from pieces of equal diameter, placed vertically or horizontally till they break.

2. Measure the different degrees of cohesion in silk, thread, horse-hair, &c. by weights suspended by cords of each, placed vertically or horizontally.

The result of sundry experiments, made by professor Musschenbroek, to show the cohesive power of different solids, may be seen in the following table. In estimating the absolute cohesion of solid bodies, he applied weights to separate them according to their length: the pieces of wood which he used were parallelopipedons, each side of which was $\frac{27}{100}$ths of an inch, and the metal wires made use of were $\frac{1}{10}$th of a Rhinland inch in diameter, and they were drawn asunder by the following weights:

	lb.		lb.
Fir	600	Copper	$299\frac{1}{4}$
Elm	950	Brass	360
Alder	1000	Gold	500
Oak	1150	Iron	450
Beech	1250	Silver	370
Ash	1250	Tin	$49\frac{1}{4}$
		Lead	$29\frac{1}{4}$

From the experiments of Guyton Morveau, the following are the utmost weights, which wires of 0.787 of an English line in diameter can support without breaking:

		lb. Av.
A wire of Iron supports	- - - -	549.25
Copper	- - - - - - -	302.278
Platinum	- - - - -	274.32
Silver	- - - - - -	187.137
Gold	- - - - - -	150.753
Zinc	- - - - - -	109.54
Tin	- - - - - -	34.63
Lead	- - - - - -	27.621

PROP. III. The attraction of cohesion takes place between particles of the same fluid.

Exp. 1. A drop of water, at the end of a small cylinder of wood, will hang in a spherical form The drop is spherical, because each particle exerts an equal power in every direction, drawing other particles towards it on every side as far as its power extends.

2. Two globules of mercury, on meeting, unite, which union can arise only from their strong attraction. Drops of water will do the same.

PROP. IV. The attraction of cohesion takes place between two solid bodies of the same kind; and the more perfect the contact, the greater is the attracting force.

Exp. 1. Two glass bubbles, floating near each other on water, rush together.

2. A glass bubble floating on water in a glass vessel moves towards the side of the vessel.

3. Two circular pieces of cork, placed upon water, and brought near each other, will be attracted.

4. Two plates of glass laid together, though perfectly dry, will cohere.

5. Two leaden balls, having each a flat surface of a quarter of an inch in diameter, scraped smooth, on being forcibly put together, will cohere so strongly as sometimes to require a weight of nearly 100lb. to separate them.

6. Two polished plates of brass, smeared with oil, will cohere strongly.

M. Musschenbroek found that the adhesion of polished planes, about two inches in diameter, heated in boiling water, and smeared with grease, required the following weights to separate them:

		Cold Grease.	Hot Grease.
Planes of Glass	- - - -	130lb.	300lb.
Brass	- - - -	150	800
Copper	- - - -	200	850
Marble	- - - -	225	600
Silver	- - - -	150	250
Iron	- - - - -	300	950

PROP. V. The attraction of cohesion takes place between solids and fluids.

Exp. 1. A plate of glass, or metal, will retain drops of water, or mercury, when inverted.

2. If a plate of glass be in part immersed in a vessel of water, the water which lies contiguous to the glass will rise above the level.

3. Water rises above its level between two parallel plates of glass at a small distance from each other, and in a glass tube having a fine bore, called a capillary tube.

4. The fluid will rise between parallel plates, and in capillary tubes, in *vacuo*. Hence it appears, that the ascent of fluids in capillary tubes is not owing to the pressure of the air.

5. Human blood will rise to a great height in a tube having an exceedingly fine bore.

6. Water will ascend in the cavities of sponge, sugar, and other porous bodies.

7. If a drop of oil be poured upon a plate of glass laid horizontally, and another plate of glass be so placed as to meet the first plate at one edge, and be at such a distance from it at the other, as just to touch the drop of oil; this drop, because its touching surface is continually enlarging, will move, with increasing velocity, towards that edge. If the planes be lifted up on the side where they meet, the motion will be retarded, stopped, or reversed, according to the degree of elevation.

8. The same phenomenon takes place in a tube of unequal bore.

9. A circular piece of ice, two inches and a half in diameter, exactly balanced and brought to touch the surface of some mercury, will be so strongly attracted by the mercury, as to require more than nine pennyweights, in the opposite scale, to restore the equilibrium.

10. A piece of wood having a smooth and plane surface, suspended from a beam and balanced, on touching a surface of water, will be attracted; and it will require an additional weight in the opposite scale to separate them.

Scholium 1. As it is by the attraction of cohesion that the parts of a body are kept together; so when a body is broken, this attraction is only overcome. Hence the reason of the soldering of metals, gluing of wood, &c. Hence also may be explained why some bodies are *hard*, others *soft*, and others *fluid*, which properties may result from the different figures of the particles, and the greater or less degree of attraction consequent thereupon. *Elasticity* may arise from the particles of a body, when disturbed, not being drawn out of each other's attraction; as soon therefore, as the force upon it ceases to act, they restore themselves to their former position.

Scholium 2. Solids are dissolved in menstruums from the particles of the solid being more attracted to the fluid than to themselves. Precipitation arises from a like cause; for if to the solution of any solid in a fluid, some other solid or fluid be added, the particles of which are attracted by the fluid with a greater force than those of the solid which was dissolved, the solid falls to the bottom in a fine powder. Thus silver dissolved in aqua fortis is precipitated by copper.

PROP. VI. The heights to which a fluid arises between parallel plates of glass are inversely as the distances of the plates.

The absolute attractive force of the plates will always remain the same, whatever be the distance of the plates. The same weight of fluid must, therefore, at different distances of the plates, be supported. But the quantity of fluid supported can only continue the same, when the height of the column supported is reciprocally as its base; that is, when as much as the height is increased the base is diminished, and the reverse. Now, the length of the base remaining unvaried, the base can only be made greater or less, by increasing or diminishing the distance between the plates. Therefore, the force, and the quantity of fluid supported, remaining the same, the height will be greater as the distance of the plates is less, and the reverse.

Let H, B, D, express the height, base, and distance, when the plates are at any given distance, and h, b, d, express the same when they are brought nearer: from what has been shown, $H : h :: b : B$; but $b : B :: d : D$; therefore $H : h :: d : D$.

Exp. Let two parallel plates of glass be immersed, at different distances from each other, in a vessel of coloured water.

PROP. VII. The suspension of the fluid, in capillary tubes, is owing to the attraction of the ring of glass contiguous to the upper surface of the fluid.*

Every ring of glass below the surface attracts the water above it as much downwards, as it attracts the water below it upwards, and consequently can contribute nothing towards the support of the column: and the action of the lowest ring upon all the fluid in the tube, within its surface of attraction, must either concur with the force of gravity to bring the fluid downwards, or, acting upon it at right angles, can have no effect in suspending it within the tube. The fluid therefore can only be supported by the ring of glass contiguous to its upper surface, which, attracting upwards, opposes the action of gravitation by which the fluid endeavours to descend. This reasoning may be applied to the fluid raised between parallel plates of glass.

Exp. Let a capillary tube be composed of two parts, the bore of one of which is wider than that of the other: immerse its wider orifice in water, till it is filled to any height less than the length of the wider part; the fluid will only rise to the height to which it would rise if the tube were throughout of the same bore with the wider part: but immerse the tube till the fluid enters the smaller part, and the whole column will be suspended, provided its length do not exceed that of the column which a tube of the smaller bore is capable of supporting.

Hence it is manifest, that the water is sustained by the attraction of the narrower part of the tube, for the wider part could not sustain so long a column: it is also manifest, that it is sustained by the ring contiguous to the upper surface; for if it were sustained by the ring at the lower surface, no reason could be assigned why this should now support the greater column in both parts of the tube, when it was before only able to sustain a column which filled a part of the wider tube.

Next, let the tube be inverted, and the water be raised into the lower extremity of the wider part; when the suspended column is of greater length than that which a tube of the same bore with the wider part is capable of sustaining, it will immediately sink: whence it is manifest, that the suspension of the column in this case depends upon the attraction of the wider part of the tube; for the narrower part could sustain a larger column: and also, that it is sustained by the ring contiguous to the upper surface;

* This proposition has been disputed. Dr. Hamilton, in his second lecture, supposes that the suspension arises from the attraction of the annulus lying just within the lower orifice of the tube. But Mr. Parkinson rejects both suppositions, and concludes, that the fluid is sustained by the immediate attraction of the glass. See Parkinson's Hydrostatics, p. 39.

for if it were sustained by the ring at the lower surface, it has been seen that this ring could support a much longer column.

Schol. The reason why the narrower or wider ring sustains a column of the same length in the unequal tube above described, as in a tube throughout of the same diameter as the upper ring, is that the moving forces of the columns are in both cases the same; as will be more fully shown hereafter. Book III. Pr. iv. Schol.

PROP. VIII. In capillary tubes, the heights to which the fluid rises are inversely as the diameter of the bores.

The fluid being suspended (Prop. VII.) by the ring of glass contiguous to the upper surface, and the distance to which the attracting force of glass reaches being unvaried; the attracting force which sustains the fluid will be as the number of attracting particles, that is, as the circumference, or diameter of the ring, or of the tube. Let Q, q, then, represent the quantities of fluid to be raised in two tubes of different bores; D, d, the diameters of their bores; and H, h, the heights to which fluids rise in the tubes; because Q, q, represent two cylinders of the fluid, from the properties of the circle and cylinder (El. XII. 2. II, and 14) $Q : q :: DDH : ddh$; and from the nature of this attraction, which is as the diameters of the tubes, $Q : q :: D : d$; therefore $DDH :: ddh :: D : d$; and consequently $D : d :: h : H$.

Cor. From this proposition it appears, that in any glass capillary tube, the height to which it will elevate water, and keep it sustained, multiplied into the diameter of the tube, is a given quantity; this is found by experiment to be .053 part of an inch; by means of this value the diameter of a capillary tube being given, the height to which it will elevate water will be known, for it will be equal to .053, divided by the diameter; thus suppose the diameter is $\frac{1}{20}$ of an inch, the height to which the water will be elevated $= .053 \times 20 = 1.06$.

Exp. Let two tubes of different bores be immersed in a vessel of coloured water; it will be found, that the water will rise as much higher in the smaller tube, as the diameter of its bore is less than that of the larger tube.

PROP. IX. Between two glass plates, meeting on one side, and kept open at a small distance on the other, water will rise unequally; and its upper surface will form a curve, in which the heights of the several points above the surface of the fluid will be to one another reciprocally, as their perpendicular distances from the line in which the plates meet.

Let AE be the surface of the fluid; AF the line in which the plates meet; HL the curve formed by the surface of the raised fluid; GB, IC, KD, LE, perpendicular to AE, expressing the heights of the respective points G, I, K, L, in the curve, above the surface of the fluid, and AB, AC, AD, AE, perpendiculars to AF, expressing the distances of the same points from the line in which the plates meet: these heights and distances are reciprocally proportional. For let the lines GB, IC, KD, LE, represent pillars of fluid of an equal but exceedingly small breadth; those portions of the glass plates, which by their attraction, support these pillars being equal, will sustain equal quantities of fluid; that is, the pillars will be equal. But the pillars may be considered as parallelopipeds, which (El. XI. 34) are equal when their bases and altitudes are reciprocally proportional. And the bases, being equal in breadth, are as their lengths, that is, as the intervals between the plates: and since the intervals continually increase as the distance from the line in which the plates meet increases, these intervals, at the points B, C, D, E, are as their distances AB, AC, AD, AE, from the line AF. Since, then, the heights of the pillars are reciprocally as the intervals, the heights GB, IC, &c. are reciprocally as the distances AB, AC, &c. This is the property of an hyperbola, whose asymptotes are AE and AF.

Plate I. Fig. 3.

Exp. Let coloured water rise between two glass plates (their inner surfaces being first moistened) meeting on one side according to the proposition.

PROP. X. Some bodies appear to possess a power the reverse of the attraction of cohesion, called repulsion.

Exp. 1. If a piece of iron be laid upon mercury, the surface of the mercury near the iron will be depressed.

2. A fine needle laid upon water will swim.

3. Two circular plates of tinfoil being placed upon water, and pressed down by a small additional weight upon their surface, repelling the water, will have a cavity round them: but when they are brought near each other, they will rush together; the reaction of the water on the outer side of the plates being greater than the reaction on the inner side, where the two cavities produced by repulsion are united.

4. Mercury, poured into a recurved glass tube, having the bore on one side exceedingly fine, and on the other large, will not rise so high in the narrow, as in the wide bore : water will rise higher.*

5. Melted glass dropped into water, forms globules with a stem, (called Prince Rupert's drops) which on breaking the stem will burst with great violence, and fall into powder.

PROP. XI. All bodies on or near the surface of the earth tend towards its centre, by the attraction of gravitation.

A stone or other heavy body, let fall, will move towards the earth till it meet with some other body to obstruct its course. And bodies move in lines perpendicular to the surface, because the point to which they ultimately tend is the centre of the earth, and the line of direction produced coincides with the radius, and is at right angles with the surface, which is nearly spherical. Some bodies ascend, because they are acted upon by a force greater than the attraction of gravitation, and in a contrary direction. Vapours, smoke, &c. do not descend, because they are lighter than the air, and supported by it.

Exp. 1. Smoke or steam will descend in an exhausted receiver.

2. Any boiling fluid being placed in a scale and balanced, the balance will be destroyed by evaporation.

Schol. 1. When we speak of attracting powers, we do not attempt to explain their nature, or assign their causes. Having derived general principles, or laws of nature, from phenomena, we only give a name to these principles, in order to explain other appearances by them.

Schol. 2. The tendency of all bodies towards the earth really results from their tendency towards the several parts of the earth. For by an experiment made by Dr. Maskelyne upon the side of the mountain Schehallien, he found the attraction of that mountain sufficient to draw the plumb-line sensibly from the perpendicular. See Phil. Trans. Vol. LXV. or Sir J. Pringle's Discourses.

* The phenomena exhibited by the four preceding experiments are rather to be considered as examples of *cohesion*, modified by circumstances. See American Journal of Science, &c. Vol. III. p. 133.

BOOK II.

OF MECHANICS, OR THE DOCTRINE OF MOTION.

CHAPTER I.

Of the General Laws of Motion.

PROPOSITION I.

EVERY body will continue in its state of rest, or of uniform motion in a right line, until it is compelled, by some force, to change its state.

Any body at rest on the surface of the earth will always continue so, if no external force be impressed upon it to give it motion, and if the obstacle which hinders the attraction of gravitation from carrying it towards the centre be not removed. A body being put into motion by some external impulse, if all external obstructions were removed, and the attraction of gravitation suspended, would move on forever in a right line; for there would be no cause to diminish the motion, or to alter its direction. This cannot be fully established by experiment, because it is impossible entirely to remove all obstructions; but, since the less obstruction remains the longer motion continues, it may be reasonably inferred, that if all obstacles could be removed, motion once communicated to any body would never cease.

Exp. 1. A body at rest requires some degree of force to put it in motion: and when in motion, it will continue to move longer on a smooth surface than on a rough one; instances of which occur in the use of friction rollers; in the exercise of skating, &c.

2. If a stone be whirled round in a string, on being set at liberty it will continue to move with the force which it has acquired.

3. If a vessel containing a quantity of water be moved along upon a horizontal plane, the water, resisting the motion of the vessel, will at first rise up in the direction contrary to that in which the moving force acts: when the motion of the vessel is communicated to the water, it will persevere in this state; and if the vessel be suddenly stopped, resisting the change from motion to rest, it will rise up on the opposite side. In like manner, if a horse which was standing still, suddenly start forwards, the rider will be in danger of being thrown backwards; if the horse stop suddenly, the rider will be thrown forwards.

Schol. This proposition suggests two methods of distinguishing between *absolute* and *apparent* motions. (1.) Absolute motion, or change of absolute motion, may sometimes be distinguished from apparent, by considering the *causes* which produce them. When two bodies are absolutely at rest, they are relatively so; and the appearance is the same when they are moving at the same rate, and in the same direction: a relative motion, therefore, can only arise from an absolute motion in one or both of the bodies, which (by the Prop.) cannot be produced but by force impressed. Hence, then, if we know that such a cause exists, and acts upon one of the bodies, and not upon the other, we may conclude that the relative motion arises from a change in the state of rest, or absolute motion of the former; and that with respect to the latter, the effect is merely apparent. Thus when a person on board a ship observes the coast receding from him, he knows the appearance arises from the motion of the ship upon which the wind or tide is acting.

(2.) Absolute motion may sometimes be distinguished from apparent by the *effects* produced. A body in absolute motion endeavours to proceed in the line of its direction: if the motion be only apparent, there is no such tendency. It is in consequence of the tendency to persevere in a rectilinear motion, that a body, revolving in a circle, constantly endeavours to recede from the centre. This effort is called a *centrifugal force;* and as it rises from absolute motion only, whenever it is observed, we are convinced that the motion is real.

Exp. Let a bucket, partly filled with water, be suspended by a string, and turned round till the string is considerably twisted; then let the string untwist itself. At first the water remains at rest, but

as it acquires the motion of the bucket, the surface grows concave to the centre, and the water ascends up the sides, thus endeavouring to recede from the axis of motion; and this effect increases till the water and bucket are relatively at rest. When this is the case, let the bucket be suddenly stopped and the absolute motion of the water will be gradually diminished by the friction of the vessel; and at length, when it is again at rest, the surface becomes plane. Thus the centrifugal force does not depend upon the relative, but upon the absolute motion, with which it begins, increases, decreases, and disappears.

PROP. II. The change of motion produced in any body is proportional to the force impressed, and in the direction of that force.

Effects are proportional to their adequate causes. If, therefore, a given force will produce a given motion, a double force will produce the double of that motion. If a new force be impressed upon a body in motion, in the direction in which it moves, its motion will be increased proportionally to the new force impressed: If this force act in a direction contrary to that in which the body moves, it will lose a proportional part of its motion: If the direction of this force be oblique to the direction of the moving body, it will give it a new direction.

Exp. Let one clay ball, suspended by a string, strike another clay ball suspended in the same manner, at rest or in motion, it will communicate a degree of motion greater or less in proportion to the force of the striking body: In the opposite direction, motion will be destroyed in the same proportion.

Cor. Since the effect produced by two bodies upon each other, depends upon their *relative* velocity, it will always be the same whilst this remains unaltered, whatever be their *absolute* motions.

PROP. III. To every action of one body upon another, there is an equal and contrary re-action: Or, the mutual actions of bodies on each other are equal and in contrary directions, and are always to be estimated in the same right line.

Whatever quantity of motion any body communicates to another, or whatever degree of resistance it takes away from it, the acting body receives the same quantity of motion, or loses the same degree of resistance in the contrary direction: the resistance of the body acted upon producing the same effect upon the acting body, as would have been produced by an active force equal to, and in the direction of, that resistance.

Cor. 1. Hence it appears, that one body acting upon another, loses as much motion as it communicates; and that the sum of the motions of any two bodies in the same line of direction cannot be changed by their mutual action.

Cor. 2. This proposition will explain the manner in which a bird, by the stroke of its wings, is able to support the weight of its body. For if the force with which it strikes the air below it, is *equal* to this weight, then the re-action of the air upwards is likewise equal to it; and the bird being acted upon by two equal forces, in contrary directions, will rest between them. If the force of the stroke be *greater* than its weight, the bird will rise with the *difference* of these two forces: And if the stroke be *less* than its weight, then it will sink with the *difference*.

Exp. Let a clay ball in motion strike another equal to it at rest: The striking body will lose half its quantity of motion, which will be communicated to the other body.

Schol. These three laws of motion may be illustrated by experiments, but their best confirmation arises from hence, that all the particular conclusions drawn from them agree with universal experience. They were assumed by Sir Isaac Newton as the fundamental principles of mechanics; and the theory of all motions deduced from them, as principles, being found to agree, in all cases, with experiments and observations, the laws themselves are considered as mathematically true.

CHAPTER II.

Of the Comparison of uniform Motions.

PROP. IV. The quantities of matter in bodies are in the compound ratio of their magnitudes and densities.

If the magnitudes of two bodies be given, the quantities of matter will be as the densities: If their densities be given, the matter will be as the magnitudes: therefore the matter is universally in the compound ratio of the magnitudes and densities. For example; If A and B be two balls equal in mag-

nitude, the quantity of matter in A will be to that in B, as the density of A is to that of B: if both be of the same density, their quantities of matter will be as their magnitudes.

PROP. V. The velocities, with which bodies move, are directly as the spaces they describe, and inversely as the times in which they describe these spaces.

It is manifest, that the degree of velocity increases as the space a body passes over in a given time increases, and as the time in which it passes over a given space decreases; and the reverse. For example; If one ball A move over six feet, and another ball B over three feet in the same time, A has double the velocity of B; but if the ball A passes over six feet in two seconds of time, and the ball B passes over six feet in one second, the velocity of B is double of that of A.

PROP. VI. The spaces which bodies describe are in the compound ratio of their times and velocities.

It is evident, that the longer time any body continues to move, and the greater velocity it moves with, the greater space it will pass through; and the reverse. If, for example, the body A move for one second, and the body B move for two seconds, both moving with the same velocity; A will move through half as much space as B: If A move with two degrees of velocity, and B with one degree of velocity; A will, in the same time, pass over twice as much space as B.

PROP. VII. The times in which bodies move are directly as the spaces, and inversely as the velocities.

The greater space any body passes through, and the less degree of velocity it moves with, the greater must be the portion of time taken up in the motion; and the reverse. For example; If the ball A move with the same velocity with the ball B, but pass over double the space, A will move twice as long as B; If A move over the same space with B, and with half the velocity, it must, in this case also, move twice as long as B.

PROP. A. If bodies be acted upon by different constant forces, the velocities communicated will vary in a ratio compounded of the forces and times.

Let F, V, T, represent force, velocity, and time, and be supposed variable; it is evident that the velocity will be increased and diminished in the same ratio with both force and time, and these being independent of each other, V will be as $F \times T$.

Cor. If, therefore, F be compared with any other known force f capable of generating a velocity equal to v in the time t, then $V : v :: F \times T : f \times t$.

PROP. VIII. The power required to move a body at rest is as the quantity of matter to be moved.

Each particle of matter in any body resisting motion, a force must be exerted upon each particle to overcome this resistance; if, therefore, two bodies containing different quantities of matter are to be moved, the greater body will require the greater force.

Def. I. The *momentum* of any body is its quantity of motion.

PROP. IX. In moving bodies, if the quantities of matter be equal, the momenta will be as the velocities.

It is manifest, that if the body A be equal to the body B, but A have twice the velocity of B, A has twice as much motion as B.

PROP. X. The velocities of two bodies being equal, their momenta will be as their quantities of matter.

The bodies A and B moving with equal velocities, since every portion of matter in A has as much motion as an equal portion of B, it is evident, that if A have twice the quantity of matter in B, it must have twice as much motion.

PROP. XI. The momenta of moving bodies are in the compound ratio of their quantities of matter and velocities.

The greater quantity of matter there is in any body, and the greater velocity it moves with, the

greater will evidently be its quantity of motion; and the reverse. If, for example, the body A be double of the body B, and move with twice its velocity, the momentum of A will be quadruple of that of B: For it will have twice the momentum of B from its double velocity, and also twice the momentum of B from its double quantity of matter.

Cor. Hence, if in two bodies the product of the quantities of matter and velocities are equal, their momenta are equal.

PROP. XII. The velocities of moving bodies are as their momenta directly, and their quantities of matter inversely.

The greater momentum any body has, and the less quantity of matter it contains, the greater must be its velocity. For example; If the body A be half of B, and their momenta be equal, A will move with twice the velocity of B; and if A and B are equal, and the momentum of A is double of that of B, its velocity will also be double.

PROP. XIII. The force, or power of overcoming resistance, in any moving body, is as its momentum.

Since a body having a certain degree of motion is able to overcome a certain degree of resistance, it is manifest, that with an increased momentum, it will be able to overcome a greater resistance.

Cor. Hence the momentum of any body is measured by its power of overcoming resistance.

Schol. Let Q, q, denote the quantities of matter in any two bodies, D, d, their densities, B, b, their bulk or magnitude, V, v, their velocities, T, t, the times of their motion, S, s, the spaces over which they pass, P, p, the moving powers, M, m, their *momenta*, and F, f, their force. The preceding propositions may be thus expressed:

Prop. IV.	$Q : q :: BD : bd$
V.	$V : v :: \frac{S}{T} : \frac{s}{t}$
VI.	$S : s :: TV : tv$
VII.	$T : t :: \frac{S}{V} : \frac{s}{v}$
A.	$V : v :: F\times T : f\times t$
VIII.	$P : p :: Q : q$
IX.	$M : m :: V : v$ if $Q=q$.
X.	$M : m :: Q : q$ if $V=v$.
XI.	$M : m :: QV : qv$
XII.	$V : v :: \frac{M}{Q} : \frac{m}{q}$
XIII.	$F : f :: M : m$

CHAPTER III.

Of the Composition and Resolution of Forces.

Def. A. Equable motion is either simple or compound. *Simple* motion is that which is produced by the action, or impressed force, of one cause. *Compound* motion is that which is produced by two or more conspiring powers, i. e. by powers whose directions are neither opposite nor coincident.

PROP. XIV. A body acted upon by two forces united, will describe the diagonal of a parallelogram, in the same time in which it would have described its sides by the separate action of these forces.

Plate I. Fig. 4. If in a given time, a body, by the single force M impressed upon it at a point A, would be carried from A to B; and by another single force N impressed upon it at the same point, would be carried from A to C; complete the parallelogram ABDC; and with both forces united, the

body will be carried in the same time through the diagonal of the parallelogram from A to D. For since the force N acts in the direction of the right line AC parallel to BD, this force (by Prop. II.) has no effect upon the velocity with which the body approaches towards the line BD. by the action of the force M. The body will therefore arrive at the line BD in the same time, whether the force N is impressed upon it or not; and at the end of that time will be found somewhere in the line BD. For the same reason at the end of the same time it will be found somewhere in the line CD; therefore it must be found at the point D, the intersection of these two lines. And (by Prop. I.) it will move in a right line from A to D.

Exp. Two equal leaden weights, suspended at the end of a triangular frame of wood to give them a steady motion, and let fall at the same instant from equal heights, striking a ball suspended by a cord at the point in which their lines of direction meet, will carry it forwards in the diagonal of the parallelogram of those lines produced.

Cor. 1. Hence, the velocity produced by the joint action of two forces is to that with which the body moves by the action of each force singly, as the diagonal of the parallelogram to either side; for the diagonal is described in the same time with either side.

Cor. 2. If two sides of a triangle represent the directions and quantities of two forces, the third side will represent the direction and quantity of a force equivalent to both acting jointly: For the third side may be considered as the diagonal of a parallelogram.

Cor. 3. A body may be moved through the same line by different pairs of forces. In plate 1. fig. 4. AD is the diagonal both to the parallelogram ABDC, and to the parallelogram AEDF; and consequently expresses a force equal to AB, AC, and to AF, AE.

Cor. 4. Hence we learn why any heavy body let fall perpendicularly from the top of a mast, when a ship is under full sail, will fall to the bottom, in the same manner as if it had been at rest.

Scholium. This proposition may be farther illustrated. If two men sit upon the opposite sides of a boat under full sail, and toss a ball to each other, they will catch the ball in their turn, just as they would have done if the boat had been at rest. The ball is here acted upon by two forces: (1.) it partakes of the motion of the boat, which is common to the ball, the boat, and the men: (2.) the other force is that with which the man throws it across the boat. By these two forces together, the ball will describe the diagonal of a parallelogram, one of whose sides is the line that the boat has described whilst the ball was flying across; and the other side is a line drawn from one man to the other.

PROP. XV. The velocity produced by two joint forces, when they act in the same direction, will be as the sum of the forces, and when they act in opposite directions, will be as their difference; and the velocity will be the greater, the nearer they approach to the same direction, and the reverse.

In the parallelograms ABCD, in which AB, AC, express the direction and quantity of two joint forces, the side AB, being placed at different angles with AC, it is manifest, that as AB approaches towards AC, the diagonal increases, till at length it becomes equal to AC+CD, that is, to AC+AB, and the velocity is as the sum of the forces, since they act in the same direction. Plate 1. Fig. 5.

In the parallelograms ABDC, as AB recedes from CD, the diagonal increases, till at length it vanishes with the angle, and the two sides AB, AC, constitute one right line, the parts of which, AB, AC, representing forces acting in opposite directions, if the forces be equal, they will destroy each other; if unequal, the velocity will be as their difference. Plate 1. Fig. 6.

PROP. XVI. Any single force or motion may be resolved into two forces or motions; and the directions of these may be infinitely varied: also any two forces may be compounded into single forces.

A body moving in the line AD, may be considered as receiving its direction and velocity from two forces acting jointly in the directions AB, AC, or from two other forces expressed by AF, AE. For (Prop. XIV. Cor. 3) each pair would produce the same effect. In like manner the direction and quantities of the forces will be diversified with every change of the sides of the parallelogram, the diagonal remaining the same. Plate 1. Fig. 4.

It is also manifest, that any two joint forces may be compounded into one, being expressed by the sides of a parallelogram, or its diagonal.

PROP. XVII. If a body is acted upon by three forces, which are proportional to, and in the direction of, the three sides of a triangle, the body will be kept at rest.

Let a body placed at D be acted upon by three forces AD. GD, FD, proportional to, and in Plate 1. Fig. 7.

the direction of, the three sides of the triangle GED: complete the parallelogram GEFD; and make AD equal to, and in the direction of, the diagonal ED.

If the body at D be acted upon by the forces AD, ED, equal and in opposite directions, it will be kept at rest. But the force ED (Prop. XVI.) is equivalent to the two forces DG, DF, that is, DG, GE; therefore the body acted upon by the three forces AD, DG, DF, that is, by three forces proportional to, and in the direction of, the sides of the triangle GED, will be at rest.

Exp. Let three weights in the proportions of 3, 4, 5, be suspended by cords, which pass over pulleys and meet in a point; if the directions of the cords be parallel to the sides of a triangle (drawn in a plane parallel to the plane of the cords) whose sides are to each other as the weights, a ball at the point in which the cords meet will be kept at rest.

Cor. The body will be at rest if the three forces are proportional to the three sides of a triangle drawn perpendicular to the direction of the forces; for such a triangle is similar to the former. Draw A *g*, C *d*, and B *e*, perpendicular to the sides GE, GD, DE, forming a triangle *g e d*, which is equiangular to GED; hence, the sides about their equal angles being proportional, the forces which are proportional to the lines GE, GD, and DE, are also proportional to *g e*, *g d*, and *d e*.

Scholium. A boy's kite, as it rests in the air, is an instance of a body resting whilst three forces act upon it. For the kite is acted upon by the wind; by its own weight; and by the string that holds it.

PROP. XVIII. The force of oblique percussion is to that of direct or perpendicular action, as the sine of the angle of incidence to radius.

Plate 1. Fig. 8. Let a body strike upon the plane AD, at the point D, in the direction BD: the line BD expressing the force of direct impulse may be resolved into two others, BC, BA, the one parallel, the other perpendicular to the plane. Of these, the force BC, parallel to the plane, cannot affect it. The whole force upon the plane may therefore be expressed by BA. But BA is to BD as the sine of the angle of incidence BDA is to radius.

Schol. If the surface to be struck be a curve, let AD be made tangent to the curve at D, and the proof will be the same.

PROP. XIX. The force of oblique action produced by percussion is to that of direct action, as the cosine of the angle, comprehended between the direction of the force and that in which the body is to be moved, to radius.

Plate 1. Fig. 9. Let FD represent a force acting upon a body at D, and impelling it towards E; but let DM be the only way in which it is possible for the body to move. The force FD may be resolved into two forces FG, FH, or GD; of which only the force GD impels it towards M. And, FD being radius, GD is the cosine of the angle FDG, or MDE, comprehended between the direction of the force, and that in which the body is to be moved.

CHAPTER IV.

Of Motion, as communicated by Percussion in Non-Elastic and Elastic Bodies.

Def. II. Bodies are non-elastic, which, when one strikes another, do not rebound, but move together ofter the stroke.

Cor. Hence their velocities after the stroke are equal.

Def. III. Bodies are elastic, which have a certain spring, by which their parts, upon being pressed inwards by percussion, return to their former state, throwing off the striking body with some degree of force; when the elasticity is perfect, the body restores itself with a force equal to that with which it is compressed.

Exp. The existence of this property is visible in a ball of wool, cotton, or sponge compressed.

PROP. XX. When one non-elastic body in motion, strikes upon another at rest, or moving with less velocity in the same direction, the sum of their momenta remains the same after the stroke as before.

For (Prop. III. Cor. 1.) as much motion as the striking body communicates, so much it loses; whence, if the motions of the bodies are in the same direction, whatever is added to the motion of the preceding body will be subducted from that which follows, and the sum will remain the same.

PROP. XXI. When two non-elastic bodies, moving in opposite directions, strike upon each other, the sum of their momenta, after the stroke, will be equal to the difference of their momenta before the stroke.

For (from Prop. III. Cor. 1.) that body which had the least motion will destroy a quantity equal to its own in the other; after which they will move together with the remainder, that is, the difference.

Exp. Let two cylinders filled with clay, A, B, be of equal weight, and suspended by cords from equal heights; let two other cylinders of the same kind, C, D, but in weight as 2 to 1, be suspended from the same height. The heights from which they are let fall, in the arc formed by the motion of the cylinder (from the nature of the pendulum, afterwards to be explained) will be the measure of their velocity; and (by Prop. XI.) their momenta will be as their velocities multiplied into their quantities of matter; whence the cases of the two preceding propositions may be established by the following experiments. N. B. Quantity of matter is expressed by q, velocity by v, and momentum by m.

No. 1. Prop. XX. Case 1. Let the cylinder A fall from the height of 18 inches, upon the cylinder B at rest. The momentum of A before the stroke (by Prop. XI.) is 18; for the quantity of matter is 1, and the velocity 18; whence $q\ 1 \times v\ 18 = m\ 18$. After the stroke, the quantity of matter being (Def. II.) 2, and the velocity of each cylinder 9, the momentum will be 18; $q\ 2 \times v\ 9 = m\ 18$.

No. 2. Case 2. Let A fall from 18 inches, and B from 9, in the same direction; their momenta before the stroke are $18 + 9 = 27$; after the stroke, the quantity of matter will be 2, and the velocity $13\frac{1}{2}$; whence $v\ 13\frac{1}{2} \times q\ 2 = m\ 27$.

No. 3. Prop. XXI. Case 1. Let the equal cylinders A and B fall in opposite directions, from the height of 12 inches; the momenta being equal and opposite, the motion of both will be destroyed.

No. 4. Case 2. Let A fall from the height of 12 inches, and meet B falling in the opposite direction from 6 inches; their velocity after the stroke being 3, and quantity of matter 2, the momentum will be 6; $q\ 2 \times v\ 3 = m\ 6$.

No. 5. Prop. XX. Case 3. Let the cylinder C, double of the cylinder D, fall from 12 inches on D at rest. Before the stroke, the quantity of matter in C is 2 and its velocity is 12; whence its momentum is 24; $q\ 2 \times v\ 12 = m\ 24$. After the stroke, the velocity will be 8, and quantity of matter 3; whence $q\ 3 \times v\ 8 = m\ 24$.

No. 6. Case 4. Let C fall from 12 inches, and D from 6 inches in the same direction. Before the the stroke, the velocity of C is 12, and quantity of matter 2; whence its momentum is 24; $q\ 2 \times v\ 12 = m\ 24$; and the velocity of D is 6, and its quantity of matter 1; whence $q\ 1 \times v\ 6 = m\ 6$; therefore the whole momentum is 30. After the stroke, the velocity of the whole is 10, and the quantity of matter 3; whence $q\ 3 \times v\ 10 = m\ 30$.

No. 7. Prop. XXI. Case 3. Let C fall from 6 inches, and D from 12, in opposite directions, the quantity of matter in C being 2, and its velocity 6; and the quantity of matter in D being 1, and its velocity 12, their momenta will be equal, and being opposite, will destroy each other. C $q\ 2 \times v\ 6 = m\ 12$; D $q\ 1 \times v\ 12 = m\ 12$.

No. 8. Case 4. Let C fall from 3 inches, and D from 12, in opposite directions: Before the stroke, the momentum of C is 6; $q\ 2 \times v\ 3 = m\ 6$, and the momentum of D is 12; $q\ 1 \times v\ 12 = m\ 12$; whence the difference of their momenta is 6. After the stroke, the velocity is 2, and quantity of matter 3; whence the momentum is 6; $q\ 3 \times v\ 2 = m\ 6$.

PROP. XXII. When one elastic body strikes upon another of the same kind, the one loses, and the other gains, twice as much momentum, as if the bodies had been void of elasticity.

For since (by Def. III.) perfectly elastic bodies, on percussion, restore themselves with a force equal to that with which they are compressed, whatever momentum is gained by one body, or lost by the other, on percussion, from the law of re-action, the same must be gained, or lost, from the power of elasticity.

Cor. 1. If one of the bodies, considered as non-elastic, would lose more than half its momentum, as elastic, it loses more than all, that is, acquires a negative momentum in a contrary direction.

Exp. The following experiments may be made with ivory balls suspended by strings; they correspond with the preceding experiments on non-elastic bodies.

Let A and B be equal balls; and let C be a ball double of the ball D.

No. 1. A, falling from 18 inches on B at rest, has 18 degrees of momentum before the stroke; therefore, after the stroke, supposing the balls non-elastic, the same momentum belonging to the two equal balls together, each has 9 degrees of momentum, and A has lost and B gained 9. This being doubled, A, as elastic, will lose 18, and B will gain 18 degrees of momentum: whence A will be at rest, and B will move with 18 degrees of momentum.

No. 2. A, falling from 18 inches, and B from 9 in the same direction; as non-elastic after the stroke, each has $13\frac{1}{2}$ momentum, or A has lost $4\frac{1}{2}$, and B gained $4\frac{1}{2}$. As elastic, after the stroke, A loses 9, and B gains 9; therefore A rises to 9 inches, B to 18.

3. A and B, falling in opposite directions from 12 inches, as non-elastic, would lose all their momentum; as elastic, each loses 24 degrees of momentum; that is, gains 12 in the contrary direction.

No. 4. A, falling from 12 inches, and B in the opposite direction from 6, as non-elastic, the momentum of each, after the stroke, will be in the direction of A; whence A loses 9, and B loses 9, moving 3 degrees in the contrary direction. As elastic, A loses 18, or has 6 in the contrary direction, and B loses 18, or gains 12 in the contrary direction.

No. 5. C, double of D, falling from 12 inches on D at rest, the momentum of C, before the stroke, being 24, and of D nothing; as non-elastic, C, after the stroke, having its momentum 16, and moving with the velocity 8, will have lost 4 degrees of velocity, and 8 of momentum: and D will have gained 8 of each. As elastic, therefore, C will lose 8 degrees of velocity, or (Prop. XI.) 16 of momentum, and D will gain 16 of each; that is, C will move with 4 degrees of velocity, and D with 16.

No. 6. C, falling from 12 inches, and D from 6 in the same direction, before the stroke, the velocity of C is 12, and its momentum 24; and the momentum of D 6. After the stroke, as non-elastic, the momentum of C is 20, because $q\,2 \times v\,10 = m\,20$; and the momentum of D is 10, because $q\,1 \times v\,10 = m\,10$; therefore C has lost 4 degrees of momentum, or 2 degrees of velocity, and D gained 4 of each. If, therefore, the gain or loss be doubled on account of the elasticity of C and D, C will lose 8 degrees of momentum, or 4 of velocity, and D will gain 8 of each; that is, C will move with 8 degrees of velocity, and D with 14.

No. 7. C, falling from 6 inches, and D from 12, in opposite directions, their momenta, being equal, would destroy each other without elasticity: Therefore, being elastic, each will acquire the momentum of 12 in opposite directions; that is, D will return to 12 and C to 6.

No. 8. C, falling from 3 inches, and D from 12 in opposite directions; since the momentum of C, before the stroke, is 6, and of D 12, as non-elastic bodies they would, after the stroke, move in the direction of D, with the velocity of 2; whence C would move in the direction contrary to its first motion with 4 degrees of momentum, and lose 10; and D would lose 10: Therefore, being elastic, C will lose 20 degrees of momentum, and also D 20; whence C will move in the contrary direction with 14 degrees of momentum; that is, will return to 7; and D will return to 8.

Cor. 1. If the sum of two conspiring momenta, or the difference of two contrary momenta, be divided by the sum of the quantities of matter in both the moving bodies, the quotient will give the common velocity after the stroke.*

Schol. Let A and B be two spherical bodies, moving with their centres in the same line; and let their velocities be a and b. The momentum of A, before the stroke, is $A\,a$, and that of B is $B\,b$; their sum, or their difference, is $A\,a + B\,b$, or $A\,a - B\,b$. Therefore (by Prop. XX. and XXI.) the momentum, after the stroke, is expressed by $A\,a \pm B\,b$, and their common velocity by $\frac{A\,a \pm B\,b}{A + B}$. Hence the momentum of A, after the stroke, is $\frac{AA\,a \pm AB\,b}{A + B}$; and that of B is $\frac{AB\,a \pm BB\,b}{A + B}$.

Next, suppose the bodies perfectly elastic. Subtract the momentum of A considered as non-elastic, after the stroke, $\frac{AA\,a \pm AB\,b}{A + B}$, from its momentum, before the stroke, $A\,a$; and the remainder, $\frac{AB\,a \mp AB\,b}{A + B}$, will express the momentum in that case lost by A, and gained by B. Subtract this remainder, $\frac{AB\,a \mp AB\,b}{A + B}$, from the momentum of A, as non-elastic, after the stroke, $\frac{AA\,a \pm AB\,b}{A + B}$; and add the same remainder to the momentum of B, after the stroke, $\frac{AB\,a \pm BB\,b}{A + B}$: the difference, $\frac{AA\,a \pm 2\,AB\,b - AB\,a}{A + B}$, will express the momentum of A, after the stroke, and the sum

$$\frac{2\,AB\,a \pm BB\,b \mp AB\,b}{A + B}$$

* [This corollary belongs to the preceding proposition; for it is true only of non-elastic bodies.]

will express the momentum of B, after the stroke, supposing them perfectly elastic. And $\frac{A a \pm 2 B b - B a}{A + B}$, and $\frac{2 A a \pm B b \mp A b}{A + B}$, will express their respective velocities.

Cor. 2. If there be any number of elastic, equal, and spherical bodies, whose centres are placed in the same line, and the first body strikes upon the second in the direction of that line, all the bodies will be at rest except the last, which will move off with the velocity of the first.

Exp. Several equal ivory balls, being so suspended as to have their centres in a right line, if the first be let fall upon the second, the last will fly off, to the height from which the first fell.

Cor. 3. When the striking ball is less than the quiescent, there will be an increase of momentum.

Exp. Let the ball D fall from 12 inches upon C, double of D, at rest. If they were non-elastic, they would proceed together, and, their velocity being the same, C, after the stroke, would have double the momentum of D; that is, C would have 8 degrees, and D 4; whence D would have communicated more than half its momentum to C. The effect being doubled by the elasticity of the bodies, D communicates to C 16 degrees of momentum, and loses as much itself, or returns with 4 degrees af momentum in the contrary direction: while C moves forwards with 4 degrees more momentum than D had at the first. Thus the whole sum of momentum is increased from 12 to 20 degrees; but as much as the momentum is increased in the direction in which D first moved, so much is given to D in the contrary direction. In this manner may momentum be continually increased by a series of bodies.

Cor. 4. If a non-elastic body strike upon an immoveable obstacle, it will lose all its motion; an elastic body will return with a force equal to the stroke.

Exp. Let a leaden ball, and an ivory ball, strike upon any fixed plane.

CHAPTER V.

Of Motion, as produced by the Attraction of Gravitation.

SECTION I.

Of the Laws of Gravitation in Bodies falling without Obstruction.

PROP. XXIII. The motion of a body, falling freely by the attraction of gravitation, is uniformly accelerated, or its velocity increases equally in equal times.

A new impression being made upon the falling body, at every instant, by the continued action of the attraction of gravity, and the effect of the former (by Prop. I.) still remaining, the velocity must continually increase. Suppose a single impulse of gravitation, in one instant, to give it one degree of velocity; if, after this, the force of gravitation were entirely suspended, the body would continue to move with that degree of velocity, without being accelerated or retarded. But, because the attraction of gravitation continues, it produces as great a velocity in the second instant as in the first; which being added to the first, makes the velocity in the second instant double of what it was in the first. In like manner, in the third instant, it will be tripled; quadrupled in the fourth; and in every instant, one degree of velocity will be added to that which the body had before; that is, the motion will be uniformly accelerated.*

Cor. The velocities of falling bodies are as the times in which they are acquired.

PROP. XXIV. The force of the attraction of gravitation acting upon any body is as its quantity of matter.

For each particle of matter in any body being acted upon by gravitation, the greater number of particles are contained in any body, the greater force must be exerted upon it; that is, the force increases as the quantity of matter increases.

Exp. Let two unequal balls, suspended by threads of the same length, be let fall at the same time from points equally distant from the lowest points of the arcs in which they move: The vibrations of each

* All bodies descending in vacuo by gravity, whether great or small, dense or rare, are found to fall through 16.1 feet in one second, and to acquire a velocity in falling which would carry them uniformly through 32.2 feet in the next second, and an increase of velocity, equal to this, is found to be added to every succeeding second of time.

will be performed in equal times, and consequently their velocities will be equal; whence the momenta (Prop. XI.) will be as the quantities of matter; but (Prop. XIII.) the force producing motion, is as the quantity of motion, or momentum produced: Therefore the force of gravitation is as the quantity of matter; that is, as much greater force is exerted upon the larger body than upon the less, as its quantity of matter is greater than that of the less.

Cor. 1. The weight of any body is as its quantity of matter; for weight is the degree of force with which any body is acted upon by gravitation.

Cor. 2. If the attraction of gravitation were increased in any ratio, the weight of a given body would be increased in the same ratio. Substituting, therefore, W, Q, F, for the weight, quantity of matter, and force of gravity, respectively, and supposing them to be variable; W will be as $Q \times F$.

PROP. XXV. The velocities of bodies falling from the same height, without resistance, are equal.

If two bodies of different quantities of matter fall from the same height, the attracting force which acts upon the greater body, will (Prop. XXIV.) exceed that which acts upon the less, as much as the greater body exceeds the less in quantity of matter; whence they must move with equal velocities.

Exp. A guinea, and a feather, or other light body, in the exhausted receiver of an air-pump will fall through the same space in the same time.

PROP. XXVI. The spaces described by falling bodies are as the squares of the times from the beginning of the fall, and also as the squares of the last acquired velocities; or in the ratio compounded of the times and velocities.

Plate I. Fig. 10. In the triangle ABC, let AB express the time in which a body is falling, and BC the velocity which it has acquired at the end of the fall; let AF, AD, be parts of the time AB; and through F, D, draw FG, DE, parallel to BC.

Because the triangles ABC, ADE, are similar, AB is to AD as BC to DE; but AB and AD, express times of descent, and BC expresses the velocity acquired in the time AB; therefore since (Prop. XXIII. Cor.) the velocities are as the times, DE expresses the velocity acquired in the time AD. In like manner GF, any other right line parallel to BC, expresses the velocity acquired in the time AF. Therefore the sum of the lines which may be supposed drawn parallel to CB in the triangle ADE; that is, the whole triangle ADE, will represent the sum of the several velocities with which the falling body moves in the time in AD. For the same reason, the triangle ABC will represent the sum of the velocities with which the falling body moves in the time AB. Since therefore it is manifest, that the space which a body passes through in any moment of time is as the velocity with which it moves at that moment; and consequently, that the spaces through which it passes in any times whatsoever, are as the sums of the velocities with which it moves in the several moments of those times; the spaces passed through in the times AD, AB, are to each other as the triangles ADE, ABC. But the triangle ADE (El. VI. 19.) is to the triangle ABC in the duplicate ratio of the homologous sides AD, AB, and also of DE, BC; that is, the spaces are as the squares of the times, and also as the squares of the last acquired velocities; consequently the spaces described are in the compound ratio of the times and the velocities.

Exp. Let there be two pendulums, one of which vibrates twice as fast as the other, a ball let fall from such a height above the ball of the shorter pendulum as to reach it in one vibration, must fall from four times this height, to reach the longer pendulum in one of its vibrations.

Cor. 1. Hence, if the forces are variable, the spaces described are as the forces and squares of the times; or as the squares of the velocities directly, and forces inversely. For by the Prop. (calling S, V, and T, the space, velocity, and time) S is as $T \times V$, and (by Prop. A. p. 9.) V is as $F \times T$ $\therefore$ S is as $F \times T^2$; also, S is as $\frac{V^2}{F}$; for T is as $\frac{V}{F}$ $\therefore$ S is as $V \times \frac{V}{F}$ or as $\frac{V^2}{F}$.

Cor. 2. The times in which bodies fall from unequal heights, and their last acquired velocities, are as the square roots, or in the subduplicate ratio of their heights. Since TT is as S, T will be as $\sqrt{S}$; and since VV is as S, V will be as $\sqrt{S}$.

Cor. 3. If the time of the fall of a body be divided into equal parts, the spaces through which it falls in each of these parts, taken separately, will be as the odd numbers 1, 3, 5, &c. The spaces being as the squares of the times or velocities, if the times be as the numbers 1, 2, 3, 4, the spaces will be as 1, 4, 9, 16; whence, in the first time the space will be as 1, in the second time, the space passed over will be as 3, in the third, as 5, &c.

Plate I. Fig. 10. Cor. 4. The space passed through during any portion of the time a body is falling, is always proportional to the difference of the squares of the velocities at the beginning and end of that portion of time.

For (by the Prop.) $AFG : Afg :: FG^2 : fg^2$; therefore $AFG - Afg$ or $fFGg : FG^2 - fg^2 :: AFG : FG^2$, and $ADE : Ade :: DE^2 : de^2$; therefore $ADE - Ade$ or $dDEe : DE^2 - de^2 :: ADE : DE^2$; but $AFG : FG^2 :: ADE : DE^2$; therefore $fFGg : FG^2 - fg^2 :: dDEe : DE^2 - de^2$.

SCHOL. Since S is as T^2, and as in the first second of time a body freely descending by the force of gravity falls through 16.1 feet, we easily find the space described in any given number of seconds; for $S = 16.1 \times T^2$. Thus in 5″ a body will fall through 402 feet; for $16.1 \times 25 = 402$. Again, the spaces fallen through in the 1st, 2d, 3d, seconds, are 16.1; 16.1×3; 16.1×5, respectively.

PROP. XXVII. The space which a body passes over in any given time from the beginning of the fall, is half that which it would pass over in the same time, moving with the last acquired velocity.

For the triangle ABC (by Prop. XXVI.) expresses the space passed over in the time AB when the motion is uniformly accelerated; the last acquired velocity is expressed by BC; and the rectangle of AB, BC, rightly expresses the space passed through in the time AB with the equable velocity BC; since therefore the triangle ABC is half of the rectangle AB, BC, the proposition is manifest. Plate 1. Fig. 10.

PROP. XXVIII. The motion of a body thrown upwards is uniformly retarded by gravitation: the time of its rise will be equal to that in which a body falling freely acquires the same degree of velocity with which it is thrown up; and the height to which it will rise will be as the square of the time, or first velocity.

The same force which accelerates a falling body, acting in an opposite direction upon one thrown upwards, must retard it: and, since the action of gravitation is uniform, in whatever time it generates any velocity in a falling body, it must in the same time destroy the same velocity in a rising body: through whatever space the falling body must pass to acquire any velocity, the rising body must pass through the same to lose it; whatever ratio the spaces bear to the velocities and times in the one case, must take place in the other: the effect of gravitation in rising bodies being in all respects the reverse of its effect upon falling bodies.

SCHOL. As the force of gravity near the surface of the earth is constant, and known by experiment, and as the spaces described by falling bodies vary as the squares of the times (T^2), or as the squares of the velocities (V^2); hence every thing relating to the descent of bodies, when accelerated by the force of gravity; and to their ascent, when they are retarded by that force, may be deduced from the foregoing propositions.

(1.) When a body falls by the force of gravity, the velocity acquired in any time, as T″, is such as would carry it uniformly over 2 FT in 1″; where F = 16.1 feet.

EXAM. The velocity acquired in a falling body in $6'' = 32.2 \times 6$, or such as would carry it uniformly through 193 feet in 1″.

(2.) The space fallen through to acquire the velocity V is $\frac{V^2}{4F}$. For $S : F :: V^2 : \overline{2F}|^2$ or $S = \frac{V^2}{4F}$.

EXAM. If a body fall from rest till it acquire a velocity of 20 feet per second, the space fallen through is $\frac{\overline{20}|^2}{4 \times 16.1} = 6.2$ feet.

From these three expressions, $V = 2FT$; $S = \frac{V^2}{4F}$; and $S = FT^2$ (Cor. 1. Prop. XXVI.); any one of the quantities S, T, V, being given, the other two may be found.

EXAM. 1. To find the time in which a body will fall 400 feet; and the velocity acquired.

Since $S = FT^2 \therefore T = \sqrt{\frac{S}{F}} = \sqrt{\frac{400}{16.1}} = 5''$ nearly, and V being equal to $2\,FT = 32.2 \times 5'' = 161$ feet = velocity acquired.

EXAM. 2. If a body be projected perpendicularly downwards, with a velocity of 20 feet per second, to find the space described in 4″.

The space described in 4″ by the first velocity is 4×20, and the space fallen through by the action of gravity is 16.1×4^2, therefore the whole space described is 337.6 feet.

EXAM. 3. To what height will a body rise in 3″, which is projected perpendicularly upwards with a velocity of 100 feet per second?

The space described in 3″ by the first velocity is 300 feet, and the space through which the body would fall by gravity in 3″ is $16.1 \times 3^2 = 144.9$ feet; therefore the height required is $300 - 144.9 = 155.1$ feet.

SECTION II.

Of the Laws of Gravitation in Bodies falling down inclined Planes.

Def. IV. An inclined plane is a plane which makes an acute or obtuse angle with the plane of the horizon.

PROP. XXIX. The motion of a body, descending down an inclined plane, is uniformly accelerated.

In every part of the same plane, the accelerating force has the same ratio to the force of gravitation acting freely in a perpendicular direction, and is therefore (El. V. 9.) equally exerted in every instant of the descent; whence (as was shown concerning bodie s alling freely, Prop. XXIII.) the motion must be uniformly accelerated.

Cor. Hence, whatever has been demonstrated concerning the perpendicular descent of bodies, is equally applicable to their descent down inclined planes, the motion in both cases being uniformly accelerated by the same power of gravitation.

PROP. XXX. The force, with which a body descends by the attraction of gravitation down an inclined plane, is to that with which it would descend freely, as the elevation of the plane to its length; or as the sine of the angle of inclination to radius.

Plate 1. Fig. 11. Let AB be the length of an inclined plane, and AC its elevation, or perpendicular height. If the force of gravitation with which any body descends perpendicularly be expressed by AC, and this force be resolved into two forces, AD, DC, by drawing CD perpendicular to AB; because the force CD is destroyed by the reaction of the plane, the body descends down the inclined plane only with the force AD. And (El. VI. 8. Cor.) AD is to AC, as AC is to AB; that is, the force of gravitation down the inclined plane is to the same force acting freely, as the elevation of the plane is to its length, or as the sine of the angle of inclination ABC is to the radius AB.

Cor. 1. Hence, the force necessary to sustain a body on an inclined plane, is to the absolute weight of a body, as the elevation of the plane to its length, for the force requisite to sustain a body must be equal to that with which it endeavours to descend; which has been shown to be to that with which it would descend freely, as the elevation of the plane to its length.

Cor. 2. If H be the height of an inclined plane, L its length, and the force of gravity be represented by unity; the accelerating force on the inclined plane is represented by $\frac{H}{L}$. For by the Prop. the accelerating force is to the force of gravity (1) as H is to L $\therefore$ the accelerating force $= \frac{H}{L}$.

Cor. 3. Hence $\frac{H}{L}$ varies as the sine of the angle of inclination.

Cor. 4. If a body fall down an inclined plane, the velocity V generated in T″ is such as would carry it uniformly over $\frac{H}{L} \times 2\,F\,T$ feet in 1″, where, as before, F is equal to 16.1.

For (by Prop. A. p. 9.) the velocity varies as the force and time, i. e. as $\frac{H}{L} \times T$, and the velocity generated by the force of gravity in one second is 2 F, therefore $V = \frac{H}{L} \times 2\,F\,T$.

Ex. If L : H : : 2 : 1, a body falling down the plane will, at the end of 4″, acquire a velocity of $\frac{1}{2} \times 32.2 \times 4 = 64.4$ feet per second.

Cor. 5. The space fallen through in T″ from the state of rest, is $\frac{H}{L} \times FT^2$, for (Prop. XXVI.) the spaces described vary as the squares of the times.

Ex. 1. If $H = \frac{L}{2}$, the space through which a body falls in 5″ is $\frac{1}{2} \times 16.1 \times 25 = 201\frac{1}{4}$ feet.

Ex. 2. To find the time in which a body will descend 40 feet down this plane. Since $S = \frac{H}{L} \times FT^2$, therefore $T = \sqrt{\frac{S \times L}{H \times F}} = \sqrt{\frac{40 \times 2}{1 \times 16.1}} = 2.2$ seconds.

COR. 6. The space through which a body must fall, from a state of rest, to acquire a velocity V, is $\frac{L}{H} \times \frac{V^2}{4F}$. For (Cor. 1. Prop. XXVI.) S is as $\frac{V^2}{F}$, therefore the space through which the body falls by the force of gravity, is to the space through which it falls down the plane, as the square of the velocity directly, and as the force inversely in the former case, is to the same in the latter; and if F (16.1) be the space fallen through by gravity, 2 F is the velocity acquired in 1″; hence $F : S :: \frac{\overline{2F}|^2}{1} : \frac{L}{H} \times V^2$,

$$\text{and } S = \frac{L}{H} \times \frac{V^2}{4F}.$$

EX. 1. If $L = 2H$, and a body fall from a state of rest till it has acquired a velocity of 40 feet per second, the space described is $\frac{2}{1} \times \frac{40^2}{64.4} = 50$ feet nearly.

EX. 2. If a body fall 40 feet from a state of rest down this plane, to find the velocity acquired. $V^2 = 4FS \times \frac{H}{L} = 64.4 \times 40 \times \frac{1}{2} = 1288$, and $V = 35.8$ feet per second.

PROP. XXXI. The space described in any given time by a body descending down an inclined plane, is to the space through which it would fall perpendicularly in the same time, as the elevation of the plane to its length.

Let AC represent the force with which a body would fall perpendicularly; CD being drawn from C perpendicular to AB; AD, as was shown (Prop. XXX.), will represent the force with which the body descends down the inclined plane AB. And, since the spaces through which bodies fall in any given time must be as the forces which move them, the space through which the body will fall down the inclined plane AB, is to that through which it will fall perpendicularly in the same time, as the force AD, to the force AC. But AD is to AC (El. VI. 8. Cor.) as AC the elevation to AB the length of the plane; therefore the space through which the body will fall in a given time down the inclined plane AB, will be to the space through which it would fall perpendicularly in the same time, as the elevation of the plane to its length. Plate 1. Fig. 11.

COR. 1. A body would fall down the inclined plane from A to D, in the same time in which it would fall perpendicularly from A to C. For, the spaces passed through in any given time are as AC to AB, that is, (El. VI. 8. Cor.) as AD to AC; consequently, if AC is the space passed through in any given time by the body falling freely, AD will be the space passed through in the same time, down the inclined plane AB.

COR. 2. Having the space through which a body falls in a perpendicular direction, we can easily find the space which a body will describe in the same time, on planes differently inclined, by letting fall perpendiculars, as CD, on those planes respectively.

PROP. XXXII. The velocity, acquired in any given time by a body descending down an inclined plane, is to the velocity acquired in the same time by a body falling freely, as the elevation of the plane to the length.

In an uniformly accelerated motion, the velocities produced in equal times are as the forces which produce them; but (by Prop. XXX.) the force with which a body descends down an inclined plane, is to that of its perpendicular descent, as the height of the plane to its length; therefore the velocities produced in equal times are in the same ratio.

PROP. XXXIII. The time, in which a body moves down an inclined plane, is to that in which it would fall perpendicularly from the same height, as the length of the plane to its elevation.

The square of the time in which AB is passed over, is to the square of the time in which AD is passed over (compare Prop. XXVI. with Prop. XXIX. Cor.) as AB to AD; that is, since AB, AC, AD, (El. VI. 8. Cor.) are continued proportionals, as the square of AB to the square of AC. Therefore the times themselves are as the lines AB, AC; that is, as the length of the plane to its elevation. Plate 1. Fig. 11.

COR. Hence, if several inclined planes have equal altitudes, the times in which those planes are described by bodies falling down them, are as the lengths of the planes. For the time of the descent down AC is to the time of the fall down AB, as AC to AB; and the time of the fall down AB is to the time of the descent down AG, as AB to AG; therefore (El. V. 11.) the time of the descent from A to C is to the time of the descent from A to G, as AC to AG; that is, the times are as the lengths of the planes. Plate 1. Fig. 12.

PROP. XXXIV. A body acquires the same velocity in falling down an inclined plane, which it would acquire by falling freely through the perpendicular elevation of the plane.

Plate 1. Fig. 11. The square of the velocity which a body acquires by falling to D, is (by Prop. XXVI. compared with Prop. XXIX. Cor.) to the square of the velocity it acquires by falling to B, as the space AD is to the space AB, that is (El. VI. 8. Cor.) as the square of AD is to the square of AC; and consequently the velocity at D is to the velocity at B, as AD is to AC. But, because AD and AC (Prop. XXXI. Cor.) are passed over in the same time, the velocity acquired at D is (by Prop. XXXII.) to that which is acquired at C, as AD to AC. Since then the velocity at D has the same ratio to the velocities at B, and at C, namely, the ratio of AD to AC, the velocities at B and C (El. V. 9.) are equal.

Plate 1. Fig. 12. Cor. 1. Hence the velocities acquired by bodies falling down planes differently inclined are equal, where the heights of the planes are equal. The velocities acquired in falling from A to C, and from A to G, are each equal to the velocity acquired in falling from A to B, and therefore equal to one another.

Cor. 2. Hence if bodies descend upon inclined planes whose heights are different, the velocities will be as the square roots of their heights. For (Fig. 8 and 9) the velocity in D is equal to that in A, and the velocity in D is equal to that in G. Therefore the velocity in D (Fig. 8.) is to that in D (Fig. 9.) as $\sqrt{AB}$ is to $\sqrt{FG}$ (by Cor. Prop. XXVI.)

PROP. XXXV. A body falls perpendicularly through the diameter, and obliquely through any chord of a circle, in the same time.

Plate 1. Fig. 12. In the circle ADB, let AB be a diameter, and AD any chord; draw BC a tangent to the circle at B; produce AD to C, and join DB. Because ADB (El. III. 31.) is a right angle, a body (by Prop. XXXI. Cor.) will fall from A to D on the inclined plane in the same time in which it will fall from A to B perpendicularly. In like manner let the chord AE be produced to G; and because AEB is a right angle, a body will fall from A to E on the inclined plane in the same time in which it would fall from A to B.

Cor. 1. Hence all the chords of a circle are described in equal times.

Cor. 2. Hence also the velocities, and accelerating forces, will be as the lengths of the chords.

PROP. XXXVI. If a body descends along several contiguous planes, the velocity which it acquires by the whole descent, provided it lose no motion in going from one to another, is the same which it would acquire if it fell from the same perpendicular height along one continued plane; and this velocity will be the same with that which would be acquired by the perpendicular fall from the elevation of the planes.

Plate 1. Fig. 13. Let AB, BC, CD, be several contiguous planes; through the points A and D, draw HE, DF, parallel to the horizon, and produce the contiguous planes CB, CD, to G and E. By Prop. XXXIV. Cor. the same velocity is acquired at the point B, whether the body descends from A to B, or from G to B. Therefore, the line BC being the same in both cases, the velocity acquired at C must be the same, whether the body descends through AB, BC, or along GC. In like manner, it will have the same velocity at D, whether it falls through AB, BC, CD, or along ED; that is, (by Prop. XXXIV.) its velocity will be equal to the velocity acquired by the perpendicular fall from H to D.

Cor. Hence, if a body descend along any arc of a circle, or any other curve, the velocity acquired at the end of the descent is equal to the velocity acquired by falling down the perpendicular height of the arc; for such a curve may be considered as consisting of indefinitely small right lines, representing contiguous inclined planes.

Schol. The velocity of a body, passing from one inclined plane to another, is diminished in the ratio of radius, to the cosine of the angle between the directions of the planes. Let BC, or B m (Fig. 20.) represent the velocity acquired at B, and resolve BC into B n and C n, by letting fall the perpendicular C n; mn will be the velocity lost; therefore the velocity at B is to the velocity diminished by passing from AB to BD as BC to B n, or as radius to the cosine of the angle between the directions of the planes.

PROP. XXXVII. If two bodies fall down two or more planes equally inclined, and proportional, the times of falling down these planes will be as the square roots of their lengths.

Plate 1. Fig. 14. Let the inclined planes be AB, BC, DE, EF; let AG, DH, be lines drawn parallel to the horizon; let AB, DE, be equally inclined to the plane of the horizon, and also BC, EF; let AB be to DE as AG to DH and as BC to EF, and draw GB, HE.

Because ABG, DEH, are similar triangles, AB is to DE (El. VI. 4.) as BG to EH, and $\sqrt{AB}$ to $\sqrt{DE}$ as $\sqrt{BG}$ to $\sqrt{EH}$; also AB is to DE as BG + BC is to HE + EF, and $\sqrt{AB}$ to $\sqrt{DE}$ as $\sqrt{BG + BC}$, or $\sqrt{GC}$, is o $\sqrt{HE + EF}$, or $\sqrt{HF}$.

And since (by construction) AB is to DE as BC to EF, AB is to DE as AB + BC is to DE + EF, and $\sqrt{AB}$ to $\sqrt{DE}$, as $\sqrt{AB + BC}$ to $\sqrt{DE + EF}$. But AB, DE, being planes equally inclined, the accelerating force of gravitation will be the same upon each, and the bodies descending upon them may be considered as falling down different parts of the same plane. Hence, (Prop. XXVI. Cor. 2. and XXIX. Cor.) the time of descent along AB is to that along DE, as $\sqrt{AB}$ to $\sqrt{DE}$; and the time of descent along GC is to that along HF, as $\sqrt{GC}$ to $\sqrt{HF}$; that is, as $\sqrt{AB}$ to $\sqrt{DE}$. Again, the time of descent along GB is to that along HE as $\sqrt{BG}$ $\sqrt{EH}$; that is, as $\sqrt{AB}$ to $\sqrt{DE}$. Since, therefore, the time of descent along the whole plane GC is to that along the whole plane HF, as $\sqrt{AB}$ to $\sqrt{DE}$, and that of the part GB is also to that of the part HE, as $\sqrt{AB}$ to $\sqrt{DE}$, the time of descent along the remainder BC is to that along the remainder EF (El. V. 19.) as $\sqrt{AB}$ to $\sqrt{DE}$. Consequently, the time of descent down BA + BC is to that down DE + EF, as $\sqrt{AB}$ to $\sqrt{DE}$; that is, as $\sqrt{AB + BC}$ to $\sqrt{DE + EF}$.

COR. Hence, if bodies descend through arcs of circles, the times of describing similar arcs will be as the square roots of the arcs. For such similar arcs may be considered as composed of an equal number of proportional sides, or planes, having the same inclination to each other, and their elevations equal; whence, by this proposition, the times of descent will be as the square roots of the lengths of the arcs.

PROP. XXXVIII. If a body be thrown up along an inclined plane, or the arc of a curve, it will, in the same time, rise to the same height, from which, with equal force, it would have descended; and any velocity will be lost in the same time in which it would, in descending, have been acquired.

For the force of gravitation has, in every respect, the same efficacy to retard the motion of bodies ascending, as to accélerate them descending on an inclined plane or curve.

SECTION III.

Of the Pendulum and Cycloid.

DEF. V. A pendulum is a heavy body, hanging by a cord or wire, and moveable with it upon a centre.

PROP. XXXIX. The vibrations of a pendulum are produced by the force of gravitation.

Let the ball A, suspended from the centre B by the cord BA, be drawn up to C and let fall from thence: it will descend by the force of gravitation to A, from whence (being prevented from falling farther by the cord) it will proceed (by Prop. XXXVI. Cor.) with a velocity equal to that which it would have acquired in falling perpendicularly from E to A, which will carry it on the opposite side to the height from which it fell. Being brought back again towards A by the force of gravitation, it will acquire a new velocity which will carry it towards C; and in this manner it will vibrate by the force of gravitation, till the resistance of the air, and the friction of the string, stop its motion. Plate 1. Fig. 15.

PROP. XL. The same pendulum, vibrating in small unequal arcs, performs its vibrations nearly in equal times.

In the circle CGA, the small arcs CA, EA, will differ little from their respective chords in length or declivity. But (by Prop. XXXVI. Cor.) the times in which the chords are passed over are equal; therefore the times of describing the arcs CA, EA, and also (by Prop. XXXVIII.) of describing their doubles CAD, EAF, will be nearly equal. Plate 1. Fig. 16 and 17.

EXP. Two equal pendulums, vibrating in small, but unequal arcs, will, for a long time, keep pace in their vibrations.

PROP. XLI. If a pendulum vibrate through small arcs of circles of different lengths, the velocity it acquires at the lowest point, is as the chord of the arc which it describes in its descent.

Plate 1. Fig. 16 and 17. Let BA be the pendulum, and CAD, EAF, the arcs through which it vibrates; and draw the horizontal lines EK, CH. The velocity acquired in falling from H to A is (by Prop. XXXVI. Cor.) to that acquired in falling from G to A, as $\sqrt{HA}$ to $\sqrt{GA}$; that is, (by El. VI. 8. Cor.) as CA to GA. For the same reason, the velocity acquired in falling from G to A, is to that acquired in falling from K to A, as GA to EA. Consequently, *ex æquali*, the velocity acquired in falling from H to A is to that acquired in falling from K to A, as CA to EA. But (by Prop. XXXVI. Cor.) the velocity acquired in falling from H to A is equal to that from C to A; and the velocity acquired in falling from K to A is equal to that from E to A. Therefore the velocity acquired in descending through the arc CA is to that through EA, as the chord CA is to the chord EA; and the same may be shown concerning the remaining half of the vibrations, AF, AD.

Fig. 17. Cor. Hence the lengths of the chords of arcs, through which pendulums move, are measures of velocity.

PROP. XLII. The time of the descent and ascent of a pendulum, supposing it to vibrate in the chord of a circle, is equal to the time in which a body, falling freely, would descend through eight times the length of the pendulum.

For the time of the descent of a body upon the chord is (by Prop. XXXV.) equal to that of the fall through the diameter of the circle, which is twice the length of the pendulum; but in double that time, that is, in the descent and ascent, or whole vibration, the body would fall (by Prop. XXVII.) through four times the space, that is, through eight times the length of the pendulum.

PROP. XLIII. The times in which pendulums of different lengths perform their vibrations, are as the square roots of their lengths.

Plate 1. Fig. 18. Let the two pendulums, AB, CD, be of different lengths. The time in which the first, AB, vibrates through a chord, is equal to that in which a body (Prop. XXXV.) would fall freely through twice AB, the diameter of the circle of which AB is radius. In like manner, the time in which CD vibrates is equal to that in which a body would fall through twice CD. But the times in which a body would fall through these different spaces are (Prop. XXVI. Cor. 2.) as the square roots of the spaces; that is, as the square roots of AB and CD, the lengths of the pendulums; therefore the vibrations are in the same ratio.

Fig. 19. Cor. The times in which pendulums of unequal lengths vibrate, are as the square roots of the similar arcs through which they move. Let BA, BC, be pendulums of different lengths, vibrating in the similar arcs FG, DE. Since the times of vibration are as the square roots of the lengths BA, BC, and similar arcs are as the diameters, the times of vibration are as the square roots of the arcs, FA, DC, or of their doubles, FG, DE.

Exp. Two pendulums, the lengths of which are as 1 to 4, will perform their vibrations in times as 1 to 2; that is, the shorter pendulum will make two vibrations, whilst the longer makes one; for $T : t :: \sqrt{L} : \sqrt{l}$.

PROP. XLIV. The squares of the times in which a pendulum of a given length performs its vibrations, are inversely as the accelerating forces, or gravities.

By Prop. XXVI. where the accelerating force is given, the space described is as the square of the time in which it is described. And since, in any given moving body, the velocity is as the accelerating force (Prop. A. p. 9.) where the square of the time, or the time itself, is given, (by Prop. II.) the space described will be as the accelerating force. Consequently, where neither the accelerating force, nor the square of the time, is given, the space described will be in the ratio compounded of both. If then the space described be called S, the accelerating force A, and the square of the time T^2, S will be as T^2A; whence $\frac{T^2A}{A}$, or T^2, is as $\frac{S}{A}$. But, when the spaces are equal, S is a given quantity; whence (since fractions, whose numerators are given, are inversely as their denominators,) $\frac{S}{A}$ is inversely as A. But T^2 is as $\frac{S}{A}$; therefore where S is given, T^2 is inversely as A; that is, where the spaces described are equal, the squares of the times in which they are described are inversely as the accelerating forces. And if the squares of the times of falling bodies are inversely as their accelerating forces, the squares of the times, in which pendulums vibrate, are in the same ratio, on account of the constant equality between the time of vibration and that of the descent through eight times the length of the pendulum, by Prop. XLII.

Cor. 1 Hence, if the same pendulum, at different parts of the earth, perform its vibrations in different times, the forces of gravitation will, in those places, be inversely as the squares of those times.

Cor. 2. If the vibrations of pendulums of unequal lengths be performed in the same time, the accelerating forces will be as their lengths. For (by Prop. XLIII. and XLIV.) $T^2 : t^2 :: \frac{L}{A} : \frac{l}{a}$; therefore, when $T = t$, $A : a :: L : l$. Hence, as it is known by experiment, that the lengths of pendulums that vibrate seconds are diminished in approaching the equator, the force of gravity must also decrease.

Ex. At the equator a pendulum vibrating seconds is $\frac{1}{10}$th of an inch shorter than such a pendulum in the latitude of London, and the length of this pendulum in London is 39.2 inches; therefore gravity under the equator is to gravity here as 391 is to 392.

LEMMA I.

If, from X *as the centre, with any distance* XA, *a quadrant of a circle* ADB *be described, and in the right line* AX *a body descends with such force, that its velocity in any points* M, N, *&c. shall be always as* MD, NP, *&c. the sines of the arcs* AD, AP: *the time in which the body will descend from* A *to* X, *will be equal to the time in which it would describe the whole arc* ADB, *with the uniform velocity, expressed by* XB, *acquired by the falling body when it arrives at* X; *also, the time of the fall through any space* AM, *will be to the time of the fall through any other space* AO, *as the arc* AD *to the arc* AQ; *and the force, with which the body is accelerated in any place* M, *will be as* MX, *the distance of that place from the centre.* Plate 2. Fig. 1.

Let DP be a part of the circumference taken indefinitely small, and therefore not assignably differing from a right line; join DX; and draw DL perpendicular to NP. Because the triangles MDX, LDP, are similar (having each a right angle, and, the angles MDX, LDP, whose common complement is LDX, equal) MD will be to DX as LD or MN to DP. But, by the hypothesis, MD is as the velocity of the descending body at the point M, that is, as the velocity with which the indefinitely small line MN is described; and XD is as the velocity last acquired by the falling body at X, that is, as the uniform velocity with which the arc DP is described. The velocity therefore of the body descending through the indefinitely small line MN will be to the velocity of the body moving along the arc DP, as MN to DP. Wherefore, since the velocities are proportional to the spaces passed over, the times wherein those spaces MN, DP, are described, will be equal. After the same manner it may be proved, that any other indefinitely small portion of the circumference, PQ, may be described with the velocity XB, in the same time in which the corresponding line NO will be described with the corresponding velocity NP; and consequently, by composition, the falling body will descend through all the indefinitely small portions of the perpendicular AX, that is, through the whole line, in the time in which all the corresponding parts of the circumference, that is, the whole quadrant ABD, is described with a uniform velocity as XB.

Moreover, the time in which the falling body descends from A to M, is equal to the time in which the arc AD is passed over; and the time in which it descends from A to O is equal to the time in which the arc AQ is described; but the time in which the arc AD is passed over, is to that in which the arc AQ is passed over, (since they are both described with the same velocity) as the arc AD to the arc AQ; therefore the time of descent from A to M, will be to the time of descent from A to O, as the arc AD to the arc AQ; and consequently, by division, the time of descent through AM will be to the time of descent through MO, as the arc AD to the arc DQ.

Lastly, let the arcs DP, PQ, be equal; join XP, and from P let fall PS perpendicular to OQ. The time of descent through MN will be equal to that through NO; and, since the triangles LDP, MDX, are similar, and also SPQ, NPX; LP will be to DP or PQ, as MX to XD or XP; also PQ is to SQ as XP to XN; and consequently (El. V. 11.) LP will be to SQ as XM to XN. But LP is as the increment of the velocity acquired while the body is passing over MN, and SQ is as the increment of the velocity acquired in passing over in an equal time the indefinitely small line NO; and the forces with which the body is accelerated at M and N, are as the increments of the velocities generated in equal times; the accelerating forces at M and N, will therefore be as the lines LP, SQ; that is, the force with which the body is impelled at M is to that at N, as the distance XM to the distance XN, or the accelerating forces are as their distances from the centre.

Cor. Hence, conversely, if a body, descending from A to X, is impelled by a force which is as its distance from the centre X, and the force at the beginning of the motion is expressed by the right line CE (the arc AE being taken indefinitely small), the velocities of the same body in any places M, O, will be expressed by the sines MD, OQ; and the times by the arcs AD, AQ; and the increments of the velocities, or, if the arcs increase equally, the accelerating forces, will be expressed by the increments of the sines.

Plate 2. Fig. 2.

LEM. II. *If a body, moving along the line* AX, *be impelled by forces proportional to its distance from the point* X; *from whatever height it falls, it will arrive at the point* X *in the same time; and this time will be to the time in which it would move over the line* AX *with the velocity which it acquires by falling from* A *to* X, *as half the circumference of a circle to its diameter.*

Let two bodies be let fall from the points A and P at the same time; and let them be impelled by forces proportional to their distances from the point X; these bodies will come to X at the same time. From X as a centre, with the radii XA and XP, describe the two quadrants AB, PQ; and let the force by which the body A is impelled, or, which is the same thing, its velocity at the beginning of motion, be represented by RS, the sine of the indefinitely small arc AS. It is manifest (from the Cor. of the preceding lemma) that its velocity, after the fall to X, will be properly expressed by XB. But, by hypothesis, the force by which the body at A is accelerated, is to that by which the body at P is accelerated, as AX is to PX, that is, (since the arcs AS and PN are similar) as RS to MN. As therefore RS expresses the first velocity of the body moving from A, MN will express the first velocity of the body moving from P; and consequently (by the Cor. to the last lemma) XQ will express the velocity of the body moving from P, when it arrives at X. Farther, the time of the fall from A to X (by Lemma I.) is equal to the time in which the arc AB would be described with a velocity as XB; and the time of the fall from P to X is equal to the time in which the arc PQ would be described with a velocity as XQ. But (because the line XQ is to the line XB as the arc PQ to the arc AB, and the spaces passed over are proportional to the velocities) the time in which the arc AB is described with the velocity XB is equal to the time in which the arc PQ is described with the velocity XQ. Wherefore the time of the fall from A to X will be equal to the time of the fall from P to X.

Again, since (by Lem. I.) the time in which a body would fall from A to X is equal to the time in which it would move over the arc AB, with its last acquired velocity at X; and since it is evident, that the time in which a body would move over the arc AB with the velocity at X is to the time in which it would move over AX with the same velocity, as AB is to AX; the time in which a body would fall from A to X is to the time in which it would move over AX with the last acquired velocity as AB to AX. But AB is to AX, as twice AB to twice AX; that is, as half the circumference of a circle is to its diameter. Therefore the time in which a body would fall from A to X is to the time in which it would move over AX with its last acquired velocity, as half the circumference of a circle is to its diameter.

Plate 2. Fig. 3.

Def. VI. If a circle, as FCH, be rolled along the line AB, till it has turned once round; the point C in its circumference, which at first touched the line at A, will describe the curve line ACXB, which curve is called a *Cycloid.* The right line AB is its *base;* the middle point X is its *vertex;* a perpendicular, as XD, let fall from thence to the base, is its *axis;* and the circle FCH, or any other, as XGD, equal thereto, is called the *generating circle.*

LEM. III. *If on* XD, *the axis of the cycloid, as a diameter, the generating circle* XGD *be described; and if from a point in the cycloid, as* C, *the line* CIK *be drawn parallel to the base, the portion of it* CG *will be equal to the arc* GX.

Because the generating circles FCH, DGX, are equal (the diameter HF being drawn), KG is equal to CI; whence, adding GI to both, KI will be equal to CG; and KI, by construction, is equal to DF; therefore CG is equal to FD. By the description of the cycloid, the arc CF is equal to the line AF; and by the construction the arc CF is equal to DG; therefore AF is equal to DG; but, by the description of the cycloid, AFD is equal to DGX; consequently, FD is equal to GX; and CG was proved to be equal to FD; therefore CG is equal to GX.

LEM. IV. *A tangent to the cycloid at the point* C *is parallel to* GX, *a chord of the circle* DGX.

Draw ck, parallel to the base and indefinitely near to CK, meeting the cycloid in c, the axis in k, and the circle in g. Let Cu and Gn, parallel to the axis, meet ck in u and n, and from T, the centre of the circle XGDM, draw the radius TG. Since cg is equal (Lem. III.) to gX, gk being added to both, ck will be equal to X$g + gk$; therefore cu, the excess of ck above CK, is equal to G$g + gn$, the excess of X$g + gk$ above XG + GK. And, if we suppose ck to approach towards CK, as Gg and gn vanish, the triangles Ggn and GTK become similar; for the angle gGn is then equal to the angle

TGK, since both have the the same angle nGT, or its alternate GTK, as their complement. Whence Gg is to gn as TG to TK, and (El. V. 18.) Gg + gn to gn, as TG + TK or DK to TK; but Gn is to gn as GK to TK; therefore Gg + gn is to Gn as DK is to GK, that is, (El. VI. 8.) as GK to XK. And consequently cu (shown to be equal to Gg + gn) is to Gn, or Cu, as GK to XK; and if the chord Cc be drawn, the triangles Cuc, XKG, will be similar; so that the chord Cc (as the points C and c coincide) becomes parallel to XG; therefore the tangent of the cycloid at C is parallel to XG.

LEM. V. *If from a point of the cycloid, as* L, *the line* LMK *be drawn parallel to the base* AB, *the arc* XL *of the cycloid, will be double of* XM, *the chord of the circle corresponding thereto.* Plate 2. Fig. 3.

Draw a line Sk parallel and indefinitely near to LK crossing the circle in R, and the chord XM produced, in P; join the points X and R; on MP let fall the perpendicular RO; and draw MN, XN, tangents to the circle at M and X. Then will the lines XN and kS, being each perpendicular to the diameter DX, be parallel; and the triangles MNX, MPR, having their angles at M vertical, and at P and X alternate, will be similar. But the tangents NX and NM are equal; (El. III. 36.) whence the lines PR and RM are also equal; the triangle RMP is therefore isosceles; and RO being perpendicular to its base MP, MO (El. I. 26.) is equal to OP; whence MP is equal to twice MO. The indefinitely small arc LS of the cycloid will not assignedly differ from a portion of a tangent drawn through the point L. LS may therefore (Lem. IV.) be said to be parallel to MP, and consequently (from the parallelism of ML and PS) equal to it; it is therefore equal also to twice MO. But LS is the difference between the cycloidal arcs XL and XS; and MO is the difference between the chords XM and XR; for since XO and XR are indefinitely near to each other, RO, which is perpendicular to one of them, may be considered as perpendicular to both; the indefinitely small difference therefore between any two arcs of the cycloid is twice that which is between the two corresponding chords of the circle; and the same is true when the magnitude of the difference is assignable, because such difference is compounded of indefinitely small parts. Now, any arc whatsoever may be considered as a difference between two arcs, and consequently any arc, as XL, is double of the corresponding chord XM.

Cor. Since when the arc XL becomes XB, the corresponding chord XM becomes XD, the diameter of the circle DMX; it is obvious, that the semicycloid BX, or AX, is equal to twice DX, the diameter of the generating circle DMX.

LEM. VI. *If a body descend in a cycloid, the force of gravity, so far as it acts upon the body in causing it to descend along the cycloid, will be proportional to the distance of the body from the lowest point of the cycloid.*

Let the cycloid be AXB, whose base is AB, and its axis DX; on which last, as a diameter, describe the generating circle DQX; draw the chords OX and QX; through the points O and Q, and parallel to the base AB, draw the lines LS and MR; draw also the tangents LV and MY. Then because (by Lem. IV.) the tangent LV is parallel to OX, and the tangent MY parallel to QX, it is obvious that gravity exerts the same power upon a body descending in the cycloid at L, (because it then descends in the tangent LV) as it would do upon the same body descending along the chord OX; and, for the like reason, it exerts the same force upon it when it comes to M, that it would do if it were descending along QX; but (from Prop. XXXV.) the power or force of gravity upon bodies descending along the chords OX and QX, are as the lengths of those chords; that is, by Lem. V. (halves being proportional to their wholes) as the length of the cycloidal arcs LX and MX. The force therefore of gravity upon a body descending in the cycloidal at the point L is to its force upon the same when at M (as may be said of any other corresponding points) as the space or distance it has to move over in the former case, before it reaches the lowest point X, to that which it has to pass over in the latter, before it arrives at the same point. Plate 2. Fig. 4.

PROP. XLV. If a pendulum be made to vibrate in a cycloid, all its vibrations, however unequal in length, will be performed in equal times.

The force of gravity, (by Lem. VI.) so far as it causes a body to descend in a cycloid, is proportional to the distance of that body from the lowest point. Imagine then that body to be a pendulum vibrating in the cycloid, and from whatever point it sets out, it will (by Lem. II.) come to the lowest point in the same time; and consequently, since the same may be easily inferred in its ascending from that point, all its vibrations, be they large or small, will be performed in the same time.

Schol. This proposition is demonstrated only on the supposition that the whole mass of the pendulum is concentrated in a point, for it cannot otherwise take place, because as the string varies in its length,

the centre of oscillation of a body will vary. On this account, therefore, pendulums vibrating in circular arcs are now always used, for the same arcs will be always described in the same time.

PROP. XLVI. To make a pendulum vibrate in a given cycloid.

Plate 2. Fig. 4. Let AXB be the given cycloid; its base AB, its axis DX, and its generating circle DQX, as before; produce XD to C, till DC is equal to DX; through C draw the line EF parallel to AB, and take CE and CF, each equal to AD or DB; and on the line CE as a base, and with the generating circle AGE equal to DQX, describe the semicycloid CTA, whose vertex will therefore touch the base of the given cycloid in A. And on the line CF also as a base, describe an equal semicycloid CB. Let the semicycloids CA, CB, represent thin plates of metal bent to their figure, and on the point C, hang the pendulum CTP by a flexible line equal in length to the line CX. The upper part of its string (as CT, in its present situation in the figure) as it vibrates, will then apply itself to the cycloidal cheeks CA and CB, and a ball at P will oscillate in the given cycloid AXB.

Draw TG and PH each parallel to the base AB, and draw AG and DH. Then (Lem. V. Cor.) AC is equal to twice AE; and by construction, twice DC; that is, twice AE is equal to CX; therefore AC is equal to CX. Also, by construction, CTP is equal to CX; that is, to ATC; whence, taking away CT, AT is equal to TP. By Lem. IV. GA is parallel to TP; and, by construction, AK is parallel to GT; therefore GA is equal to TK, and GT to AK; but (Lem. V.) GA is half TA; therefore TK is equal to half TA; since therefore it has been proved that TA is equal to TP, TK is equal to half TP; that is, to KP. Hence it is manifest, that the parallel lines GT, PH, are equally distant from AD, the arc GA equal to the arc DH, the chords GA and DH parallel, and GE equal to HX. And because GA has been shown to be parallel to TK, and also to DH, KP and DH are parallel; whence KD is equal to PH. But (Lem. III.) GT, that is, AK is equal to the arc AG; and by the description of the semicycloid CTA, AKD is equal to AGE; therefore KD is equal to EG; that is, PH is equal to HX. And (by Lem. III.) if PH be equal to HX, P is a point in the cycloid AXB. The ball of the pendulum therefore, being at that point, is in the given cycloid.

Plate 1. Fig. 18. Schol. 1. It is easy to conceive, that in a pendulum there must be some one point, on each side of which the momenta of the several parts of the pendulum will be equal, or in which the whole gravity of the pendulum might be collected without altering the time of its vibrations. This point, which is called the *centre of oscillation*, is different from the centre of gravity; for if a plane perpendicular to the string of the pendulum AB, be conceived to pass through the centre of the ball B, bisecting it; the velocity of the lower half, and consequently its momentum, will, in vibration, be greater than that of the upper half; consequently the centre of oscillation must be farther from A than the centre of gravity is; and a plane passing through the centre of oscillation will divide the ball into two unequal parts, so that the greater quantity of matter above it shall compensate for the greater velocity below it, and the momenta on each side be equal. If the pendulum be an inflexible rod, every where of equal size, it is found, that the distance of the centre of oscillation from the point of suspension is two thirds of the length of the rod.

If, whilst a pendulum is in motion, it meets with an obstacle at its centre of oscillation sufficient to stop it, the whole motion of the pendulum will cease at once, without any jarring; for the obstacle resists equal momenta above and below this point; which is therefore also called the *centre of percussion*.

Schol. 2. The vibrations of pendulums are subjected to many irregularities, for which no effectual remedy has yet been devised. These are owing partly to the variable density and temperature of the air, partly to the rigidity and friction of the rod by which they are suspended, and principally to the dilatation and contraction of the materials of which they are formed. The metalline rods of pendulums are expanded by heat, and contracted by cold; therefore clocks will go slower in summer and faster in winter. The common remedy for this inconvenience is the raising or lowering of the bob of the pendulum (by means of a screw) as the occasion may require. By the last scholium it appears, that a pendulum consisting of a tube of glass or metal, every where uniform, filled with quicksilver, and 58.8 inches long, will vibrate seconds; for $\frac{2}{3}$ of 58.8 is equal to 39.2. Such a pendulum will be expanded and contracted at the same time; for when the tube is extended by heat, the mercury will also be expanded, and by rising in the tube, will raise the centre of oscillation, so that its distance from the point of suspension will be diminished, and the vibrations of the pendulum, which would have been rendered slower by the expansion of the tube, will become quicker by the expansion of the mercury; and, by adjusting the tube and mercury in such a manner, that these contrary effects may be the same, a clock with such a pendulum would admit of little or no variation for a long time. Phil. Trans. No. 392, p. 40.

SECTION IV.

Of the Centre of Gravity.

PROP. XLVII. In every body there is a *centre of gravity,* or a point about which all its parts balance each other.

Let AB be an inflexible rod, throughout uniform and of the same density; let it be supported at the point C, equally distant from its extreme points A and B, by the prop C. Let A and B be indefinitely small and equal portions of the rod AB. These portions, A and B, tend towards the centre of the earth with equal forces of gravitation. They would likewise, without obstruction, move with equal velocities; for if the rod AB be moved on its prop till it come into the position DE, the velocities of the parts A, B, or F, will be as the spaces over which they pass in the same time; that is, as the arcs AD, EB, or FG, which arcs are as their respective circumferences, or as their diameters or *radii;* whence the velocity of the part B is to the velocity of the part A, or F, as AC, or FC. And the quantities of matter in A and B are by supposition equal. Therefore, if the parts A and B were in motion, they would have equal momenta; that is, the efforts which A and B make to descend towards the earth are equal. But these efforts counteract each other; for, whilst the portion A endeavours, with a certain force, to draw down one arm of the rod, the other portion B endeavours with the same force to draw down the other arm; that is, since the rod is inflexible, to raise the portion A. Therefore the portion A is acted upon by two equal forces in contrary directions, and consequently must be at rest. For the same reason, the portion B will be at rest. And the same may be shown concerning any other equal portions, at equal distances from C, in the rod AB. Therefore the rod will be at rest; that is, the parts on each side of the point will balance each other, and C will be the *centre of gravity.* Plate 2. Fig. 5. Fig. 6.

If the rod were placed oblique to the prop C, indefinitely small and equal parts being taken, as before, at equal distances from C, and resolving each oblique force into a horizontal and perpendicular force (as in Prop. XVI.) it might be shown, by a similar manner of reasoning, that they would tend towards the earth with equal forces, and consequently, that an equilibrium would be produced.

And if, instead of equal portions of the rod, portions of matter were placed at different distances, which should be to each other inversely as those distances, as at F and B, the equilibrium would still be preserved; for the forces with which such portions of matter, so situated, would endeavour to descend, would be equal, when the quantities of matter, multiplied into the velocities with which they are endeavouring to move, that is, into their distances (Prop. XI. Cor.) are equal; as will be more fully shown, in treating of the Mechanical Powers. Fig. 6.

Since, therefore, all the parts of any irregular body may be referred to some one of the above cases, it is manifest, that there is in every body a certain point, the parts on each side of which balance each other.

PROP. XLVIII. If the centre of gravity in any body be supported, the whole body is supported; if this centre be not supported, the body will fall.

For when the centre of gravity is supported, the body rests on a prop on which the parts on each side, acting with equal force against each other, will (Prop. XLVII.) be in equilibrio, and neither side will move; but when this centre is not supported, but the body has a prop under some other point, the parts of the body on one side of that other point will overbalance the parts on the other side, and the body will fall.

Cor. Whenever a body moves by the power of gravitation, or falls, its centre of gravity descends; for if this centre do not descend, it must be supported; and if the centre be supported, the whole body is sustained or kept from falling.

Exp. 1. Let a board of a circular form be sustained perpendicularly on its centre of gravity, it will be at rest in any position.

2. A beam, turning on an axis which passes through its centre of gravity, will rest in the same manner.

3. A beam, whose axis passes through a point which is directly above the centre of gravity, will be at rest only when the beam is parallel to the plane of the horizon, because the centre of gravity will be then fallen as low as possible.

4. A cylinder, which has its centre of gravity near one of its sides, will roll up an inclined plane, if the side nearest the centre of gravity be placed towards the upper part of the plane; for this centre, endeavouring to descend, will carry the cylinder forward in the ascending direction of the plane.

5. Let a body, consisting of two equal and similar cones united at their bases, be placed upon the

edges of two straight and smooth rods, which at one end meet in angle, and rest upon a horizontal plane, and at the other are raised a little above the plane, the body will roll towards the elevated end of the rules, and appear to ascend, while its centre of gravity descends; as may be seen by applying a string horizontally above the path of the base of the cones.

PROP. XLIX. If the line of direction come within the base on which any body is placed horizontally, the body will be sustained; otherwise it will fall.

Plate 2. Fig. 7. In the body ABDE let C be the centre of gravity. The line of direction CO (that is, the line drawn from the centre of gravity towards the centre of the earth) being within the base DE, the body will be supported, because the weight presses upon the base. Also since the body cannot fall towards K without turning round on the point E, the point C must in the motion ascend towards F, contrary to Prop. XLVIII. Cor. But in the position of the body *a b d e*, *c o* the line of direction falling out of the base, *c* in its motion towards *k* descends, and the body will fall.

Our own motions and actions are subject to this rule. When a man stands upright, his centre of gravity falls between his feet, and he is supported; but if he lean forward he throws the line of direction without his base, and he would fall if he did not put forward one of his feet so as to cause it to fall within. Hence a porter, with a load on his back, leans forward that the load may not throw the line of direction out of his base behind; and by an artful adjusting of this point it is that such wonders are performed in horsemanship, and on the tight and slack rope, &c.

Exp. 1. Any body of a cylindrical or other regular form, so placed upon its base, that its line of direction does not come within the base (which may be seen by a cord and weight suspended from the centre of gravity) will fall; otherwise it will not fall.

2. Let two bodies be laid upon an inclined plane, the one a cube, the other a figure with many sides, and let the line of direction of the former fall within the base, and that of the latter without the base, the former will *slide*, the latter *roll* down the plane.

Def. VII. The centre of motion is the point about which a body moves.

PROP. L. A heavy body suspended on a centre of motion will be at rest, if the centre of gravity is directly under, or above, the centre of motion; otherwise it will move.

Plate 2. Fig. 8. If a heavy body E, hangs by a string on a centre of motion C, the action of gravitation at E is in the direction EL, contrary to the direction in which the string acts to prevent the body from falling. In this position, therefore, the opposite forces being equal and in contrary directions, destroy each other, and the body is at rest. But if the body is at *p*, one of the forces acts in the direction *p* C, and the other in the direction *p* L; that is, in directions oblique to each other; whence the body will move in the diagonal of the parallelogram formed by *p* C, *p* L. And in all cases, since (without the aid of mechanical powers afterwards explained) the force which sustains any body must be equal to its weight, the centre of gravitation can only be at rest when these forces are in the same line of direction, that is, when the centre of gravity is directly under, or directly above the centre of motion.

Exp. A circular board, sustained at a point above or below the centre of gravity, will only be at rest when the centre of gravity is at the lowest point, that is, in the line of direction; or when the centre of gravity is in the same line above the centre of motion.

Schol. If two or more bodies be united, they may be considered as one, and have a common centre of gravity.

Exp. 1. Let two unequal balls be fixed upon the ends of a wire, they will have a common centre of gravity.

2. A board, which of itself would fall from a table (its centre of gravity lying beyond the edge of the table) may be made, in the same position, to support a vessel of water, hanging upon it *near the table;* if a stick, fixed with one end at the bottom of the vessel, and the other in a hole in the horizontal board, be long enough to push the vessel a little out of the perpendicular; that is, to bring the centre of gravity of the whole under the table.

Plate 2. Fig. *a*. Schol. The common centre of gravity of any number of bodies may be thus found. Let C be the common centre of gravity of two bodies, dividing their distances (see Prop. XLVII.) in such a manner, that AC is to CB, as B to A; whence $A \times AC = B \times BC$, and consequently if the point C is supported, the bodies A and B balance each other. Suppose a third body, equal to the sum of A and B, placed in their common centre of gravity C; from the point C draw a right line to the centre of a third body D, which divide in O, so that OD may be to OC, as $A + B$ is to D; then is O the common centre of the three bodies, A, B, D. In the same manner may be found the common centre of any number of bodies.

PROP. LI. If any number of bodies move uniformly in right lines, whether in the same or different directions, their common centre of gravity is either at rest, or moves uniformly in a right line.

If two bodies, A and B, move towards each other in the same right line, having their common centre of gravity C, and their momenta equal, the velocity of A will be to that of B, as the body B to the body A, that is (as was shown, Prop. XLVII.) as AC to BC. Whence (Prop. VI.) whilst A passes through AC, B will pass through BC, and the bodies will meet in C, which is their centre of gravity during their motion, and at the time of concourse; therefore the point C remains at rest. Plate 2. Fig. 5.

In the same manner, it may be shown, that if the bodies recede from each other with uniform motions, the centres of gravity will be at rest.

Next, suppose that two bodies, A and B, move in different directions AC, BD, describing equal spaces AC, CE, and BD, DF, in equal times; their common centre of gravity L will move uniformly in a right line. Produce CA, DB, till they meet in G; make AG to GH, as AC is to BD; draw the line AH; and through C and E, draw CI, EK, parallel to AH. AC is to HI (El. VI. 2.) as AG to GH; that is, as AC to BD. Therefore (El. V. 9.) HI is equal to BD, and adding IB to each, HB is equal to ID. In like manner, CE is to IK, as AG to GH; that is, as AC to BD, or CE to DF; therefore (El. V. 9.) IK is equal to DF, and adding KD to each, ID is equal to KF; but ID was proved to be equal to HB; therefore, KF is equal to HB. From L, the common centre of gravity of the bodies A and B, draw LM parallel to BD; draw GM, and produce it till it cut CI, EK, in the points N and O; and through these points draw NP, OQ, parallel to BD. AL is to LB (El. VI. 2.) as AM to MH; and CP to PD, (as CN to NI, that is) as AM to MH; therefore (El. V. 11.) CP is to PD, as AL to LB; that is, (because L is the common centre) as B to A. Consequently, P will be the common centre of the bodies when they are found in C and D; and, in like manner, it may be shown, that Q will be their common centre, when they are in E and F. But, since ML is to HB, as AM to AH; that is, as CN to CI, that is, as NP to ID, and that HB has been proved to be equal to ID, ML is equal to NP; and, in like manner, NP equal to OQ; whence (ML, NP, OQ, being parallel to one another) the line LPQ is equal to the line MNO, and the points, P, Q, (any points of the line in which the common centre of gravity is found as the bodies are moving from A to E, and from B to F) will be in a right line. Moreover, since (El. VI. 2.) AC is to CE, as MN to NO, or LP to PQ, and that AC is equal to CE, LP will be equal to PQ. Therefore the common centre of gravity of the bodies A and B, is always in the same right line, and moves uniformly, or passes over equal spaces in equal times. Plate 2. Fig. 9.

In like manner, the common centre of these two bodies and any third body, or of the three bodies and a fourth, &c. being found, it may be proved that it moves uniformly in a right line.

Cor. 1. Hence it is manifest, that any forces acting upon a system of bodies, must affect the motion of the common centre of gravity of that system, in the same manner as if the same force were similarly applied to a body equal to the sum of all the bodies, placed in the common centre of gravity. And the mutual actions of the parts of a system upon each other, producing (by Prop. III.) equal momenta in contrary directions, cannot change the state of motion or rest of their common centre of gravity. Consequently the law of a system of bodies, as to motion or rest, is the same as that of one body, and is rightly estimated from the motion of its centre of gravity.

Cor. 2. Hence the centre of gravity of a system of bodies will not be disturbed by their mutual attractions, as the motions thus communicated are always equal and opposite. Hence the centre of gravity of our system of planets is either at rest, or moves uniformly in a straight line. The latter is supposed by Dr. Herschel to be the case.

CHAPTER VI.

Of Motion as directed by certain instruments, called Mechanical Powers.

Def. VIII. That body, which communicates motion to another, is called *the Power*.

Def. IX. That body, which receives motion from another, is called *the Weight*.

Def. X. The *Lever* is a bar, moveable about a fixed point, called its *fulcrum*, or prop. It is in theory considered as an inflexible line without weight. It is of three kinds; the first, when the prop is between the weight and the power; the second, when

the weight is between the prop and the power; the third, when the power is between the prop and the weight.

Exp. Let the three kinds of the lever be shown, as in Plate 2, Fig. 12, 13, 14.

PROP. LII. A power and weight acting upon the arms of a lever will balance each other, when the weight is to the power, as the perpendicular distance of the line in which the power acts, from the fulcrum, is to the perpendicular distance of the line in which the weight acts, from the fulcrum.

Plate 2. Fig. 10. Case. 1. When the power acts perpendicularly; Let AB be the lever, C the prop, P the power, W the weight. The force with which any body moves being as its momentum, (Prop. XIII.) and its momentum as the quantity of matter multiplied into the velocity, (by Prop. XI. Cor.) the force with which the weight W would move in the first instant of its motion, if no other body counteracted it, would be as its quantity of matter multiplied into its velocity. But because the weight W is suspended from the lever AB at the point B, it would move with the same velocity as this point; which (as was shown in Prop. XLVII.) is as the distance of the point B from the prop C, or D. The force therefore, with which the weight W would move without any counteracting force, is as its quantity of matter multiplied into the distance of the point of suspension B from the prop C, or D. But the weight will be prevented from descending, if a force equal to that with which it would descend without obstruction, acts upon it in the contrary direction; that is, if a force be applied to raise the point B of the lever AB, equal to to that with which the weight W would draw it downwards. Let the power P be suspended from the other extremity of the lever at the point A; and let the quantity of matter in the power P multiplied into the distance of A, its point of suspension from C, or D, be equal to the quantity of matter in the weight W multiplied into the distance of B from C, or D; it appears from what has been said concerning the weight, that the force with which the power P, without obstruction, would descend and draw down the point A, is equal to the force with which the weight W would descend and draw down the point B. But, as much force as the power P exerts to draw down the point A, it exerts to raise the point B. Therefore equal and opposite forces are exerted to raise and depress the point B; and consequently it will continue at rest, and the weight and power will balance each other.

Case 2. When the power acts obliquely;

Plate 3. Fig. 1. Let the weight A hang freely from one end of a balance, so as to have its line of direction DA perpendicular to the arm of the balance; and let another weight as B, be hung at the other end E, in such manner that its line of direction EC, by passing over a pulley at C, may be oblique to the arm of the balance. If the whole force of gravity in the weight B acting in the direction EC, be denoted by the line EC, it may be resolved into two forces denoted by EF and FC, acting in the directions of these lines; of which two forces, the latter only, which acts in the direction FC perpendicular to the arm of the balance, resists the force of gravity in the weight A, the other force FE acting in the direction of the line of the lever. Since, therefore, that part of the weight B which acts in opposition to the weight A, is to the who le weight B, as FC to EC; it is manifest, that in order to make the weight B balance the weight A, it must exceed the weight A, in the same ratio that the line EC exceeds the line FC. If from G, the centre of motion, be let fall GH perpendicular to EC produced, that line will be the perpendicular distance of the direction EC from G; and EG, equal to DG, the perpendicular distance of the direction DA; but the triangles EFC and EHG are similar, consequently (El. VI. 4.) as EC is to CF, so is EG to HG; but the weight B is to the weight A, as EC to FC; B is therefore to A, as EG, or DG, to HG.

Plate 2. Fig. 11. Or more generally, Let C be the centre of motion in the lever KL; let A and B be any two powers applied to it at K and L, acting in the directions KA and LB. From the centre of motion C, let CM and CN be perpendicular to those directions in M and N; suppose CM to be less than CN, and from the centre C, at the distance CN, describe the circle NHD, meeting KA in D. Let the power A be represented by DA, and let it be resolved into the power DG, acting in the direction CD, and the power DF perpendicular to CD, by completing the parallelogram AFDG. The power DG, acting in the direction CD from the centre of the circle, or wheel, DHN towards its circumference, has no effect in turning it round the centre, from D towards H, and tends only to carry it off from that centre. It is the part DF only that endeavours to move the wheel from D towards H and N, and is wholly employed in this effort. The power B may be conceived to be applied at N as well as at L, and to be wholly employed in endeavouring to turn the wheel the contrary way, from N towards H and D. If, therefore, the power B be equal to that part of A which is represented by DF, these efforts, being equal and opposite, must destroy each other's effect; that is, when the power B is to the power A, as DF to DA, or (because of the similarity of the triangles AFD, DMC) as CM to CD, or as CM to CN, then the

powers must be in equilibrio; and those powers will sustain each other, which are inversely as the distances of their directions from the centre of motion.

Schol. It is evident that in the first kind of lever, either the weight may exceed the power or the power may exceed the weight; but in the second kind, the weight must exceed the power, and in the third, the power must exceed the weight. The second is adapted to produce a slow motion by a swift one; and the third serves to produce a swift motion of the weight, by a slow motion of the power. See Fig. 12, 13, and 14.

To the *first* kind of lever may be reduced several sorts of instruments; such as the steelyard, whose arms are unequal; the false balance, whose arms are imperceptibly unequal; the common balance, whose accuracy depends on its possessing the following properties; (1.) The arms must be equal in length and weight. (2.) The centre of motion must be a little above, and directly over the centre of gravity. (3.) The points from which the scales are suspended should be in a right line, passing through the centre of gravity of the beam. And (4.) the friction of the beam on the centre of motion should be as little as possible. Scissars, pincers, snuffers, &c. are formed of two levers, the fulcrum of which is the pin which rivets them.

To the *second* kind of lever may be reduced oars and rudders of ships; cutting knives fixed at one end; doors moving on hinges, &c.

To the *third* kind, we may refer the action of the muscles of animals, ladders fixed at one end, and raised against a wall by a man's arms, &c.

Def. XI. The *wheel and axis* is a wheel turning round together with its axis; the power is applied to the circumference of the wheel, and the weight to that of the axis, by means of cords.

PROP. LIII. An equilibrium is produced in the wheel and axis, when the weight is to the power, as the diameter of the wheel to the diameter of the axis.

Let AB be the diameter of the wheel, DE that of the axis, W the weight, and P the power, suspended from the points D and B. When the wheel has performed one revolution, the power P has drawn off as much chord from the wheel as is equal to its circumference, and has therefore moved through a space equal to that circumference. In the same time the weight W is raised through a space equal to the circumference of the axis, upon which the cord, by which the weight is suspended, is once turned round. Therefore the velocity of the power exceeds the velocity of the weight, as much as the circumference, that is, the diameter of the wheel exceeds that of the axis. If then the weight exceeds the power as much as the velocity of the power exceeds that of the weight, that is, as much as the diameter, or semidiameter of the wheel, AB, or CB, exceeds the diameter, or semidiameter of the axis, DE, or CE, the momenta will be equal, and the power and weight will balance each other. Plate 3. Fig. 2.

Or thus; The axis and wheel is a lever of the first kind; in which the centre of motion is in C, the centre of the axis; the weight W, sustained by the rope DW, is applied at the distance DC, the radius of the axis; and the power P, acting in the direction PB perpendicular to CB, the radius of the wheel, is applied at the distance of that radius; therefore, Prop. LII. there is an equilibrium, when the power is to the weight, as the radius of the roller to the radius of the wheel.

Cor. 1. Hence it is evident, that by increasing the diameter of the wheel, or diminishing that of the axis, a less power may sustain a given weight.

Cor. 2. The thickness of the rope to which the weight is suspended, ought not to be neglected.

Schol. To the wheel and axle we may refer the capstan, mills, cranes, &c. A drawing and description of a safe and truly excellent crane, invented by Mr. James White, may be seen in the 10th volume of the Transactions of the Society for encouraging Arts and Sciences, in London.

Def. XII. The *pulley* is a small wheel, moveable about its axis, by means of a cord, which passes over it.

PROP. LIV. When the axis of the pulley is fixed, the pulley only changes the direction of the power; if moveable pulleys are used, an equilibrium is produced, where the power is to the weight as one to the number of ropes applied to them. If each moveable pulley has its own rope, each pulley will double the power.

If the pulley ED be fixed upon the beam A, the power and weight, in equilibrio, will be equal. But, if one end of the rope be fixed in B, and the other supported by the power P, it is evident, that in order to raise the weight W one foot, the power must rise two; for both the ropes BC and CP will be Plate 3. Fig. 3. Fig. 4.

shortened a foot each; whence the space run over by the power will be double of that of the weight; if therefore the power be to the weight as 1 to 2, their momenta will be equal. For the same reason if there be four ropes passing from the upper to the lower pulleys, the velocity of the power will be quadruple of that of the weight, or as 4 to 1, &c. In all cases, therefore, when the power is to the weight, as 1 to the number of ropes passing from the upper to the lower pulleys, there will be an equilibrium.

Fig. 5. Or thus; Every moveable pulley hangs by two ropes equally stretched, which must bear equal parts of the weight; and therefore when one and the same rope goes round several fixed and moveable pulleys, since all its parts on each side of the pulleys are equally stretched, the whole weight must be divided equally amongst all the ropes by which the moveable pulleys hang. Consequently, if the power which acts on one rope be equal to the weight divided by that number of ropes, the power must sustain the weight.

Fig. 6. If each moveable pulley has its own cord, the first, as appears from what has been said, doubles the velocity of the power; and therefore if the power be half of the weight, the momenta will be equal, and the balance will be produced. In like manner, the second pulley causes the weight to move with half the velocity with which it would move, if suspended from the first moveable pulley, that is, makes the velocity of the power quadruple of that of the weight; and so of the rest.

Plate. 12 Fig. 1. If in the solid block B, grooves be cut, whose radii are 1, 3, 5, 7, &c. and in the block A other grooves be cut, whose radii are 2, 4, 6, 8, &c. and a string be fastened to A and passed round these grooves, the grooves will answer the purpose of so many distinct pulleys, and every point in each, moving with the velocity of the string in contact with it, the whole friction will be removed to the two centres of motion in the blocks A and B, which is a great advantage over the common pulleys. This pulley was invented by Mr. James White.

PROP. LV. In the inclined plane the power and weight balance each other, when the power is to the weight, as the sine of the inclination of the plane is to the sine of the angle, which the line of the direction of the power makes with the perpendicular to the plane.

Plate 3. Fig. 7. Let a weight be supported on the inclined plane CA by a power acting in any given direction PD. Let the whole force, whereby the weight would descend perpendicularly, be represented by BP; and resolving PB into two forces, one of which, BD, is perpendicular to the plane CA, and the other, PD, is in the direction of the power; the force BD is destroyed by the reaction of the plane, and the force PD will be sustained by an equal power, acting in the direction PD. Therefore, when there is an equilibrium, the power is to the weight, as PD to PB; that is, as the sine of the angle PBD, or (El. VI. 8.) its equal CAB, to the sine of the angle PDB.

When PD is in the direction of the plane, this ratio becomes that of CD to CB, or of the height of the plane CB, to CA its length.

When the direction of the power PD is parallel to the base of the plane, the ratio of the power to the weight becomes that of ED to EB; or (El. VI. 8. Cor.) of CB, the height of the plane, to BA, the base.

When the direction of the power coincides with the perpendicular BD, the ratio of the power to the weight becomes that of the sine of a finite angle, to the sine of an angle indefinitely diminished. From which it appears, that no finite power is sufficient to support a weight upon an inclined plane, if that power acts in a direction perpendicular to the plane.

Def. XIII. The *screw* is a cylinder, which has either a prominent part, or a hollow line, passing round it in a spiral form, so inserted in one of the opposite kind, that it may be raised or depressed at pleasure, with the weight upon its upper, or suspended beneath its lower, surface.

PROP. LVI. In the screw the equilibrium will be produced, when the power is to the weight, as the distance between two contiguous threads, in a direction parallel to the axis of the screw, to the circumference of the circle described by the power in one revolution.

Plate 3. Fig. 8. While the screw is made to perform one revolution, the weight W may be considered as raised up an inclined plane $c\,q$, whose height $c\,p$ is the interval between two contiguous spirals, whose base $p\,q$ is the periphery of the cylinder, and whose length $c\,q$ is the spiral line, by a power acting parallel to the

base of the plane; for such an inclined plane, involved about a cylinder, will form the spiral line of the screw. A power at p, acting parallel to the base, is in equilibrio with the weight W to be raised, when the power is to the weight, as the height of the inclined plane, to the base; or in this case as $p\,c$, the interval between the spirals, to the circumference described by p; but a power applied at P, which is to that applied at p, as the circumference described by p, to the circumference described by P, has the same effect; therefore there is an equilibrium, when the power applied at P is to the weight to be raised, as $p\,c$, the interval between two contiguous spirals, to the circumference described by the power P.

Def. XIV. The *wedge* is composed of two inclined planes, whose bases are joined.

PROP. LVII. When the resisting forces, and the power which acts on the wedge, are in equilibrio, the weight will be to the power, as the height of the wedge, to a line, drawn from the middle of the base to one side, and parallel to the direction in which the resisting force acts on that side.

Let the equilateral triangle ABC represent a wedge, whose base, or back, is AC, whose sides are the lines AB and CB, and whose height is the line BP, which bisects the vertical angle ABC, and also the base perpendicularly in P. Let E and F represent two bodies, or two resisting forces acting on the sides of the wedge perpendicularly, and whose lines of direction EP and FP meet at the middle point of the base, on which the power P acts perpendicularly, then will EP and FP (El. I. 5, and 26) be equal. Let the parallelogram ENFP be completed; its diagonals PN and EF will bisect each other perpendicularly in H. Now when these forces (which act perpendicularly on the sides and base of the wedge) are in equilibrio, they will be to each other (Prop. XIV.) as the sides and diagonal of this parallelogram; that is, the sum of the resisting forces will be to the power P, as the sides EP and FP to the diagonal PN, or as one side EP to half the diagonal PH; that is, (from the similarity of the right-angled triangles BEP, EHP) as BP, the height of the wedge, to EP, the line which is drawn from the middle of the base to the side AB, and is the direction in which the resisting force acts on that side. Plate 3. Fig. 9.

From the demonstration of this case, in which the resisting forces act perpendicularly on the sides of the wedge, it appears that the resistance is to the power which sustains it, as one side of the wedge AB is to the half of its breadth AP; because AB is to AP, (El. VI. 8.) as BP is to EP.

It appears also from hence, that if PN be made to denote the force with which the power P acts on the wedge, the lines PE and PF, which are perpendicular to the sides, will denote the force with which the power P protrudes the resisting bodies in directions perpendicular to the sides of the wedge.

Let us now suppose, in the second case, that the resisting bodies E and F act upon the wedge in directions parallel to the lines DP and OP, which are equally inclined to its sides, and meet in the point P. Draw the lines EG and FK perpendicular to DP and OP; then making PN denote the force with which the power P acts on the wedge, PE and PF will denote the forces with which it protrudes the resisting bodies in directions perpendicular to the sides of the wedge, as was observed before; now each of these forces may be resolved into two, denoted respectively by the lines PG and GE, PK and KF, of which GE and KF will be lost, as they act in directions perpendicular to those of the resisting bodies; and PG and PK will denote the forces by which the power P opposes the resisting bodies, by protruding them in directions contrary to those in which they act on the wedge; therefore, when the resisting forces are in equilibrio with the power P, the former must be to the latter, as the sum of the lines PG and PK, is to PN, or as PG is to PH. But (El. VI. 4.) PG is to PE, as PE to PD; and PH is to PE, as PE to PB; whence (El. VI. 16.) both the rectangle PG × PD and the rectangle PH × PB, are equal to the square of PE; these rectangles are therefore equal to one another; whence their sides (El. VI. 14.) are reciprocally proportional; that is, PG is to PH, as PB to PD. Whence it follows from what was shown above, that, in equilibrio, the resisting forces are to the power, as PB to PD; that is, as the height of the wedge to the line drawn from the middle of the base to one side of the wedge, and parallel to the direction in which the resisting force acts on that side.

From what has been demonstrated, we may deduce the proportion of the power to the resistance it is able to sustain in all the cases in which the wedge is applied. First, when in cleaving timber the wedge fills the cleft, then the resistance of the timber acts perpendicularly on the sides of the wedge; therefore in this case, when the power which drives the wedge, is to the cohesive force of the timber, as half the base, to one side of the wedge, the power and resistance will be in equilibrio.

Secondly; When the wedge does not exactly fill the cleft, which generally happens because the wood splits to some distance before the wedge; let ELF represent a cleft into which the wedge ABC is partly driven; as the resisting force of the timber must act on the wedge in directions perpendicular

to the sides of the cleft, draw the line PD in a direction perpendicular to EL, the side of the cleft, and meeting the side of the wedge in D; then the power driving the wedge and the resistance of the timber, when they balance, will be to each other as the line PD to PB, the height of the wedge.*

Thirdly; When a wedge is employed to separate two bodies that lie together on a horizontal plane, for instance, two blocks of stone; as these bodies must recede from each other in horizontal directions, their resistance must act on the wedge in lines parallel to its base CA; therefore the power which drives the wedge will balance the resistance when they are to each other as PA, half the breadth of the wedge, to PB its height.

Schol. 1. Since in all the mechanical powers, an equilibrium is produced, when the power is to the weight as the velocity of the weight is to the velocity of the power, in all compound machines there will be an equilibrium, when the sum of the powers is to the weight, as the velocity of the weight is to the sum of the velocities of the powers.

Schol. 2. In the theory of mechanical powers, we suppose all planes and bodies perfectly smooth; levers to have no weight; cords to be perfectly pliable, and the parts of machines to have no friction. (See Schol. 3.) Allowances, however, must be made for the difference between theory and practice. Mr. Ferguson observes, that there are but few compound machines, but what, on account of friction, will require a third part more to work them, when loaded, than what is sufficient to constitute an equilibrium between the weight and the power.

Plate 2. Fig. 15. Exp. 1. Let A, B, C, be a compound lever, consisting of three levers, in the first of which, A, the velocity of the weight is to that of the power, as 1 to 5; in the second, B, as 1 to 4; in the third, C, as 1 to 6. The velocity of the weight will be to that of the power, as 1 to $5 \times 4 \times 6 = 120$; and if the power be to the weight, as 1 to 120, they will balance each other.

Fig. 16. 2. Let GC and LF be the levers fixed to the supporters RA, SE, and let their shorter arms be kept in equilibrio with the longer respectively by the weights fixed at G and L. Let NH be a bar screwed to the fixed parts to keep them steady. If the power C be ten times farther from A the prop, than the weight P, they will be in equilibrio when the power C is to the weight P, as 1 to 10. In like manner, the distance ME being ten times DE, if the power M be $\frac{1}{10}$ of the weight C suspended from D, they will be in equilibrio; whence M, 1, will balance P, 100.

3. Exhibit models or draughts of different compound machines, as mills, cranes, the pile-driver, &c.

Schol. 3. The inequality of the surface on which any body moves occasions an attrition, called friction, which prevents the accurate agreement of many experiments in mechanics, with theory. On this subject the very accurate experiments of Mr. Vince should be consulted, the object of which was, to determine, (1.) Whether friction be an uniformly retarding force. (2.) The quantity of friction. (3.) Whether friction varies in proportion to the pressure or weight. And (4.) whether the friction be the same, on whichever of its surfaces a body moves. After a great variety of experiments made with the utmost care and attention, Mr. Vince deduces the following conclusions, which may be considered as established facts.

1. That friction is an uniformly retarding force in hard bodies, not subject to alteration by the velocity; except when the body is covered with woollen cloth, &c. and in that case the friction increases a little with the velocity.

2. Friction increases in a less ratio than the weight of the body, being different in different bodies. It is not yet sufficiently known for any one body, what proportion the increase of friction bears to the increase of weight.

3. The smallest surface has the least friction, the weight being the same. But the ratio of the friction to the surface is not accurately known.

See a full account of these experiments, Vol. lxxv, Phil. Trans.

Schol. 4. Wheel carriages are used, to avoid friction as much as possible. A wheel turns round upon its axis, because the several points of its circumference are retarded in succession by attrition, whilst the opposite points move freely. Large wheels meet with less resistance than smaller from external obstacles, and from the friction of the axle, and are more easily drawn, having their axles level with the horses. But in uneven roads, small wheels are used, that in ascents the action of the horse may be nearly parallel with the plane of ascent, and therefore may have the greatest effect; small wheels are also more conveniently turned. The greater part of the load should be laid on the hinder part of a wheel carriage.

* In estimating the lateral cohesion of woody fibres when separated by a wedge, the pressure on only one side of the wedge should be reckoned, so that on *this account* the cohesion should be estimated at only half what it is in the text; but on *another account*, not mentioned in the text, it should be reckoned much more, for the sides of the cleft are actually levers, in which the pressure of the wedge is the power, the point where the cohesion is just giving way is the place of weight, while the fulcrum is at some distance further from the wedge, more or less, according to the rigidity or flexibility of the timber; so that, notwithstanding the cohesive force is erroneously doubled in the text, it is probably much underrated.

CHAPTER VII.

Of Motion as produced by the united Forces of PROJECTION *and* GRAVITATION.

SECTION I.

Of Projectiles.

PROP. LVIII. Bodies thrown horizontally or obliquely, have a curvilinear motion, and the path which they describe is a parabola; the air's resistance not being considered.

If a body be thrown in the direction AF, and acted upon by the projectile force alone, it will continue to move on uniformly in the right line AF, and would describe equal parts of the line AF in equal times, as AC, CD, DE, &c. But if, in any indefinitely small portion of time, in which the body would by the projectile force move from A to C, it would, by the force of gravity, have fallen from A to G; by the composition of these forces (Prop. XVI.) it will at the end of that time, be found in H, the opposite angle of the parallelogram ACGH. In two such portions of time, whilst it would have moved from A to D by the projectile force, it would (Prop. XXVI.) by gravitation fall through four times AG, that is, AM; and therefore, these forces being combined, it will be found at the end of that time in I, the opposite angle of the parallelogram DM. In like manner, at the end of the third portion of time, it would by the projectile force be carried through three equal divisions to E, and by the force of gravitation over nine times AG to N; and consequently, by both these forces acting jointly, it will be carried to K, the opposite angle of the parallelogram EN. Therefore the lines CH, DI, EK; that is, AG, AM, AN, which are to each other as the numbers 1, 4, 9, are as the squares of the lines AC, AD, AE; that is, GH, MI, NK, which are as 1, 2, 3. And because the action of gravitation is continual, the body in passing from A to H, &c. is perpetually drawn out of the right line in which it would move if the force of gravitation were suspended, and therefore moves in a curve. And H, I, and K are any points in this curve in which lines let fall from points equally distant from A in the line AB meet the curve. Therefore the body moves in a parabola, the property of which is (*Simpson's Conic Sections*, Book I. Prop. XII. Cor.) that the *abscissæ* AG, AM, AN, are to each other as the squares of the ordinates GH, MI, NK. Plate 3. Fig. 10, 11.

REMARK. Very dense bodies moving with small velocities describe the parabolic track so nearly, that any deviation is scarcely discoverable; but with very considerable velocities the resistance of the air will cause the body projected to describe a path altogether different from a parabola, which will not appear surprising when it is known that the resistance of the air to a cannon ball of two pounds weight, with the velocity of 2000 feet per second, is more than equivalent to 60 times the weight of the ball. See *Hutton's* Dict. Art. *Resistance.*

PROP. LIX. The path which a body thrown perpendicularly upward describes in rising and falling is a parabola.

A stone lying upon the surface of the earth, partaking of the motion of the earth (here supposed) round its axis, this motion which it has with the earth will not be destroyed by throwing it in a direction perpendicular to the surface of the earth. After the projection, therefore, the stone will be moved by two forces, one horizontal, the other perpendicular, and will rise in a direction which may be shown, as in the last proposition, to be the parabolic curve; in which it will continue till it reaches the highest point, from whence it might be shown, as in the last proposition, that it will descend through the other side of the parabola.

PROP. LX. The velocity with which a body ought to be projected to make it describe a given parabola, is such as it would acquire by falling through a space equal to the fourth part of the parameter belonging to that point of the parabola from which it is intended to be projected.

The velocity of the projectile at the point A (by Prop LVIII.) is such as would carry it from A to E, in the same time in which it would descend by its gravity from A to N. And the velocity acquired in falling from A to N (by Prop. XXVII.) is such as in the same time by an uniform motion would carry the Plate 3. Fig. 10, 11.

body through a space double of AN. Therefore the velocity which is acquired by the body in falling to N is to that with which the body is projected at A, and uniformly carried forward to E, as twice AN is to AE. But since, from the nature of the parabola (*Simpson's Conic Sections*, Book I. Prop. XIII.) $\frac{AE^2}{AN}$ is equal to the parameter of the point A, one fourth part of this parameter will be expressed by $\frac{\frac{1}{4}AE^2}{AN}$. And because the velocities acquired by falling bodies are (by Prop. XXVI. Cor. 1.) as the square roots of the spaces they fall through, the velocity acquired by a body in falling through AN is to the velocity acquired in falling through $\frac{\frac{1}{4}AE^2}{AN}$ or one fourth part of the parameter of A, as the square root of AN to the squore root of $\frac{\frac{1}{4}AE^2}{AN}$; that is, as $\sqrt{AN}$ to $\frac{\frac{1}{2}AE}{\sqrt{AN}}$, or AN to $\frac{1}{2}$ AE, or twice AN to AE. Therefore the velocity acquired by a body in falling from A to N has the same ratio to the velocity with which the body is projected or the line AE described, and to the velocity acquired by a body in falling through a fourth part of the parameter belonging to the point A; consequently (El. V. 11.) these velocities are equal.

Plate 3. Fig. 13. Cor. Hence may be determined the direction in which a projectile from a given point, with a given velocity, must be thrown to strike an object in a given situation.

Let A be the place from which a body is to be thrown, and K the situation of the object. Raise AB perpendicular to the plane of the horizon, and equal to four times the height from which a body must fall to acquire the given velocity. Bisect AB in G; through G draw HG perpendicular to AB; at the point A raise AC perpendicular to AK, and meeting HG in C; on C as a centre with a radius CA describe the circle ABD; and through K draw the right line KEI perpendicular to the plane of the horizon, and cutting the circle ABD in the points E and I. AE, or AI, will be the direction required.

For, drawing BI, BE, since AK is a tangent to the circle, and BA, IK, are parallel to each other, the angle ABE (El. III. 32.) is equal to the angle EAK; and the alternate angles BAE, AEK, are equal; therefore the triangles ABE, AEK, are similar; and AB is to AE, as AE to EK. Therefore $AB \times EK = AE^2$; and $AB = \frac{AE^2}{EK}$. In like manner, the triangles BAI, KAI, being similar, BA is equal to $\frac{AI^2}{IK}$. Since, then, AB is equal to four times the height from which a body must fall to acquire the velocity with which it is to be thrown; $\frac{AE^2}{EK}$ (or $\frac{AI^2}{IK}$ its equal) is the same. Consequently (by this Prop.) the point K will be in the parabola which the body will describe, which is thrown with the given velocity in the direction AE, or AI, and the body will strike an object placed at K.

Schol. If the velocity with which a projectile is thrown be required, it may be determined from experiments in the following manner. By the help of a pendulum or any other exact chronometer, let the time of the perpendicular flight be taken; then, since the times of the ascent and descent are equal, the time of the descent must be equal to one half of the time of the flight, consequently, that time will be known; and, since a heavy body descends from a state of rest at the rate of 16.1 feet in the first second of time, and that the spaces through which bodies descend are as the squares of the times; if we say, as one second is to 16.1 feet, so is the square of the number of seconds which express the time of the descent of the projectile, to a fourth proportional, we shall have the number of feet through which the projectile fell, which being doubled, will give us the number of feet which the projectile would describe in the same time with that of the fall, supposing it moved with a uniform velocity, equal to that which it acquired by the end of the fall; which last found number of feet, being divided by the number of seconds which express the time of the projectile's descent, will give a quotient, expressing the number of feet, through which the projectile would move in one second of time with a velocity equal to that which it acquired in its descent, which velocity is equal to the velocity with which the projectile was thrown up; consequently, this velocity is discovered.

PROP. LXI. The squares of the velocities of a projectile in different points of its parabola, are as the parameters belonging to those points.

For (by the last Prop.) the velocities in the several points of the parabola, are equal to the velocities acquired in falling through the fourth parts of the parameters of the points. Therefore the squares of these velocities being (by Prop. XXVI.) as the spaces described, the squares of the velocities in the several points of the parabola are as the fourth parts of the parameters of those points; but the whole parameters are as their fourth parts; therefore the squares of the velocities at the several points of the parabola are as the parameters of those points.

Cor. Hence, setting aside any difference which may arise from the resistance of the air, a projectile will strike a mark as forcibly at the end as at the beginning of its course, if the two points be equally distant from the principal vertex; for, the parameters belonging to these points being equal, the velocities in these points must also be equal.

PROP. LXII. When a body is thrown obliquely with a given velocity, if the space through which it must have fallen perpendicularly to acquire that velocity is made the diameter of the circle, the height to which the body will rise is equal to the versed sine of double the angle of elevation.

Let a body be thrown in the direction BE, with the same velocity which any body would acquire by falling perpendicularly through AB; if AB be made the diameter of a circle, the greatest height to which it will rise will be BD. Plate 3. Fig. 12.

Let IL be a right line drawn in the plane of the horizon, touching the circle in B, and making with the line BE, which is the direction in which the body is thrown, the angle IBE, or angle of elevation. Because IL touches the circle, and EB drawn in the circle meets it in the point of contact, (El. III. 32.) the angle EBI is equal to the angle EAB. And ECB is double of EAB, (El. III. 20.) therefore ECB is double of EBI, the angle of elevation. And BD is the versed sine of ECB; that is, of double the angle of elevation.

Let BE represent the velocity with which the body is thrown. Then since this velocity is, by supposition, such as might be acquired by falling down AB, if the body was thrown perpendicularly upward with the same velocity BE, it would rise to the height BA. Let the oblique motion BE be resolved into two others, one in the direction BD perpendicular to the horizon, and the other in the direction DE parallel to it; then the ascending velocity will be to the horizontal velocity, as BD to DE, and to the whole velocity, as BD to BE. But the part of the velocity BD is the only part which is employed in raising the body, since the other part DE is parallel to the plane of the horizon. Now, the height of a body ascending perpendicularly with the whole velocity BE, will be to the height when it ascends with the part BD (compare Prop. XXVI. and Prop. XXVIII.) as the square of BE to the square of BD. But because (El. VI. 8.) the triangle EDB is similar to the triangle AEB, BD is to EB, as EB is to BA; and BD, BE, BA, being continued proportionals, BD is to BA, as the square of BD is to the square of BE. And the perpendicular heights to which the velocities BE and BD will make the body ascend have been shown to be as the square of BE to the square of BD; the heights are therefore as BA to BD. Since therefore the first velocity BE would make the body ascend through BA, the other velocity BD, which is the part of the whole velocity which acts to make the body thrown in the direction BE to ascend, will carry it to the height BD, which is the versed sine of double the angle of elevation. The same might be shown in any other direction of the body, as BF, or BG.

Def. XV. The *Random* of a projectile is the horizontal distance to which a heavy body is thrown.

PROP. LXIII. When a body is thrown obliquely with a given velocity, if the space through which it must have fallen perpendicularly to acquire that velocity is made the diameter of a circle, the random will be equal to four times the sine of double the angle of elevation.

If EBI be the angle of elevation, and ECB double that angle, DE will be the sine of double the angle of elevation. Let a body be thrown from the point B in the direction BE, with the velocity which it would acquire in falling through AB; the random, or horizontal distance at which the body will fall, is equal to four times DE. Plate 3. Fig. 12.

For, since (as in the last Prop.) the velocity BE being resolved into BD, DE, the ascending velocity is BD, and the horizontal DE, if these two velocities were to continue uniform, the spaces described in equal times (Prop. V.) would be as the velocities, and in the same time in which the body by the ascending velocity would rise through BD, by the horizontal velocity it would be carried forward through DE. Of these velocities, the horizontal one DE is uniform, because the force of gravity can neither accelerate nor retard a motion in this direction; but the ascending velocity is uniformly retarded; and therefore the body (compare Prop. XXIII. and XXVIII.) will be twice as long in ascending to its greatest height BD, as it would have been if the first ascending velocity had continued uniform; but on this supposition, the body would have been carried through BD and DE in the same time; therefore in double the time, that is, in the time of ascent through BD with an uniformly retarded velocity, it would be carried forward through twice DE; consequently, in the times of descent and ascent together it would move forward through four times DE. Therefore a body thrown from B in the direction

BE with such a velocity as might be acquired by falling down AB, the diameter of a circle, will fall at the distance of four times the sine of double the angle of elevation.

PROP. LXIV. **The random of a projectile will be the greatest possible, with a given velocity, when the angle of elevation is an angle of forty-five degrees.**

The velocity being given, the height from whence the body must have fallen to acquire that velocity, or (Prop. XXXV.) the diameter of the circle AB, is a given quantity. And in a given circle the greatest sine is the radius or sine of a right angle; therefore four times the radius is greater than four times any other sine; and consequently, the random which is equal to four times the radius (which by Prop. LXII. will be the case when the double angle of elevation is a right one, or the angle of elevation forty-five degrees) will be the greatest possible random.

Exp. This proposition, and the two following, may be illustrated by water spouting from a pipe.

PROP. LXV. **The random of a projectile, whose velocity is given, will be the same at two different elevations, if the one be as much above forty-five degrees as the other is below it.**

Plate 3. Fig. 12. If EBI be an angle of 30 degrees, and GBI an angle of 60 degrees, because EBI falls short of half a right angle as much as GBI exceeds it, the double of EBI will fall short of a right angle as much as the double of GBI will exceed it; therefore, from the definition of a sine, these doubles will have the same sine. Consequently, four times their sines, that is, (by Prop. LXIII.) their randoms will be equal.

PROP. LXVI. **The greatest random of a projectile, whose velocity is given, is double the height to which it would rise if it were thrown perpendicularly with the same velocity.**

Plate 3. Fig. 12. If a body be projected in the direction BF, at an angle of forty-five degrees, and its velocity be equal to that which a body would acquire in falling down AB, (by Prop. LXIV.) the random will be the greatest possible, and will be equal to four times CF, or twice BA. But the body cast perpendicularly upwards with the same velocity would (by Prop. XXVIII.) rise to the height BA. Therefore the greatest random, with a given velocity, is double the height to which the body, thrown perpendicularly with the same velocity, would rise.

PROP. LXVII. **The randoms of projectiles, whose elevations are given, are as the squares of their velocities.**

Plate 3. Fig 12. If a body be thrown in any direction BE, its random (Prop. LXIII.) will be equal to four times DE, or four times the sine of double the angle of elevation, in a circle whose diameter AB is the height from which the body must fall to acquire the velocity with which it is projected. And because, in the triangle EDC, the angle at D being a right angle is always invariable, and that the angle ECD, which is double of EAD, that is, (El. III. 32.) of the given angle of elevation EBI, is given, the triangle ECD in every variation of AB, is always equiangular and similar to itself, and ED is always as EC; but EC being a radius, is as AB; therefore ED, the sine of twice the given angle of elevation, is as AB, the diameter. Consequently four times the sine ED, that is, the random, is as AB. But the height AB, from which a body must fall to acquire any velocity, is (by Prop. XXVI.) as the square of that velocity. Therefore the random is as the square of the velocity.

SECTION II.

Of Central Forces.

PROP. LXVIII. **A body which is constantly drawn or impelled toward any point, may be made to describe, round that point as a centre, a curve returning into itself.**

Plate 3. Fig. 14. Let T be the centre of the earth, and GDEI its surface. Let a body be projected in any direction GH, which does not pass within the surface of the earth. The projectile force, together with the force of gravity, will make it describe a curve, which, as the projectile force is increased, will recede farther from the perpendicular GE, as GB, GC, GD. It is manifest that the projectile force may be increased, till the body shall pass beyond the surface CDKF and move in the path GML, GNV, or some larger curve.

First; Suppose the projectile force to be such, that the body will be carried in the semicircle GN, it will continue in the curve of that circle till it returns to G. For, when a body moves in the circumference of a circle (as in fig. 15.) the projectile force, acting in a line which is a tangent to the circle, as GB, acts (El. III. 18.) in a direction which is perpendicular to the direction BA, in which it is impelled towards the centre. And since if the force which impels the body towards the centre ceased to act in any point, as C, the body would move forward in the right line CF, the projectile force in every point of the circumference, acts in a direction perpendicular to the force of gravitation; consequently, these two forces remaining the same, and acting always in the same direction with respect to each other, the velocity of the body must remain the same; whence, at the point M, it will have the same power to recede from the centre as at G; and, retaining this power through every remaining part of its course, it will proceed in the circumference, till it arrive at G, and will continue to revolve in the circle. Plate 3. Fig. 15. Fig. 14.

Next; Let the body be projected from G with a force less than that which is required to carry it round in the circumference of the circle GNV; and let the curve in which it moves be an ellipse, having the earth in its remoter focus. Because the force of projection, as the body proceeds in the first half of its orbit, acts in the direction of a tangent to the curve, whilst the force of gravitation acts in the direction of a right line from the body to the centre of the earth, the directions of these two forces make an acute angle with one another, and consequently, through this part of the course of the body, the force of gravitation *conspiring with* the force of projection, the velocity of the body must be increased, and at the same time it must be continually *drawn downward* toward the earth. At the point in which the forces act in directions perpendicular to each other, the force of gravitation does not conspire with that of projection to bring the body toward the earth; and afterward in the latter half of its course, the directions of the forces making an obtuse angle with each other, the force of gravitation is *opposed* by that of projection in the same degree in which the former was before aided by the latter; and therefore the body in passing toward G will fly off from the earth or *rise*, as much as it before approached to the earth or *descended*, and thus will return to the point G with the same velocity with which it set out at first, having lost as much velocity by receding from the earth in the latter part of its course, as it had gained by falling toward the earth in the former part.

Lastly; Let the body be projected from G with a force which is greater than sufficient to carry it round in the circle GNV; and let it perform its revolutions in an elliptic curve, whose greater axis is greater than the diameter of the circle GNV, setting out from G, and having the earth in the nearer focus; the effect will be the same as in the last case, except that the projectile force will oppose the force of gravitation in the *first* half of the revolution, and conspire it with the *latter*.

Exp. Let a ball revolve round the central point of a whirling table. Concerning the construction and use of this machine, see Ferguson's Lectures, Lect. II.

PROP. LXIX. A body revolving in an orbit, endeavors in every point of its course to fly off from the centre in a right line, which is a tangent to the orbit.

Let BCDL be a circle in which a body is revolving; when it is arrived at the point B, let the force which impels it toward the centre be withdrawn, and the body (by Prop. I.) would fly off from the point B in the direction BG; in like manner at C, it would fly off in the right line CF; at D, in DH; and at L, in LK. The same is manifestly true in an elliptical orbit. Now the same force with which it would fly off, if no other cause prevented it, must make it endeavour to fly off in the same manner in every point of the orbit. Plate 3. Fig. 15

Exp. Whilst a ball is revolving on a whirling table, if the cord which retains it be suddenly cut, the ball will fly off in a right line, which will be a tangent to the orbit in which it moved.

Cor. A body revolving about a centre endeavours to recede from that centre; for every point of the tangent in which it endeavours to move out of the circle, is farther from the centre, than the point in which the tangent meets the curve.

Def. XVI. The force which impels a body toward the centre, when it revolves in an orbit, is called the *centripetal* force; that by which it endeavours to recede from the centre, is called the *centrifugal* force; and these two forces are called jointly the *central* forces.

Schol. The projectile and centrifugal forces differ from each other, as the whole from the part. The projectile force is that with which a body would move forward in a tangent to its orbit, if there were no centripetal force to prevent it; the centrifugal force is that part of the projectile force which carries the body off from the centre while it is describing the tangent. Thus if the body revolved in the orbit BD, the projectile force is that which would make it describe the tangent BA, if the centripetal force were to cease acting. But in the mean time, the whole force BA does not carry the body off Plate 3. Fig. 16.

from the centre C; when it is arrived at A, it is farther from the centre than it was at B, only by the length AN, and it is that part of the projectile force which, when the whole is resolved into two forces, may be considered as acting in this line AN, which carries the body off from the centre, and is called the centrifugal force.

PROP. LXX. When bodies revolve in a circular orbit about a centre, the centripetal and centrifugal forces are equal.

Plate 3. Fig. 16. If a body revolve in the circle BD, in the time in which it describes the arc BN, it will have been impelled toward the centre through the space AN; for, by the projectile force alone it would have been carried from B to A. The line AN is then the space described by means of the centripetal force, and this force is proportional to AN. But if, when the body was at B, no centripetal force had acted upon it, instead of describing the arc BN, it would have moved along the tangent BA, and the line NA would have been the space through which it would have departed from the centre; therefore the centrifugal force is proportional to NA. Both these forces being then proportional to the same line NA, they are equal to each other.

LEMMA I.

Quantities and the ratios of quantities, which in any finite time, tend continually to equality, and, before the end of that time, approach nearer to each other than by any given difference, become ultimately equal.

If you deny it, let them be ultimately unequal; and let their ultimate difference be D. Therefore they cannot approach nearer to equality than by that given difference D; which is contrary to the supposition. If a straight and a curve line, continually diminishing, perpetually approach toward equality, and at the end of any finite time would vanish together, at the instant in which they are vanishing they are equal.

Plate 3. Fig. 8. LEM. II. *If in any figure A* a c *E, terminated by the right lines A* a, *AE, and the curve* a c *E, there are inscribed any number of parallelograms A* b, *B* c, *C* d, *&c. contained under equal bases AB, BC, CD, &c. and the sides, B* b, *C* c, *D* d, *&c. parallel to A* a *the side of the figure; and the parallelograms* a *K* b l, b *L* c m, c *M* d n, *&c. are completed; then, if the breadth of these parallelograms be diminished, and their number augmented continually, the ultimate ratios, which the inscribed figure AK* b *L* c *M* d *D, the circumscribed figure A* a l b m c n d o E, *and the curvilinear figure A* a b c d *E, have to each other, are ratios of equality.*

For the difference of the inscribed and circumscribed figure is the sum of the parallelograms K *l*, L *m*, M *n*, D *o*, that is, (because of the equality of all their bases) the rectangle under one of their bases K *b*, and the sum of their altitudes A*a*; that is, the rectangle AB *l a*. But this rectangle, because its breadth AB is diminished indefinitely, becomes less than any given rectangle. Therefore (by Lem. I.) the inscribed and circumscribed, and much more the intermediate curvilinear figure, become ultimately equal.

Plate 3. Fig. 18. LEM. III. *The same ultimate ratios are also ratios of equality, when the breadths* AB, BC, CD, *&c. of the parallelograms are unequal, and are all diminished indefinitely.*

For, let AF be equal to the greatest breadth; and let the parallelogram FA *a f* be completed. This will be greater than the difference of the inscribed and circumscribed figures; but, because its breadth AF is diminished indefinitely, it will become less than any given rectangle.

Cor. Hence the ultimate sum of the evanescent parallelograms coincides in every part with the curvilinear figure. Much more does the rectilinear figure, which is comprehended under the chords of the evanescent arcs *a b*, *b c*, *c d*, &c. ultimately coincide with the curvilinear figure. As also the circumscribed rectilinear figure, which is comprehended under the tangents of the same arcs. And therefore, these ultimate figures (as to their perimeter *a c* E) are not rectilinear, but curvilinear limits of rectilinear figures.

PROP. LXXI. The plane of an orbit in which a body revolves passes through the line of projection, and through the centre toward which the centripetal force is directed.

Let ABCF be the orbit in which the body revolves; S the centre, or point toward which the centripetal force is directed; and AV the line of projection; the plane of the orbit will pass through AV and S; or the orbit lies in the same plane, with the lines AV, B *d*, &c. (lines in which the projectile force acts in different parts of the orbit) and with the centre S. Plate 4. Fig. 2.

For, let ABCD represent a part of the orbit described by a body impelled toward S. The body beginning to move by the projectile force from A in the direction ABV would, by that force alone, be carried on uniformly in that direction. Suppose the centripetal force to act upon it by separate impulses after equal intervals of time, and that, when the body is carried by the projectile force to B, it receives one impulse from the centripetal force, drawing it out of its course toward S, so that by the action of both forces together at B, it will (by Prop. XIV.) be made to describe BC in the same time in which the projectile force alone would have made it describe B *c*. The same will take place after equal intervals at C and D. At B, the projectile force is in the direction BV, the centripetal force in the direction BS. Let B *c* and BG, taken in the direction of these forces, represent their ratio to each other, and the projectile force be to the centripetal as B *c* to BG. The body (by Prop. XIV.) will describe BC, the diagonal of a parallelogram of which B *c* and BG are the sides. But (El. XI. 1. and 2.) BC is in the same plane with B *c* and BG, that is, with AV, the line of projection, and with BS, in which is the centre or point S. The same may be proved concerning the lines CD, and DE. And if the centripetal force act continually, and not by interrupted impulses, the diagonals AB, BC, &c. will be diminished indefinitely, and the ultimate perimeter ADF (by Lem. III. Cor.) will become a curve line, which, from what has been shown, must be always in the same plane with the line of projection, and with the centre.

PROP. LXXII. A body revolving in an orbit describes, by a radius drawn to the point toward which the centripetal force acts, equal areas in equal times, and in unequal times areas proportional to the times.

Let ABCD be part of an orbit described by a body which revolves round the point S, toward which it is impelled by a centripetal force. If this force be supposed to act upon the body by separate impulses, as at B, C, D, when the body receives the impulse at B, it will be drawn out of its course toward S, and (by Prop. XIV.) will describe the diagonal BC in the same time in which the projectile force alone would have made it describe B *c*. After equal intervals, the same will take place at C, and at D. Plate 4. Fig. 2.

Since AB, BC, &c. are the lines described in equal times by the body, the areas described round S by a radius drawn from the body to S, are ASB, BSC, &c. Now AB, B *c*, expressing spaces passed over in equal times by the uniform motion of the body acted upon by the projectile force alone, are equal bases of the triangles ASB, BS *c*, which, being terminated by the same point S, are of the same altitude; these triangles are therefore (El. I. 38.) equal. And, because the body at B is by the joint action of the projectile and centripetal forces carried forward in the diagonal BC, of the parallelogram G *c*, the opposite sides thereof, GB, C *c*, are parallel; and C *c* is parallel to BS. But BS is the common base of the two triangles BSC, BS *c*. Therefore these triangles, being upon the same base, and between the same parallels, (El. I. 37.) are equal. Consequently, ASB, which has been proved equal to BS *c*, is likewise equal to BSC; that is, the areas described in equal times are equal. And, by composition, any sums of these areas ASC, ASE, are to each other as the times in which they are described; that is, universally, the areas are as the times.

Let the number of these triangles be augmented, and their breadth diminished indefinitely, and (by Lem. III. Cor.) their ultimate perimeter will be a curve line; and therefore the centripetal force will act continually; and the above reasoning still being applicable to those triangles whose breadth is indefinitely diminished, the areas will be as the times.

Cor. 1. The velocity of a body revolving freely about an immoveable centre is inversely as a perpendicular let fall from that centre on a right line that touches the orbit. For since the lines AB, BC, are described in equal times, the velocities will be as other lines, which being (by this Prop.) the bases of equal triangles, must be inversely as the heights of the triangles; therefore the velocities are inversely as these heights, which are measured by perpendiculars let fall from the common vertex, the centre S, to the bases, or the bases produced; that is, when AB, BC, &c. are indefinitely small, to a tangent to the orbit.

Cor. 2. If the chords AB, BC, of two arcs successively described in equal times by the same body moving freely, are completed into a parallelogram ABCG, and the diagonal BG, in the position which it ultimately acquires when these arcs are diminished indefinitely, be produced toward S, it will pass through S, the centre of the centripetal force; for BG is an indefinitely small part of the radius SB.

Cor. 3. If the chords AB, BC, and DE, EF, of arcs described in equal times be completed into the

parallelograms ABCG, DEFZ, the centripetal forces at B and E, will be to each other in the ultimate ratio of the diagonals BG, EZ, when those arcs are indefinitely diminished. For, the motions of the body BC, EF. (by Prop. XVI.) are compounded of the motions B c, BG, and Ef, EZ; but BG and EZ are equal to C c and F f, which, as appears from this proposition, are generated by the impulses of the centripetal force in B and E, and are therefore proportional to those impulses.

Cor. 4. The forces with which bodies are drawn into curvilinear orbits are to each other as the versed sines, $\frac{1}{2}$GB, $\frac{1}{2}$ZE, of the indefinitely small arcs AC, DF, described in equal times, which versed sines converge to the centre S, and bisect the chords when those arcs are diminished indefinitely; for such versed sines are half the diagonals of a parallelogram BG, EZ, being bisected by the diagonals AC, FD.

PROP. LXXIII. When a body describes equal areas in equal times about an immoveable point, or proportional areas in unequal times, it is impelled towards that point by the centripetal force which retains it in its orbit.

Plate 4. Fig. 2. Let AB, BC, &c. be lines described by the revolving body in equal times; and it may be proved as before, that the triangles ASB and BS c being of the same height, and having equal bases AB, B c, are equal. But, by supposition, ASB, BSC, are equal: therefore BS c and BSC are equal. And these equal triangles, being upon the same base SB, (El. I. 39.) are between the same parallels; therefore C c and BS are parallel. Because the body at B is acted upon by two forces, the projectile force in the line B c, and the centripetal force, and by supposition, these two forces together make it describe BC; BC is the diagonal of a parallelogram of which B c is one side, and the direction of the centripetal force at B is in the other side. Now, in the parallelogram whose diagonal is BC, and one of its sides B c, C c must be another: whence the opposite side, that is, the direction of the centripetal force, must be parallel to C c; but C c and BS have been proved to be parallel, when equal areas are described in equal times about the point S. Therefore, on this supposition, the body at B is acted upon by the centripetal force in the direction BS. The same may be shown at every other point C, D, &c. Therefore the centripetal force tends to that point round which the body, by a radius drawn thither, describes equal areas in equal times.

Plate 4. Fig. 1. LEM. IV. *If any arc* ACB, *of a finite curvature, is subtended by its chord* AB, *and a straight line* AD *produced both ways touch the arc in the point* A, *the arc, the chord, and the tangent, in their ultimate vanishing state, will be equal.*

The arc being supposed of a finite curvature, or such as may be measured by a circle of a finite diameter; let BAC be the circle of the curvature; draw the line AC in that circle parallel to the subtense BD, and complete the triangle BAC. Because the angle DAB (El. III. 32.) is equal to the angle ACB, and the alternate angles CAB, ABD, are equal, the triangles, CAB, DAB, are similar. Hence, (El. VI. 4.) AB is to AD, as AC is to BC. The point B approaching continually to A, let BA become less than any assignable quantity; then the finite lines AC, BC, approach nearer to the ratio of equality, than by any given distance; therefore likewise AB, AD, which are proportional to AC, BC, and much more the intermediate arc, are ultimately equal.

LEM. V. *The nascent or evanescent subtense of the angle of contact, in circles and in all curves which have a finite curvature, is as the square of the conterminous arc.*

Plate 3. Fig. 17. Let AD, in the semicircle ADC, be the given arc, AB the tangent, and the angle BAD the angle of contact. Draw BD, HG, parallel to AC, the diameter; these lines, subtending the angle BAD, are called the subtenses of the angle of contact. The arcs AD and AG, having the common term or limit, the point A, are called conterminous arcs. Draw the lines DC, GC. If these lines be conceived to turn round upon the point C as a centre, so that the two points D, G, and with them the two subtenses BD, HG, may approach toward A; it is manifest, that as these subtenses come nearer to A, they will diminish, and at last will vanish in A. At the instant of their vanishing, BD will be to HG, as the square of the arc AD is to the square of the arc AG.

Let ED be drawn parallel to AB, and FG to AH. Then, because AB is a tangent at the point A, and consequently (El. III. 18.) perpendicular to the diameter AC, ED, which is parallel to AB, is likewise perpendicular to AC; for the same reason FG is perpendicular to AC. And ADC, AGC (El. III. 31.) are right angles. Therefore (El. VI. 8.) AED is similar to ADC, and AE is to AD, as AD to AC. Therefore (El. VI. 17.) the rectangle of AE, AC, is equal to the square of AD. But AE is equal to BD; therefore the rectangle BD, AC, is equal to the square of AD. For the same reason, the rectangle of HG, AC, is equal to the square of the chord AG. Consequently, the square of the chord AD is to the

square of the chord AG, as the rectangle BD, AC, is to the rectangle HG, AC, that is (El. VI. 1.) as BD to HG. But (by Lem. IV.) the arc AD and the chord AD, are ultimately in the ratio of equality, and also the arc AG and the chord AG. Therefore the square of the arc AD is to the square of the arc AG, at the instant in which they vanish, as BD to HG; that is, the evanescent subtense of the angle of contact is as the square of the conterminous arc.

Next; let the subtenses be not parallel to the diameter, but parallel to one another. Let MN, FG, be the subtenses parallel to the diameter; and AN, OG, two subtenses parallel to each other, but not to the diameter. Because BA is a tangent at the point B, BD (El. III, 16.) is perpendicular to BA; since therefore, MN and FG are parallel to BD, they are also perpendicular to BA, and the angles OFG, AMN, are equal. But because OG and AN are parallel by construction, the angle FOG (El. I. 29.) is equal to the angle MAN. Therefore the triangles FGO, MNA, are similar, and (El. VI. 4.) AN is to OG, as MN is to FG. But it has been proved that MN is ultimately to FG, as the squares of the conterminous arcs; therefore AN is ultimately to OG, as the squares of the conterminous arcs BN, BG. Fig. 16.

Lastly; suppose both AN and OG directed to C, the centre of the circle. In this case, each of these would be a semidiameter, continued from G and N respectively to the tangent BA. In their ultimate state these lines AN, OG, must coincide in the point B, and in the same right line BC; and therefore will become parallel, and will be, from what has been shown, ultimately as the squares of the conterminous arcs. Plate 3. Fig. 16.

If GC, DC, be beginning to move from A, they are in their nascent state; and it is manifest that the subtenses in this state are the same, and therefore have the same ratio, as in the evanescent state. Fig. 17.

COR. 1. Hence, because the tangents AB, AH, the arcs AD, AG, and their sines ED, FG, become ultimately equal (by Lem. IV.) to the chords AD, AG, their squares also will be ultimately as the subtenses BD, HG.

COR. 2. The same squares are also ultimately as those versed sines of the arcs, which bisect the chords and converge to a given point. For by the second case of this proposition these versed sines AE, AF, are as the subtenses BD, HG, or AN, OG.

COR. 3. Hence these versed sines AE, AF, are as the squares of the times in which a body describes the arcs AD, AG, with given velocities. For the spaces AD, AG, described with given velocities, are as the times, and the squares of the spaces as the squares of the times; but (by last Cor.) the squares of these spaces are as the versed sines AE, AF; therefore these versed sines are as the squares of the times in which the arcs AD, AG, are described.

LEM. VI. *The nascent or evanescent subtense of the angle of contact is equal to the square of the conterminous arc divided by the diameter.*

It has been shown in the preceding Lemma, that BD is to AD, as AD is to AC. Therefore BD multiplied into AC is equal to the square of AD, and BD $=\frac{AD^2}{AC}$. But (by Lem. IV.) the arc AD is ultimately equal to the chord AD; therefore the nascent or evanescent subtense BD is equal to the square of the arc AD divided by the diameter AC. Plate 3. Fig. 17.

PROP. LXXIV. **The centripetal forces of bodies, revolving in different circular orbits about the same centre toward which they tend, are as the squares of the arcs described in the same time, divided by the radii of the circles.**

In the circular orbits BND, RLE, let bodies revolve about the centre C, toward which they tend. Let them in the same time describe the indefinitely small arcs BG, RL. Then because the projectile forces would carry them in the same time through the tangents BF, RH, and the spaces through which, at the points G and L, they have been drawn from the tangents toward the centre by the centripetal force, are FG, HL; the centripetal forces must be as FG and HL. And (by Lem. VI.) the evanescent, or nascent, subtense FG is equal to the square of the arc BG divided by BD, and the evanescent, or nascent, subtense HL is equal to the square of the arc RL divided by RE. Therefore the subtense FG is to the subtense HL as the square of the arc BG divided by BD or its half BC, is to the square of the arc RL, divided by RE, or its half RC. Therefore the centripetal forces, when the arcs are nascent, are in the same ratio; that is, as the squares of the arcs divided by the radii. Plate 3 Fig. 16.

And this is true, whatever arcs BG and RL be taken, if they be described in the same time; for the nascent arcs will be as the velocities; and any other arcs BND, RLE, described in any given time, will be also as the velocities; therefore, the arcs BND, RLE, are as the nascent arcs BG, RL, and their squares are likewise proportional. But the centrifugal forces are as the squares of the nascent arcs, BG, RL, divided by the radii BC, RC; therefore these forces are as the squares of any other arcs, BND, RLE, divided by the radii of their circles.

PROP. LXXV. **The centripetal forces of equal bodies revolving in circular orbits, are as the squares of the velocities directly, and the radii of the orbits inversely.**

Because arcs described in the same time are as the velocities, and that the centripetal forces are (by Prop. LXXIV.) as the squares of the arcs described in the same time divided by the radii, these forces are also as the squares of the velocities divided by the radii, that is, as the squares of the velocities directly, and the radii of the orbits inversely.

Cor. Hence the centripetal forces of equal bodies, at equal distances from the centre, are as the squares of the number of revolutions in any given time; for this number is as the velocity with which the body moves.

PROP. LXXVI. **The centripetal forces of equal bodies revolving in equal circular orbits are inversely as the squares of their periodical times.**

The circular orbits or spaces being equal, the times in which these are described, or the *periodical times*, are (by Prop. V.) inversely as the velocities; and therefore the squares of the periodical times are inversely as the squares of the velocities, or the squares of the velocities are inversely as the squares of the periodical times; but (by Prop. LXXV.) the centripetal forces are as the squares of the velocities; therefore these forces are inversely as the squares of the periodical times.

PROP. LXXVII. **The centripetal forces of equal bodies revolving in unequal circular orbits, if the periodical times are equal, are as the radii of the circles.**

Plate 3. Fig. 16. Let one body revolve in the circular orbit BND, and another, in the same time, in the circular orbit RLE. Because the periodical times are equal, each body in any given part of its periodical time will describe an equal number of degrees in its respective orbit, that is, will describe similar arcs. The arcs BN, RL, being similar, will be described in equal portions of the periodical time; therefore (by Prop. LXXIV.) the centripetal forces will be as the squares of the similar arcs BN, RL, divided by the radii BC, RC; that is, as $\frac{BN^2}{BC}$ to $\frac{RL^2}{RC}$. But because similar arcs are to each other as the circumferences, or radii, of circles, BN is to RL as BC to RC, and consequently, BN^2 to RL^2 as BC^2 to RC^2. Therefore $\frac{BN^2}{BC}$ is to $\frac{RL^2}{RC}$ as $\frac{BC^2}{BC}$ is to $\frac{RC^2}{RC}$; that is, as BC to RC. But the centripetal forces (Prop. LXXIV.) are as $\frac{BN^2}{BC}$ to $\frac{RL^2}{RC}$; therefore these forces are as BC to RC; that is, as the radii of the orbits in which the bodies move.

PROP. LXXVIII. **The centripetal forces of equal bodies revolving in circular orbits, are as the radii of the orbits directly, and the squares of the periodical times inversely.**

If the periodical times are equal, and the radii unequal, the forces are (by Prop. LXXVII.) as the radii. If the radii are equal, and the periodical times unequal, the forces by (Prop. LXXVI.) are inversely as the squares of the periodical times. Therefore, if both the radii and periodical times are unequal, the forces will be in the compound ratio of both, or as the radii directly, and the squares of the periodical times inversely.

PROP. LXXIX. **When bodies revolve round the same centre, if the squares of their periodical times are as the cubes of their distances from the centre, the centripetal forces will be inversely as the squares of their distances.**

Let the distances of the two bodies be expressed by D, d; and the periodical times by P, p; then, by the supposition, $P^2 : p^2 :: D^3 : d^3$.

By Prop. LXXVIII. the centripetal forces are as the distances directly, and the squares of the periodical times inversely; that is, (taking C, c, for the centripetal forces) $C : c :: \frac{D}{P^2} : \frac{d}{p^2}$; and by supposition $P^2 : p^2 :: D^3 : d^3$; therefore, substituting D^3, d^3, for P^2, p^2, $C : c :: \frac{D}{D^3} : \frac{d}{d^3}$; that is,

$$C : c :: \frac{1}{D^2} : \frac{1}{d^2};$$

and, because where the dividend is given, the quotient is inversely as the divisor, $\frac{1}{D^2}$ is to $\frac{1}{d^2}$ inversely as D^2 to d^2. Therefore $C : c :: d^2 : D^2$; that is, the centripetal forces are inversely as the squares of the distances.

SCHOL. 1. Let C, c, express the central forces; A, a, the arcs described; V, v, the velocities with which the bodies move; P, p, the periodical times of their revolutions; D, d, the radii or distance from the centre; and N, n, the number of revolutions in a given time; the preceding Propositions may be thus expressed.

The bodies being equal,

PROP. LXXIV. $C : c :: \frac{A^2}{R} : \frac{a^2}{r}$.

LXXV. $C : c :: \frac{V^2}{D} : \frac{v^2}{d}$.

COR. $C : c :: N^2 : n^2$.

LXXVI. $C : c :: \frac{1}{P^2} : \frac{1}{p^2}$ or $p^2 : P^2$

LXXVII. $C : c :: D : d$.

LXXVIII. $C : c :: \frac{D}{P^2} : \frac{d}{p^2}$.

LXXIX. If $P^2 : p^2 :: D^3 : d^3$, $C : c :: \frac{1}{D^2} : \frac{1}{d^2}$ or $d^2 : D^2$.

SCHOL. 2. Since it was proved (Prop. LXX.) that the centripetal and centrifugal forces are, in circular orbits, equal to one another, the preceding Propositions, being demonstrated respecting the centripetal force, are also true of the centrifugal force; and it may be asserted universally, that the *central forces* are in the ratios above expressed.

These propositions may be confirmed by the following experiments, on the whirling tables.

EXP. 1. Let two equal balls be placed at equal distances from the centre of motion on the whirling tables; and let one table revolve twice whilst the other revolves once; the ball on the table whose number of revolutions is, with respect to that of the other in the same time, as 2 to 1 (or the periodical times as 1 to 2) will raise 4 times the weight raised by the other ball; that is, (according to Prop. LXXV. and Cor.) the radii being equal, $C : c :: V^2 : v^2 :: N^2 : n^2$; or (by Prop. LXXVI.) $:: p^2 : P^2$.

2. Let two equal balls be placed on tables whose number of revolutions in the same time is as 2 to 1; let the ball on the table, whose number of revolutions is 2, be placed at half the distance from the centre, at which the ball on the table, whose number of revolutions is 1, is placed; whence their velocities will be equal. The ball at the distance 1, will raise double the weight raised by the ball at the distance 2; that is, according to Prop. LXXV. the velocities being equal, $C : c :: d : D$.

3. Let two equal balls revolve on tables whose periodical times are equal; and let the distances of the balls from the centre be to each other as 2 to 1; the ball which is at the distance 2 will raise double the weight raised by the ball which is at the distance 1; that is, according to Prop. LXXVII. $C : c :: D : d$.

4. Let equal balls be placed on tables whose periodical times are as 2 to 1; let the ball on the table whose periodical time is 2, be placed twice as far from the centre as the ball whose periodical time is 1; the ball whose distance is 2, and periodical time 2, will raise half the weight raised by the ball whose distance is 1, and periodical time 1; that is, according to Prop. LXXVIII. $C : c :: \frac{D}{P^2} : \frac{d}{p^2} :: \frac{2}{4} : \frac{1}{2}$.

5. Let the equal balls be so placed on different tables, that the distance of one from the centre may be to that of the other as 2 to $3\frac{1}{6}$; let that ball which is at the least distance revolve twice in the same time in which the other ball revolves once; the periodical time of the ball at the less distance, is to that of the ball at the greater, as 1 is to 2, and the square of the periodical times will be as 1 to 4, and the cubes of the distances are 8, and 31.75; but $1 : 4 :: 8 : 32$, therefore the squares of the periodical times being in this case nearly as the cubes of the distances, the weight raised by the ball whose distance is 2, will be to that raised by the ball whose distance is $3\frac{1}{6}$, as the square of $3\frac{1}{6}$ is to the square of 2; that is, nearly as 10 to 4, or 5 to 2.

PROP. LXXX. The centrifugal forces of revolving bodies are as their quantities of matter.

For the whole centrifugal force of any body is made up of the centrifugal forces of each particle of

matter of which it consists; and therefore the more numerous the particles of matter in any body are. the greater will be its centrifugal force.

Exp. Let two glass tubes be half filled with water; into one put some leaden shot, and into the other a few small round pieces of light wood; let the orifice of each tube be closed by a cork; fasten the tubes to an inclined plane, and let the lower end of it rest upon the centre of a whirling table. On turning the table, the bodies will be carried by their centrifugal forces from the centre; and the heavier bodies will recede farther from the centre than the lighter. See Ferguson's Lectures.

Cor. Hence, when the revolving bodies are not equal, the centrifugal forces are in the ratios laid down in the preceding propositions multiplied into their quantities of matter. Thus Q, q, expressing the quantities of matter, and the other expressions remaining as in Prop. LXXIX. Schol.

$$C : c :: Q : q.$$
$$C : c :: \frac{QV^2}{D} : \frac{qv^2}{d}$$
$$C : c :: QN^2 : qn^2$$
$$C : c :: \frac{Q}{P^2} : \frac{q}{p^2}$$
$$C : c :: QD : qd$$
$$C : c :: \frac{QD}{P^2} : \frac{qd}{p^2}$$
$$\text{If } P^2 : p^2 :: D^3 : d^3,\ C : c :: \frac{Q}{D^2} : \frac{q}{d^2}$$

Cor. Hence the central forces will be equal, whenever the expressions proportional to them are equal; thus, $C = c$ if $QD = qd$.

Any of the above proportions may be confirmed by experiment; for example;

Exp. 1. Let the two balls A, B, be as 2 to 1; let the distance of the ball A be to that of the ball B from the centre, as 2 to 1, and the periodical time of the ball A be twice that of the ball B; their velocities will be equal; therefore the centrifugal force of A will be to that of B, as $\frac{Q}{D}$ is to $\frac{q}{d}$, that is, as 1 to 1, or A and B will raise equal weights.

2, 3. Let the same balls revolve about a fixed point, and have their distances reciprocally proportional to their quantities of matter, their centrifugal forces (compare Prop. LXXV. and LXXX.) will be equal, and they will balance each other. This may be shown by two balls suspended freely and united by a cord, having the point of the cord which is directly above the centre of the table at distances from the balls reciprocally as their weight; or by two balls united by a wire, and resting in equilibrio on a forked support fixed in the centre of the tables, which will continue in equilibrio when the tables are turned.

In like manner other cases may be confirmed by experiment.

LEM. VII. *If a body revolves freely in any orbit about an immoveable centre, and in an indefinitely small time describes any nascent arc; and the versed sine of the arc be drawn which may bisect the chord, and being produced may pass through the centre of force; the centripetal force, in the middle of this arc, will be as the versed sine directly, and the square of the time inversely.*

Plate 4. Fig. 3. Let two bodies revolve round their centre of force S, s; let QPM, qpm, be the nascent arcs described in any times, T, t; and let PB, pb, or QR, Aa, be the versed sines bisecting the chords, and when produced, passing through S the centre of force. Supposing the arcs QPM, ApN, to be described in the same time with different forces C, c; by Prop. LXXII. Cor. 4. QR : Aa :: C : c. Hence, supposing the forces to be equal, QR is equal to Aa described in the same time; and (by Lem. V.) QR or Aa : qr :: Ap^2 : qp^3; that is, since the motion in the arcs is uniform, Aa : qr :: T^2 t^2. Therefore, supposing both the times and forces different, and compounding these ratios, QR : qr :: C $\times$ T^2 : $c \times t^2$; whence $C : c :: \frac{QR}{T^2} : \frac{qr}{t^2}$.

Plate 4. Fig. 4. Cor. 1. If a body P, revolving about the centre S, describe a curve line APQ, and a right line ZPR touch that curve in any point P; and, from any other point Q of the curve, QR be drawn parallel to the distance SP, meeting the tangent in R; and QT be drawn perpendicular to the distance SP; the centripetal force will be reciprocally as the quantity $\frac{SP^2 \times QT^2}{QR}$, if this be taken of that magnitude

which it ultimately acquires, supposing the points P and Q continually to approach to each other. For QR is equal to the versed sine of double the arc QP, in whose middle is P; and double the triangle SQP, or SP $\times$ QT, is proportional to the time, in which that double arc is described (by Prop. LXXII.) and therefore may be used for the exponent of the time. Whence $C : c :: \frac{QR}{SP^2 \times QT^2} : \frac{qr}{sp^2 \times qt^2}$; that is, C is to c reciprocally as $\frac{SP^2 \times QT^2}{QR} : \frac{sp^2 \times qt^2}{qr}$; or the centripetal forces are reciprocally as $\frac{SP^2 \times QT^2}{QR}$.

COR. 2. Hence, if any curvilinear figure APQ is given; and therein a point S is also given, to which a centripetal force is perpetually directed; the law of centripetal force may be found, by which the body P, continually drawn back from a rectilinear course, will be retained in the perimeter of that figure, and will describe the same by a perpetual revolution. That is, we are to find the quantity $\frac{SP^2 \times QT^2}{QR}$, reciprocally proportional to this force.

PROP. LXXXI. **If equal bodies, revolving in ellipses, describe equal areas in equal times, their centripetal forces are to one another inversely as the squares of their distances from the *foci* of the ellipse toward which they tend.**

Let S be the focus; let a body P, tending toward S, describe a part of the ellipse PQ; join SP; draw QR to the tangent YZ, parallel to SP; join PC, and produce it to G. Complete the parallelogram $QxPR$, produce Qx to v, Qv is ordinately applied to GP; draw DK, a diameter parallel to YZ, and draw IH from the other focus H to SP parallel to YZ; join HP, and draw QT perpendicular to SP, as also PF to DK. Plate 4. Fig. 4.

EP is equal to the greater semiaxis AC. For, because CS is equal to CH, ES is equal to EI, (El. VI. 2.) whence EP is half the sum of PS, PI; that is, of PS, PH, for (Simson's Conic Sect. II. 11. Cor.) the angle IPR is equal to HPZ; whence (El. I. 29.) the angle PIH is equal to PHI, and PI is equal to PH; and PS, PH, together, (Simson's Conic Sect. II. 1.) are equal to the whole axis 2AC. EP therefore is equal to AC.

Putting L for the principal *latus rectum* of the ellipse, L (by definition) is equal to $\frac{2BC^2}{AC}$ (for $AC : CB :: CB : \frac{L}{2}$, whence $\frac{2CB^2}{AC} = L$.) And $L \times QR : L \times Pv :: QR : Pv$; and $QR = Px$; and $Px : Pv :: PE : PC$; whence $L \times QR : L \times Pv :: PE$ or $AC : PC$. And (El. VI. 1.) $L \times Pv : Gv \times Pv :: L : Gv$; and (Sims. II. 15.) $Gv \times Pv : Qv^2 :: PC^2 : DC^2$. And (Lem. IV.) the points Q and P continually approaching, Qv^2 is to Qx^2 ultimately in the ratio of equality. And (since the triangles QTx, EPF, are similar, for $QxT = PEF$, and QTx to EFP) Qx^2 or $Qv^2 : QT^2 :: EP^2$ or $AC^2 : PF^2$. But because (Sims. II. 20. Compare Vince's Con. Sect. II. 10. Cor. 1.) parallelograms about conjugate diameters are equal to the rectangle under the axes, the rectangle PF, DC, is equal to the rectangle ACB, whence $PF : AC :: CB : DC$, and $AC^2 : PF^2 :: CD^2 : CB^2$, wherefore $Qv^2 : QT^2 :: CD^2 : CB^2$. Compounding the following ratios,

$$L \times QR : L \times Pv :: AC : PC,$$
$$L \times Pv : Gv \times Pv :: L : Gv,$$
$$Gv \times Pv : Qv^2 :: PC^2 : CD^2;$$
$$Qv^2 : QT^2 :: CD^2 : CB^2;$$

And, striking out the equal quantities, $L \times QR : QT^2 :: AC \times L \times PC : Gv \times CB^2$.
Then substitute for $AC \times L$ its equal $2CB^2$, and

$$\begin{aligned} L \times QR : QT^2 &:: 2BC^2 \times PC : Gv \times BC^2 \\ &\text{or } BC^2 \times 2PC : Gv \times BC^2 \\ &\text{or } \quad 2PC : Gv. \end{aligned}$$

But the points Q and P continually approaching without end, 2PC and Gv are equal; wherefore $L \times QR$ and QT^2, proportional to these, are also equal. Multiply these equals into $\frac{SP^2}{QR}$ and $L \times SP^2$ will become equal to $\frac{SP^2 \times QT^2}{QR}$.

Therefore (by Lem. VII. Cor. 1 and 2.) the centripetal force is reciprocally as $L \times SP^2$; that is, since L is a given quantity, as SP^2, or in a duplicate ratio of the distance SP.

BOOK III.

OF HYDROSTATICS AND PNEUMATICS;

OR THE LAWS OF

INCOMPRESSIBLE AND COMPRESSIBLE FLUIDS.

PART I.

OF HYDROSTATICS.

CHAPTER I.

Of the Weight and Pressure of Fluids.

Def. I. A FLUID is a body, the parts of which yield to any force impressed upon them, and easily move out of their places.

PROPOSITION I.

The weight of fluids is as their quantities of matter.

Since each particle of any fluid gravitates toward the earth, the greater is the number of particles, that is, the greater the quantity of matter in any mass of fluid, the greater will be the weight of that mass.

Exp. 1. The different pressures of different columns of fluid in the same vessel at different depths, appear from the different quantities of fluid discharged, at different depths, in the same time from orifices of the same bore.

2. If the air be exhausted from a tube in part filled with water, and the tube be closed up, the solidity of the particles of water will be perceived by the sound produced by suddenly lifting up the tube.

Cor. Fluids gravitate in fluids of the same kind. For they cannot lose the property of gravity which belongs to all bodies by such a change of situation.

Exp. Suspend a stopped phial from one arm of a balance, in a vessel of water, and balance it by weights from the opposite arm of the balance; upon unstopping the phial under water, a quantity of water will rush into it, by which the weight will be increased as much as the weight of water in the phial.

PROP. II. When the surface of a fluid is level, the whole mass will be at rest.

Plate 5. Fig. 2. Let ABCD be a vessel containing water, the level surface of which is EF. Conceive the whole mass of fluid in the vessel to be divided into thin *strata*, or plates, RS, TV, XY, &c. lying horizontally one above another; and into small perpendicular columns GH, IK, LM, &c. contiguous to each other. In the stratum XY, and the columns IK, LM, let m, n, be two adjacent drops. Neither of these drops can move toward the column in which the other is, without driving that other out of its place, because the fluid is supposed incompressible. But, with whatever force the particle m endeavours to displace the particle n, this force is counterbalanced by an equal and contrary effort on the part of n; because (Prop. I.) they are equally pressed by the equal columns above them; consequently the particles will be at rest.

PROP. III. Any part of a fluid at rest presses, and is pressed, equally in all directions.

For (Def. I.) each particle is disposed to give way on the slightest difference of pressure; and, by supposition, each particle is pressed by the contiguous particles in such manner as to be kept at rest in its place; it is therefore pressed with an equal degree of force on all sides; and, consequently, (Book II. Prop. III.) it presses equally in all directions.

Cor. Hence the lateral pressure of a fluid is equal to the perpendicular pressure. This is one of the most extraordinary properties of fluids, and can be conceived to arise only from the extreme facility with which the component particles move among one another.

Exp. 1. Into several tubes, bent near their lower ends in various angles, pour a sufficient quantity of mercury to fill the lower parts of their orifices; then dip them into a deep glass vessel filled with water, keeping the orifice of the longer legs above the surface; whilst the tubes are descending, the mercury will be gradually pressed upward in the tubes, and the pressure will be equal at any given depth, whatever be the direction of the pressing column of fluid in the shorter leg of the tube. Oil may be used instead of mercury.

2. Dip an open end of a tube, having a very narrow bore, into a vessel of quicksilver; then, stopping the upper orifice with the finger, lift up the tube out of the vessel; a short column of quicksilver will hang in the lower end, which, when dipped in water lower than 14 times its own length, will, upon removing the finger, be suspended and pressed upward.

3. Let a large open tube be covered at one end with a piece of bladder drawn tight; pour into the tube a quantity of coloured water sufficient to press the bladder in a convex form; then, dip the covered end of the tube slowly into a deep vessel of water; the bladder, by the upward pressure will become first less convex, then plane, and at last concave.

4. If the like be done with several tubes, whose covered orifices are cut obliquely at different angles, the lateral pressure will be seen to increase with the depths to which the tubes are immersed.

5. Let a circular piece of brass, whose upper surface is covered with wet leather, be held close to one orifice of a large open tube, by means of a cord or wire fastened to the middle of the plate, and passing through the tube; let the plate, thus kept close to the orifice of the tube, be immersed with the tube into a large vessel of water; when the plate is at a greater depth than 8 times its thickness in the water, the cord or wire may be left at liberty, and the upward pressure of the fluid will keep the plate close to the tube.

6. Let a small bladder, tied closely about one end of an open tube, having a large bore, be filled with coloured water till the water rises above the neck of the bladder; upon immersing the bladder into a vessel of water, the bladder will be compressed on all sides, and the coloured water will be raised up in the tube in proportion to the depth to which the bladder is sunk.

PROP. IV. When a fluid flows through a tube which is wider in some parts than in others, the velocity of the fluid will, in every section of the tube, be inversely as the area of the section.

Let ADMN, a bended tube larger at IL than at FG, be filled with water to the height ADFG. Let the water be forced downward in the part ADBP, and consequently be made to rise in the other part KHMN. It is manifest, that the water which is forced out of one part of the tube, is driven into the other. Hence equal quantities pass through every section of the tube at the same time; for if less, or more water passed through the section FG than through IL in the same time, the quantity of water between FG and IL must be increased or diminished, which cannot be, since no cause is supposed which could increase or diminish it. But if equal quantities pass through unequal parts of the tube in the same time, the water must run proportionally faster where the tube is narrower, and slower where it is wider. If, for example, as much water runs through the section FG, as runs in the same time through the section IL, the water must move as much faster at FG than it moves at IL, as the tube is narrower at FG than at IL; that is, the velocity is inversely as the area of the section. Plate 5. Fig. 1.

Cor. The momentum will be the same in every section of the tube; for the quantity of water at each section is directly as the area of the section, and the velocity is inversely as the area; therefore the velocity is inversely as the quantity of matter; whence (Book II. Prop. XI.) the momentum is every where the same.

Schol. Hence we may account for the suspension of the fluid in a tube, the upper part of whose bore is capillary, and the lower of a much larger dimension, as was seen in the experiment, Book I. Prop. VII.

Let there be a tube consisting of two parts DR and RCK, of different diameters; DR, the smaller part of the tube, is able (Book I. Prop. VIII.) to raise water higher than the other; let then the height to which the larger would raise it be TC, and that to which it would rise in the lesser, if continued down to the surface of the fluid, be XH. If this compound tube be filled with water, and the larger Plate 5. Fig. 3.

Fig. 4. orifice CK be immersed in the same fluid, the surface of the water will sink no farther than XL, the height to which the lesser part of the tube would have raised it. But if the tube be inverted, and the smaller orifice XL be immersed, the water will run out till the surface falls to TF; the height to which the larger part of the tube would have raised it.

Fig. 3. Let the tube DR be conceived to be continued down to HI; and let it be supposed that the fluids contained in the tube XLHI, and the compound one XLKC, are not suspended by the ring of glass at XL, but that they press upon their respective bases, HI and CK. Let it farther be supposed that these bases are each of them moveable, and that they are raised up or let down with equal velocities; then will the velocity with which XL, the uppermost stratum of the fluid XLCK, moves, exceed that of the same stratum, considered as the uppermost of the fluid in the tube XLHI, as much as the tube RCK is wider than DR, (by this Prop.) that is, as much as the space MNKC exceeds XLIH. Consequently, the effect of the attracting ring XL, as it acts upon the fluid contained in the vessel XLCK, exceeds its effect, as it acts upon that in XLHI, in the same ratio. Since, therefore, it is able to sustain the weight of the fluid XLHI by its natural power, it is able, under this *mechanical* advantage, to sustain the weight of as much as would fill the space MNKC; but the pressure of the fluid XLCK is equal to that weight, as having the same base and an equal height (as will be shown by Prop VI.) Its pressure, therefore, or the tendency it has to descend in the tube, is equivalent to the power of the attracting ring XL, for which reason it ought to be suspended by it.

Fig. 4. Again, the height at which the attracting ring in the larger part of the tube is able to sustain the fluid is no greater than NF, that to which it would have raised it, had the tube been continued down to MN. For here the power of the attracting ring acts under a like *mechanical* disadvantage, and is thereby diminished, as much as the capacity of the tube TFNM is greater than that of HIXL; because, if the bases of these tubes are supposed to be moved with equal velocities, the rise or fall of the surface of the fluid TFXL would be so much less than that of TFMN. And, since the attracting ring TF is able, by its natural power, to suspend the fluid only to the height NF in the tube TFMN; it is in this case able to sustain no greater pressure than what is equal to the weight of the fluid in the space HIXL; but the pressure of the fluid TFXL, which has equal height, and the same base with it, is equal to that weight; and therefore is a balance to the attracting power.

From hence we may clearly see the reason, why a small quantity of water put into a capillary tube, which is of a conical form, and laid in a horizontal situation, will run toward the narrow end. For let

Plate 5. Fig. 5. AB be the tube, and CD a column of water contained within it; when the fluid moves, the velocity of the end D will be to that of the end C reciprocally as the cavity of the tube at D to that at C, (by this Prop.) that is, (El. XII. 2.) reciprocally as the square of the diameter at D, to the square of the diameter at C; but the attracting ring at D is to that at C, singly as the diameter at D to the diameter at C. Now, since the effect of the attraction depends as much upon the velocity of that part of the fluid where it acts, as upon its natural force, its effect at D will be greater than at C; for though the attraction at D be less in itself than at C, yet its loss of force upon that account, is more than compensated by the *mechanical* advantage it has arising from hence, that the velocity of the fluid in that part is more increased than the force itself is diminished at D. The fluid will therefore move towards B. See, on this subject, Mr. Vince's Principles of Hydrostatics, p. 65—9.

PROP. V. In bended cylindrical tubes, fluids at rest will be at the same height on each side.

Plate 5. Fig. 1. In the tube ADMN, filled with water to the height AD, the water cannot descend from AD, without rising toward MN. The water in each side of the vessel may therefore be considered as two forces acting upon each other in contrary directions; and consequently these two masses of fluid will only be at rest when their momenta are equal; that is, (Book II. Prop XI. Cor.) when the quantities of matter are inversely as the velocities, or (Prop. IV.) directly as the area of the section through which it flows. Thus, at the sections BP, KH, the momenta are equal, when the quantities of matter, or cylindrical masses of fluid are as the areas of the sections; that is, as the bases of the cylinders ADBP, FGHK. But cylinders are as their bases (El. XII. 11.) only when their perpendicular heights are equal. Therefore the momenta of the two cylinders of fluid will be equal, and consequently the mass will be at rest, only when the perpendicular heights of each column are equal.

Exp. 1. In a bended tube of large but unequal bore, water will rise to the same height on each side.

2. Let water spout upward through a pipe, having a small orifice inserted into the bottom of a deep vessel; it will rise nearly to the height of the upper surface of the water in the vessel. The resistance of the air, and of the falling drops, prevents it from rising perfectly to the level.

Cor. If, therefore, a pipe convey a fluid from a reservoir, it can never carry it to a place higher than the surface of the fluid in the reservoir.

Schol. In this demonstration, we do not consider the velocity with which the two columns of fluid

are moving, but the velocity with which, if they move at all, they must begin to move. And since, if their perpendicular height is the same, the velocity with which they must begin to move will be inversely as their respective quantities of matter, they cannot begin to move but with equal momenta; and their motions must be in contrary directions, because one column cannot descend without making the other ascend; therefore those equal momenta would destroy each other. These two columns then, making a continual effort to move with equal momenta in contrary directions, counterbalance each other.

PROP. VI. The pressure of fluids is proportional to the base, and the perpendicular height of the fluid, whatever be the form of the vessel or quantity of the fluid.

Case 1. Let the fluid be contained in a perpendicular cylindrical vessel. Plate 5.

In such a vessel, ABCD, because the whole weight of the fluid, and no other force, presses directly Fig. 2.
upon the bottom CD, the pressure (by Prop. I.) must be as the quantity; that is, (El. XII. 11, 14.) as the base and perpendicular height of the fluid.

Case 2. Let the fluid be contained in a perpendicular vessel, the bottom of which is equal to that of the cylinder in the last case, but its top narrower than the bottom.

Let the vessel DBLP, have the portions of its base LA, CP, each equal to OR. From Prop. I. and Plate 5.
III. it appears, that each of these portions are equally pressed by the column DBOR, as the base OR. Fig. 6.
In like manner, every portion of the base LP equal to OR is as much pressed as OR. Therefore the whole base LP is as much pressed as if the vessel was of the cylindrical form FHLP.

Or thus; because (by Prop. V.) if a tube were inserted at NT, of the diameter OR, the water, being at the height DB, would rise to the level FE, there must at NT be an upward pressure toward F sufficient to fill up the columns of fluid FELA; that is, equal to the weight of as much water as would fill the space FENT. Consequently the re-action, that is, the pressure upon ths base LA, must be equal to the weight of as much water as would fill FENT. But the base LA supports this re-action, and likewise the weight of the water NTLA, which are together equal to the weight of DBOR. The base LA, therefore, sustains a pressure equal to the weight of the column DBOR. And every equal portion of the base may, in the same manner, be shown to sustain an equal pressure. Therefore, the pressure on the base is the same in vessels of the form supposed in this case, as in cylinders of equal bases, and of the same altitude with these vessels. The same may be shown with respect to a vessel of the form of plate 5, fig. 7.

Case 3. Let the vessel be of the same base and altitude, but have its top wider than the base.

Let the fluid of the vessel be divided into strata EF, GH, IK, &c. Let us also imagine the bottom of Plate 5.
the vessel C to be moveable, that is, capable of sliding up and down the narrow part of the vessel, from Fig. 8.
C to GH. Let it further be supposed that this moveable bottom is drawn up or let down with a given velocity, while the vessel itself is fixed and immoveable; it is evident the lowest stratum, which is contiguous to the bottom, will be raised or let down with the same velocity, and will therefore have a momentum proportional to that velocity, and the quantity of matter it contains; but (by Prop. IV. Cor.) the rest of the strata will have the same momentum; consequently, the momentum of all taken together, that is, of the whole fluid, is the same as if the vessel had been no larger in any one part than it is at the bottom, for then the momentum of each stratum would also have been as great as that of the lowest. The pressure, therefore, or action of the fluid, with which it endeavours to force the bottom out of its place, is as the number of strata, that is, the perpendicular height of the fluid, and the magnitude of the lowest stratum, that is, the base.

Case 4. Let the fluid be in an inclined cylindrical vessel.

In the inclined cylindrical vessel ABNI, as much as the fluid is prevented from pressing upon the Plate 5.
base NI, by being in part supported by the side of the vessel AN, so far is the pressure upon the base Fig. 9.
increased by the re-action of the opposite side BI, which is equal to the action of the former, because the fluid, pressing every way alike at the same depth below the surface, exerts an equal force against both the sides. The base NI is therefore pressed with the same force with which it would be pressed, if the fluid contained in the vessel ABNI was included in the vessel EDIO, having an equal base, and the same perpendicular height with the vessel ABNI; that is, (by the first case) the pressure is as the base NI and altitude CN.

Since then, the pressure upon the base of vessels, either wider or narrower at the top than the bottom, and likewise the pressure upon the base of vessels inclined to the horizon, is equal to that upon the base of a cylindrical vessel of the same base and height, the sides of which are perpendicular to the horizon; and since the pressure upon the base of such a cylinder is as the base and height; the pressure upon the bottom of all vessels filled with fluid is proportional to their base and perpendicular height.

Exp. 1. Let two tubes of different forms be successively applied to the same moveable circular base, suspended by a wire, passing from the centre of the base through the tubes, to the beam of a balance;

when the different tubes are filled to the same height, it will require the same weight at the opposite end of the balance to keep the base from sinking. Hence any quantity of fluid, how small soever, may be made to balance and support any quantity how great soever, which is called the *hydrostatical paradox*.

2. Let two tubes, the one cylindrical, the other of the form of a speaking trumpet, have their bases of equal diameter, covered with bladder, and inserted in a vessel of water, as in Prop. III. Exp. 3. the bladder will become plane at the same depth in both; from whence it appears, that since the upward pressures, at the same depth, are equal, the downward pressures in the two tubes are also equal.

Cor. 1. Hence in different vessels, containing different fluids, the pressures are as the areas of the bases multiplied into the depths, and specific gravities.

Cor. 2. If a cone be filled with a fluid, and standing on its base, the pressure on its base will be equal to three times the weight of the fluid. Let B be equal to the base, H equal to the perpendicular height, then the solid content, or weight, will be equal $\frac{B}{3} \times H$, but the pressure will be $B \times H$, therefore equal to three times its weight.

Cor. 3. A small quantity of fluid may be made to press with a force sufficient to raise a great weight.

Plate 5. Fig. 6. Since (as was shown in Prop. V.) as much fluid as will fill the tube DBIV presses upward against VM, with a force equal to the weight of as much fluid as would fill the space BHVM; the base remaining the same, the space BHVM, that is, the weight which may be raised, will (by this Prop.) be as the height VB, which may be increased at pleasure.

Exp. Let two circular pieces of wood be united by leather in the manner of a pair of bellows; in the upper board insert a long tube with a large bore; through which pour water into the vessel; the upward pressure of the water, as it is poured in, will raise a great weight.

Cor. 4. From hence it may be proved, independently of the reasoning in Prop. V. that, in bended vessels, or channels of any form, fluids rise to the same height, whatever be the difference between the quantities of fluid on each side; for whatever be the form of the channels, the plane which is perpendicular to the lowest point being considered as the common base, the pressure upon it is equal, when the fluid on each side is of equal altitude; and the whole mass can only be at rest when the opposite pressures are equal.

Schol. This pressure of the fluid upon the base does not alter the weight of the vessel and fluid considered as one mass, because the action and re-action which cause it, with respect to the weight of the vessel, destroy each other; the vessel being as much sustained by the action upward, as it is pressed by the re-action downward.

PROP. VII. The pressure of a fluid upon any indefinitely small part of the side of a vessel which contains it, is equal to the weight of a column of the same fluid, whose base is the part pressed, and whose height is the distance of that part from the surface of the fluid.

Plate 5. Fig. 12. Let ABCD be a vessel filled with fluid; AB its surface; and L a point in the side of the vessel. The indefinitely small drop which lies next to the point L is pressed downward (by Prop. I.) by a force equal to the weight of a column of water whose base is L, and height LA, the distance of that part from the surface. And (by Prop. III.) this drop is pressed sideways toward L with the same force with which it is pressed downward. Whence the position is manifest concerning the point L. And the same may be proved concerning any other points M, N, C, equal to L. The same is evidently true in an inclined vessel.

PROP. VIII. The pressure of a fluid upon any plane is equal to the weight of a body which has the same density with the fluid, and is formed by raising perpendiculars upon each indefinitely small part of the plane, equal in height to the distance of that part from the surface of the fluid.

Plate 5. Fig. 12. It has been proved, in the last proposition, that the pressure upon each indefinitely small part of the line AC, in the side of the vessel ABCD, is equal to the weight of a column of fluid whose base is the part pressed, and whose height is the distance of that part from the surface AB. Hence, if from the point L a perpendicular LO be raised whose base is L, and whose length LO is equal to LA, the distance of L from the surface, if this perpendicular consisted of matter of the same density with the fluid in the vessel, the weight of this perpendicular column would be equal to the pressure upon the point L. If, in like manner, perpendiculars, consisting of matter of the same density with the fluid, were raised upon every point between A, C, they would together fill up the area of the triangle ACD; and the pressure upon the whole line AC in the side of the vessel ABCD, because it is equal to the sum of the pressures upon

all its parts, must be equal to the weight of this triangle ACD. The same may be proved concerning any other lines in the side of the vessel, as HI, EF. Consequently, the pressures upon the whole side will be equal to the weight of as many such triangles as there can be lines drawn upon it in the same manner as AC, HI, EF, are drawn. But all these triangles together would fill up the whole space, or compose a solid, CFGDAE. Therefore the pressure upon the side AECF will be equal to the weight of this solid, consisting of matter which has the same density with the fluid in the vessel; which solid is formed by raising perpendiculars upon each line of the side, respectively equal to the distance of that point from the surface of the fluid. Fig. 14.

In like manner, if AC is a line drawn in the inclined side of a vessel, in which the water reaches to the level AB, the pressure upon this line may be estimated as before. SL is the distance of L from the surface. Let therefore a perpendicular LO, equal in length to LS, be raised upon the point L; then, if this perpendicular were a column of matter of the same density with water, the weight of it would be equal to the pressure upon L. For the same reason, if a perpendicular MP is raised upon the point M, and is made equal in length to MT, the distance of M from the surface; such a perpendicular, consisting of matter of the same density with water, and being of the same size would have the same weight as the column of water MT. And since (by Prop. I.) the pressure upon M equals the weight of the incumbent water MT, it likewise equals the weight of the perpendicular MP. In like manner, the points N and C are pressed by the weight of the incumbent columns NV and XC, which is equal to the weight of the perpendiculars NQ, CR, supposing those perpendiculars to be equal in height to NV and XC, and to consist of matter whose density is the same with that of the columns NV and XC. Thus the pressure upon the whole line, being made up of the pressures upon all its parts, will be equal to the weight of as many perpendiculars, as can be raised in this manner between A and C. The sum of all those perpendiculars is the triangle ACR, whose weight therefore is equal to the pressure upon the line AC. But if as many such triangles were added together, as there are lines parallel to AC in the whole side of the vessel, all these triangles together would form a solid. And since this solid is the sum of all the pressures upon each point of the side, the weight of it, supposing it to consist of matter that has the same density as water, would be equal to the pressure upon the whole side. Plate 5. Fig. 13.

PROP. IX. The pressure upon any one side of a cubical vessel, filled with fluid, is half the pressure upon the bottom.

The bottom sustains a pressure equal to the whole weight of the fluid in the vessel. And the pressure which the side sustains is equal to the weight of the prism CFGDAE, which (El. XI. 28.) is half the cube; therefore the side sustains a pressure equal to half the pressure upon the bottom. Plate. 5 Fig. 14.

Or thus; Because the pressure upon every part of the vessel at the bottom is equal to the weight of a column whose base is the part pressed upon, and height that of a perpendicular from the bottom to the surface; if the pressure were the same every where from the top to the bottom, it would be equal to the weight of as many such columns as would correspond to all the parts of the vessel. But the pressure every where diminishes as we approach toward the surface, where it is nothing; the pressure on the side is therefore only half of that on the bottom of the vessel; a number of terms in arithmetical progression beginning from nothing being half the sum of an equal number of terms, each of which is equal to the last in the progression.

Cor. 1. The gravity of the fluid in a cubical vessel producing upon each of the four sides a pressure equal to half that upon the bottom, and upon the bottom a pressure equal to itself, produces on the whole a pressure three times as great as itself.

Cor. 2. When the area of the part pressed is given, the pressure is as the perpendicular distance of that part from the surface; where the depth of the part is given, the pressure is as the area.

Schol. There is a particular point in which the whole pressure against the side acts; it is called the *centre of pressure*, and is the same with the centre of *oscillation* of the side vibrating on the upper line of it as an axis. See Prop XLVI. Schol. 1. Book II.

CHAPTER II.

Of the Motion of Fluids.

SECTION I.

Of Fluids passing through the Bottom or Side of a Vessel.

PROP. X. The momentum with which any fluid runs out of a given orifice in the bottom or side of a vessel, is proportional to the perpendicular depth of the orifice below the surface of the fluid.

The pressure of a fluid against any given surface being (by Prop. I. and III.) proportional to the perpendicular height of the fluid above that part; if that given surface be removed, the fluid will be driven through the orifice by this pressure. The force therefore with which the fluid passes through the orifice is as the perpendicular depth of the orifice below the surface of the fluid; but the momentum is always as the moving force; therefore the momentum is also as the perpendicular depth of the orifice.

PROP. XI. The momentum with which any fluid runs out of a given orifice in the bottom or side of a vessel, is as the square of its velocity, or as the square of the quantity of matter.

The momentum (by Book II. Prop. XI.) is in the compound ratio of the quantity of matter and velocity. And it is manifest, that, since the orifice is given, the quantity of fluid discharged will always be as the velocity; therefore the momentum is as the square of the velocity, or of the quantity of fluid.

PROP. XII. The velocity with which any fluid runs out of an orifice in the bottom or side of a vessel, is as the square root of the perpendicular depth of the orifice from the surface of the fluid.

Because the momentum is as the square of the velocity, (by Prop. XI.) and as the perpendicular depth of the orifice (by Prop. X.) the square of the velocities (El. V. 11.) is as the perpendicular depth, and, consequently, the velocity as the square root of the perpendicular depth.

Cor. 1. Hence a fluid running out of a vessel which empties itself, and whose horizontal sections are all equal, flows with an uniformly retarded velocity; for the perpendicular depths are continually diminishing.

Cor. 2. Hence also the surface descends with an uniformly retarded velocity, and the spaces described by it, in equal portions of time, are (Prop. XXVIII. Book II.) as the odd numbers 1, 3, 5, 7, 9, &c. taken backward.

Cor. 3. If therefore a cylindrical vessel be divided into portions, continued to the surface of the fluid, which are as the odd numbers, 1, 3, 5, 7, &c. a clepsydra or hour-glass will be formed; for the surface will descend through these divisions in equal times.

PROP. XIII. A fluid runs out of an orifice in the bottom or side of a vessel, with the velocity which a heavy body would acquire in falling freely through a space equal to the perpendicular distance of the orifice from the surface of the fluid.

Plate 5. Fig. 10. Let ABCD be a vessel filled with any fluid, to the height FG. It is manifest, that at the beginning of the fall of each drop from the upper surface FG, it must be carried downward by its gravity with the same velocity with which any other heavy body would begin to descend. And, if an orifice be made in the vessel at L, any point below the surface, the fluid which passes through that orifice will (by Prop. XII.) move with a velocity which is as the square root of the distance from the surface. But if a body were to fall from the surface to the point L, it would acquire a velocity which would be (by Book II. Prop. XXVI. Cor. 2.) as the square root of this distance. Therefore, since the velocity with which the fluid moves is, at the beginning of its motion, equal to that of a falling body, and since at every given distance these velocities have the same ratio, namely, that of the square root of the distance from the surface, that is, (El. V. 9.) are equal, the proposition is manifest.

Cor. Supposing O, V, T, Q, to represent the area of the orifice, velocity, time, and quantity flowing out in that time, respectively; Q will vary as $O \times V \times T$, or as $O \times T \times \sqrt{H}$, (Prop. XII.) and when T is given, as $O \times \sqrt{H}$.

Schol. When a fluid spouts from a vessel, it rushes from all sides toward the orifice, which is the cause of the contraction of the stream at the distance from the orifice equal to its diameter, and is called the *vena contracta.* Now the area of the orifice is to the area of the smallest section of the stream, nearly as $\sqrt{2}$ to 1; hence (by Prop. IV.) the velocity at the vena contracta is to the velocity at the orifice as $\sqrt{2}$ to 1. Sir I. Newton found, that the velocity at the vena contracta, was that which a body acquires in falling down the altitude of the fluid above the orifice. We must, therefore, distinguish between the velocity at the orifice, and at the vena contracta, and in the doctrine of spouting fluids, it is the latter velocity which must be considered, and the point of projection must be assumed from that point.

PROP. XIV. When two cylindrical vessels have their bases, heights, and orifices equal, if one of them be always kept full, it will discharge double the quantity of fluid discharged in the same time by the other whilst it empties itself.

For (by Prop. I.) the fluid will continue through the whole time, to run with the same velocity out of the vessel that is kept full. But the fluid will run (Cor 1. Prop. XII.) with an uniformly retarded velocity out of the vessel which empties itself. And, since both vessels are full at first, the velocity which continues uniform in one vessel, will (by Prop. I.) be the same with the first velocity in the vessel in which the fluid is uniformly retarded. Therefore the quantity discharged out of the former vessel will be to the quantity discharged in an equal time out of the latter, as the space described by a body moving with an uniform velocity, to the space described by a body which sets out with the same velocity, and is uniformly retarded. But (by Book I. Prop. XXVII.) the space described by the former will be double of the space described by the latter. Therefore the quantity discharged out of the former vessel, will be double of the quantity discharged out of the latter.

PROP. XV. A stream of any fluid which spouts obliquely forms a parabola.

Each drop in a stream of fluid, spouting obliquely, is a heavy body projected obliquely by the force or pressure which drives it out of the orifice. Therefore (by Book II. Prop. LVIII.) every drop of the stream, that is, the whole stream forms a parabola.

Exp. Observe the figure formed by a fluid spouting obliquely.

Cor. Hence fluids spouting obliquely are subject to the laws of projectiles laid down, Book II. Ch. VII. Sect. 1.

PROP. XVI. When a fluid spouts horizontally from an orifice in the side of a vessel which is kept full, if a line passing through the orifice perpendicular to the horizon, and intercepted between the surface of the fluid and the horizontal plane that receives it, be made the diameter of a circle, and a line drawn horizontally from the orifice to the circumference, the distance, to which the fluid will spout, will be double of this horizontal line.

Let AB be the perpendicular; C, E, or *e*, the orifice; ADHB the semicircle drawn on the side; ED, CH, *de*, lines drawn horizontally from the orifice to the circumference. The fluid spouts at E (by Prop. XIII.) with the velocity which a heavy body would acquire in falling from A to E; and this motion, being in a horizontal direction, can neither be accelerated nor retarded by the force of gravitation, and will therefore continue uniform. But beside this, the fluid spouts with the velocity which it acquires in falling after it has passed the orifice. This velocity, when the fluid arrives at GB, is the same with that which any other heavy body would have acquired in falling through an equal space from E to B. Let this velocity be called the descending velocity, and that with which the fluid spouts at E the horizontal velocity. Then, since the horizontal velocity is the same with that which a body would acquire by falling from A to E, and the descending velocity, when the fluid arrives at the plane GB, is the same with that which a body would acquire by falling from E to B, and since (by Book II. Prop. XXVI.) the spaces AE, EB, described by falling bodies, are as the squares of the last acquired velocities of bodies falling through them; that is, (inverting the terms) the squares of these last acquired velocities, or the squares of the horizontal and descending velocities, are as the lines AE, EB. But in the triangle ADB, right-angled (El. VI. 8.) at D, DE is a mean proportional between AE, EB, and the square of ED is to the square of EB, as AE is to EB. But the square of the horizontal velocity is to the square of the last descending velocity, as AE to EB. Therefore the square of the horizontal velocity is to the square of the last descending velocity as the square ED to the square EB; whence the horizontal Plate 5 Fig. 11

velocity is to the last descending velocity as ED to EB. Now the spaces described in the same time, in uniform motions, are (Book II. PROP. VI.) as the velocities. Consequently, if the fluid had begun to fall from E with the velocity it has acquired at B, and had fallen uniformly, in the time of descent the spaces described by the horizontal and descending velocities would have been respectively as those velocities; that is, as ED to EB. Thus while the fluid was descending till it reached the plane GB, the horizontal velocity would have carried it forward through a space equal to ED, or the horizontal distance would be ED. But the descending velocity being at the first nothing, and continually increasing, the time of descent (see Book II. Prop. XXVII.) is twice what it would have been upon the supposition that it began to descend from the last acquired velocity. And the horizontal velocity is uniform, and therefore in twice the time, or the true time of descent, the fluid will be carried horizontally to twice the distance ED. Consequently, if BF be made equal to twice DE, whilst the stream is descending from E to GB, it will be carried forward to the point F. The same may be proved concerning any other points, C, *e*.

PROP. XVII. If a fluid spout horizontally out of orifices in the side of a vessel which is kept full, it will spout to the greatest distance from the orifice which is in the middle of the side, and to equal distances from orifices equally distant from the middle.

Plate 5. Let C be the orifice in the middle of the side, and E, *e*, equal orifices at equal distances from C.

Fig. 11. The distance to which the fluid will spout at C (by Prop. XVI.) is twice CH, and at E twice ED. But CH (El. III. 15.) is greater than DE, any line drawn from the diameter parallel to the radius; therefore twice CH is greater than twice ED.

Also since the horizontal distances to which the fluid will spout at E and *e*, are twice ED, or *e d*; and that ED, *e d*, being equally distant from the centre, and parallel to the radius, (El. III. 14.) are equal; the horizontal distances from E, *e*, are equal.

Hence, if in the plane of the horizon, GB be drawn perpendicular to the side AB, and GB be double of CH, and FB double of DE, or *d e*, the fluid spouting from C will fall upon G, and from E and *e*, upon F.

Cor. If the side of the jet d'eau be inclined, in any angle to the horizon, and the direction, and velocity of the spouting fluid be known, the amplitude, altitude, and time of flight, may be discovered by the rules investigated in Book II. on Projectiles.

Exp. Let water spout from the middle orifice, and from orifices equally distant from the middle, the truth of the proposition will be manifest.

Remark. In all propositions respecting the times in which vessels empty themselves, the orifice is supposed to be very small in respect to the bottom of the vessel, otherwise the experiments do not agree with the theory.

Def. II. A *river* is a stream of water which runs by its own weight down the inclined bottom of an open chanel.

Def. III. A *section of a river* is an imaginary plane, cutting the stream, which is perpendicular to the bottom.

Def. IV. A river is said to *flow uniformly* when it runs in such a manner, that the depth of the water in any one part continues always the same.

PROP. XVIII. If a river flows uniformly, the same quantity of water passes in an equal time through every section.

Plate 5. Let AB be the reservoir, BC the bottom of the river, and ZX, QR, sections of the river. Because

Fig. 15. the river flows uniformly, the same quantity of water which passes through ZX in a given time must pass through QR in the same time; otherwise the quantity of water in the space ZQXR, must in that time be increased or diminished, and consequently the depth of the water in that space altered; contrary to the supposition.

Cor. Hence if V, B, D, be the velocity, breadth, and depth respectively, $V \times B \times D$ will be a given quantity, and V will vary as $\frac{1}{B \times D}$.

PROP. XIX. The breadth of the channel being given, the water in rivers is accelerated in the same manner with any body moving down an inclined plane.

For each drop of the water moves down upon the inclined plane of the bottom, or upon the inclined plane of the sheet of water, next below it, parallel to the bottom.

PROP. XX. The breadth of the channel being given, the velocity of each drop of water in a river is the same that a body would acquire in falling from the level of the surface of the water in the reservoir, to the place of the drop.

Let AB be the depth of the reservoir, AP the level of its surface, and BC the bottom of the channel. Any drop at E, after it comes out of the reservoir at K (by Prop. XIX.) rolls down the inclined plane KE, parallel to the bottom. And this drop, when it comes out of the reservoir AB at K (by Prop. XIII.) has the same velocity which a heavy body would acquire in falling from A to K; and, in rolling down the inclined plane KE, it acquires (by Book II. Prop. XXXIV.) the same velocity which any heavy body would acquire in falling down GE, the perpendicular height of the plane. At E the drop will therefore have acquired a velocity equal to that which a body would acquire by falling through AK and GE, that is, through MGE, the perpendicular drawn from the level of the reservoir to the place of the drop.

Cor. 1. Hence the breadth of the channel being given, the velocity of each drop of water in a river is as the square root of its distance from the level of the surface of the reservoir. For, if E and R be two drops in different parts of the river, and AP the level, the velocity of the drop E is the same that a body would acquire by falling down ME, and that of R the same which a body would acquire by falling down HR. Therefore (by book II. Prop. XXVI. Cor. 2.) the velocity of the drop E is to the velocity of R, as the square root of ME to the square root of HR. Plate 5. Fig. 15.

Cor. 2. Hence the breadth of the channel being given, the water at the bottom of a river will run faster than the water at the surface.

PROP. XXI. The breadth of the channel being given, the depth of the river continually decreases as it runs.

The same quantity of water (by Prop. XVIII.) passes through each of the sections ZX, QR, in the same time. But (by Prop. XX. Cor. 2.) the water runs faster at the lower section QR, than at the upper ZX. Therefore the area of the section QR must be as much less than the area of the section ZX, as the velocity at QR is greater than the velocity at ZX. But the breadth of the sections are by supposition equal; therefore their areas are (El. VI. 1.) as their heights. Consequently the heights of the sections QR, ZX, will be inversely as the velocities at those sections; that is, the depth of the water at QR will be as much less than the depth at ZX, as the velocity at QR is greater than the velocity at ZX.

PROP. XXII. At a given distance from the reservoir, if the river flow uniformly, the velocity of the water will be inversely as the breadth of the channel.

Because the river flows uniformly, the depth at any given section ZX is always the same; and in any given time, the same quantity of water must flow through the different sections ZX, QR, as was shown in Prop. XVIII. But a given quantity of water cannot flow in a given time through any section, unless as much as the area is increased, so much the velocity is diminished, and the reverse; that is, the velocity must be inversely as the area of the section, or the depths being given, as its breadth.*

PROP. XXIII. The depth of a river being given, the pressure upon any part of the bank will be the same, whatever is the breadth of the river.

The pressure upon any given part in the bank (by Prop. I. and III.) will be as the distance of that part from the surface; which remains the same whilst the depth is the same, whatever be the breadth of the river; therefore the pressure will remain the same.

PROP. XXIV. If the breadth of a river be given, the pressure on any part of the bank will be as the depth of the river.

For the pressure on any part of the bank is (by Prop. I. and III.) as the depth of that part below the surface, which depth will increase with the depth of the river.

PROP. XXV. The pressure against any given surface in the bank of a river, if that surface reaches from the bottom to the top of the stream, is equal to the weight of a column of water whose base is the surface, and whose height is half the depth of the stream.

* This and the three preceding propositions can be applied only to straight regular canals of considerable declivity and no great length.

Plate 5. Fig. 15. Let ZQXR be a given surface in the bank, reaching from the bottom BC of the river to its top AD. The pressure upon this is (from what was shown in Prop. IX.) half the pressure on an equal surface at the bottom XR; which pressure (by Prop. I. and III.) is equal to the weight of a column of water whose base is the surface ZQ, and whose height is the depth of the stream. Therefore the pressure against the surface ZQXR is equal to the weight of a column whose base is the surface ZQ, and its height half the depth of the stream.

PROP. XXVI. When a stream which moves with the same velocity in every part strikes perpendicularly upon any obstacle, the force with which it strikes is equal to the weight of a column of the same fluid, whose base is the obstacle, and whose height is the space through which a body must fall to acquire the velocity of the stream.

Plate 5. Fig. 11. Let a stream of water flow horizontally out of the orifice *e*. If this stream were to strike upon an obstacle of the same breadth every way as the orifice or stream, placed perpendicular to the horizon, the stream must strike upon the obstacle with its whole force. But this force is equal to the weight of a column of water whose base is *e*, and height A *e*. And (by Prop. XIII.) A *e* is the height from which a body must fall to acquire the velocity with which the stream spouts from *e*. Therefore the force with which this stream would strike such an obstacle is equal to the weight of a column of water whose base is *e*, and height that from which a body must fall to acquire the velocity of the stream. And because no part of the stream, however broad, can strike the obstacle except so much as is contained within a section equal to the surface of the obstacle, no other part of the stream is to be considered in estimating this force. It is also manifest, that if the stream flow horizontally with the same velocity, in any other manner than through an orifice, as in the current of a stream, it will strike an obstacle with the same force.

PROP. XXVII. When the obstacle is given, the force with which a stream strikes upon it will be as the square of the velocity with which the stream moves.

If any stream strike upon a given obstacle, the force will (by Prop. XXVI.) be equal to the weight of a column of water whose base is the obstacle, and whose height is equal to the space through which a body must fall to acquire the velocity of the stream. Since then the base is given, the weight will be as the height of such a column. But the spaces through which bodies fall to acquire different velocities are (by Book II. Prop. XXVI.) as the squares of those velocities. Therefore the height of this column, and its weight, and consequently the force of the stream, which is equal to this weight, will be as the square of the velocity with which the stream moves.

CHAPTER III.

Of the Resistance of Fluids.

PROP. XXVIII. If a spherical body is moving in a given fluid, the resistance which arises from the reaction of the particles of the fluid is, within certain limits of the velocity, as the square of the velocity with which the body moves.

A spherical body moving in a given fluid, the number of particles which it will meet with in a given time will be as its velocity; for the space through which it will pass will be as its velocity, and the number of particles it will meet with will be as the space through which it passes. But the reaction of the particles of the fluid, and consequently the resistance, is as the number of particles or quantity of matter by which the resistance is made. Again, if a given quantity of matter is to be moved, the moving force is (by Book II. Prop. IX.) as the velocity communicated; and the resistance of that given quantity of matter is as the moving force. Therefore the resistance arising from reaction in a given number of particles of fluid is as the respective velocities with which they are moved; that is, as the velocities with which the bodies which pass through the fluid move. The resistance of the fluid being then as the velocity on a double account, first, because the number of particles moved are as the velocity of the moving body, and secondly, because the resistance of a given number of particles is as the velocity of the moving body; the resistance will be in the duplicate ratio, or as the square of this velocity.

Schol. In very swift motions, the resistance of the air increases in a greater ratio; (see Remark to Prop. LVIII. Book II.) and in other fluids the same consequence would follow for the same reason, with

respect to projected bodies. Besides, the greater the velocity is, the less will be the pressure against the back of the body which will cause a deviation in the law of resistance.

PROP. XXIX. When a spherical body moves with a given velocity in any fluid, the resistance of the fluid arising from its reaction will be as the square of the diameter of the spherical body.

A spherical body, in moving through a fluid, displaces a cylindrical column of that fluid, the height of which is the space which the sphere describes, and its base a great circle of the spherical body. Because the velocity is given, the space described in a given time, that is, the length of the column is given; whence, the quantity of fluid in the column, that is, the column will be as its base, a great circle of the sphere. And the resistance which the column of fluid makes by reaction to the motion of the sphere will be as its quantity of matter; it will therefore be as the base of the column, or as the great circle of the sphere, or (El. XII. 2.) as the square of its diameter.

PROP. XXX. If two unequal homogeneous spheres are moving in the same fluid with equal velocities, the greater sphere will be less resisted in proportion to its weight, than the lesser sphere.

The weights of spheres, or their solid contents, are (El. XII. 18.) as the cubes of their diameters; but their resistances (Prop. XXIX.) are as the squares of their diameters; and the cubes of any numbers have a greater ratio to each other than their squares. Therefore the ratio of the weights of spherical bodies is greater than that of their resistances in a given fluid; that is, the weight of the greater sphere exceeds the weight of the lesser, more than the resistance of a given fluid against the former exceeds the resistance against the latter, provided the spheres are moving with equal velocities.

SCHOL. Hence the resistance of the air may be able to support small particles of fluid, but unable to support them when they are collected into larger drops.

PROP. XXXI. The resistance of a fluid, arising from its reaction, is as the side of a body perpendicularly opposed to it.

The resistance is as the column, or quantity of fluid removed in a given time, which, as was shown, (Prop. XXIX.) is as the base of the column; that is, as the side of the body perpendicularly opposed to it.

PROP. XXXII. When equal spheres move with the same velocity in different fluids, the resistances will be as the densities of the fluids.

The resistances arising from reaction are as the momenta communicated to the fluid in a given time; that is, since the spheres move with equal velocities, as the quantities of matter moved. But because the spheres are equal, the bases of the columns to which they communicate motion are equal; and because the spheres move with equal velocity, the lengths of the columns to which they communicate motion, are equal. Hence the columns to which motion is communicated, having their bases and heights equal, are of equal magnitude; and consequently, their quantities of matter are as their densities. But it has been shown, that their momenta and resistances are as their quantities of matter; therefore their resistances are as their densities.

SCHOL. Hence drops of water may be sustained in the lower parts of the atmosphere, which cannot be sustained in the higher.

PROP. XXXIII. The retardation of bodies in a resisting fluid, where the weights of the bodies are given, is as the resistance of the fluid.

The more a body is resisted by any fluid in which it moves, the greater portion of its momentum is destroyed; but, because the weight of the body is given, its momentum is as its velocity; therefore the greater the resistance of the fluid, the greater portion of its velocity is destroyed, that is, the more it is retarded.

PROP. XXXIV. When the resistance is given, the retardation is inversely as the weights.

The same resistance will destroy an equal portion of momentum whatever is the weight of the moving body. But when the momentum is the same, the velocity is (by Book II. Prop. XII.) inversely as

the quantity of matter. Therefore the velocity destroyed, or the retardation, will be inversely as the quantity of matter in the body in which the momentum is destroyed; and the weight is as the quantity of matter; therefore the retardation is inversely as the weight.

PROP. XXXV. The retardation of spherical bodies, moving with equal velocities in the same fluid, is inversely as their diameters.

The resistance which spherical bodies meet with in a given fluid is (by Prop. XXIX.) as the squares of their diameters. The retardation, when the weight is given, is (by Prop. XXXIII.) as the resistance; and when the resistance is given, the retardation (by Prop. XXXV.) is inversely as the weight; that is, (El. XII. 18.) inversely as the cubes of the diameters. Now, when unequal spheres move with the same velocity in the same fluid, the retardations will be unequal, both because the resistances are unequal, and because the weights are unequal. The retardations will therefore be directly as the squares of the diameters, and inversely as the cubes of the diameters; that is, (compounding these ratios) inversely as the diameters.

PROP. XXXVI. When a body moves in an imperfect fluid which has tenacity, or the parts of which cohere, the resistance of any given portion of the fluid from this cause, is inversely as the velocity of the body; the resistance, when the velocity is given, is as the quantity of fluid through which the body passes; and the resistance is always as the time during which the body moves in the fluid.

Case 1. Suppose such an imperfect fluid, as soft clay, divided into thin plates; each plate having a certain portion of tenacity will continue to resist the body during the whole time in which it is passing through it; the resistance therefore will be less, the shorter time the body takes in passing through it, that is, the greater velocity the body moves with. And this is true concerning every plate which composes the fluid. Therefore the resistance arising from tenacity in a given quantity of fluid, is inversely as the velocity of the body which passes through it.

Case 2. Again, the velocity of the body being given, the resistance which the body meets with from what has been said, is also given, and will be as the number of plates or quantity of the fluid.

Case 3. Lastly, when a body moves for a given time, the resistance (by the second case) is as the number of plates, that is, as the space through which it passes in a given time, that is, (by book II. Prop. VI.) as the velocity directly. And (by the first case) the resistance is, on account of the tenacity, inversely as the velocity. Therefore as much as the resistance is increased on account of the velocity in one respect, so much it is diminished on account of the velocity in another; and consequently, whatever be the velocity of a body in such a fluid, the resistance which it meets with in a given time will be the same; whence this resistance will be as the time in which the body moves in the fluid.

CHAPTER IV.

Of the Specific Gravities of Bodies.

Def. V. The *density* of a body is its quantity of matter when the bulk is given.

Def. VI. The *specific gravity* of a body is its weight, compared with that of another body of the same magnitude.

Cor. 1. The specific gravity of a body is as its density. For the specific gravity of a body is the weight of a given magnitude, and the weight of a body (by Book II. Prop. XXIV. Cor.) is as its quantity of matter; therefore the specific gravity of a body is as the quantity of matter contained in a given magnitude, that is, as its density.

Cor. 2. The specific gravities of bodies are inversely as their magnitudes when their weights are equal. For by the last Cor. the specific gravities of bodies are as their densities, and their densities (from Def. 1.) are inversely as their magnitudes when their weights are equal. Therefore the specific gravities are also inversely as their magnitudes when their weights are equal.

PROP. A. The weight of a body varies as its magnitude and specific gravity conjointly.

For if the magnitude of any body is varied, its specific gravity remaining the same, the weight must be altered in the same ratio. And if the specific gravity vary while its magnitude continues the same, the weight must also vary in the same ratio. Therefore the weight must vary as the magnitude and specific gravity conjointly.

PROP. XXXVII. A fluid specifically lighter than another fluid will float upon its surface.

For (by Book II. Prop. XXIV.) the lighter fluid will be less powerfully acted upon by the force of gravitation than the heavier; whence, the heavier will take the lower place.

Exp. 1. Let a small and open vessel of wine be placed within a large vessel of water; the wine will ascend.

2. Let mercury, water, wine, oil, and spirits of wine, be put into a phial in the order of their specific gravities; they will remain separate.

PROP. XXXVIII. The heights to which fluids, which press freely upon each other, will rise, are inversely as their specific gravities.

Since (by Prop. VI.) the opposite parts of a homogeneous mass of fluid, in a curved tube or channel, press equally against each other when they rise to the same height; in order to preserve the pressure equal when the fluids on each side are different, that which has the least specific gravity must proportionally rise above the level to preserve the balance; and the reverse.

Exp. Into the longer arm of a recurved tube, of equal bore throughout, and open at each end, pour such a quantity of mercury, that it shall rise in each arm about half an inch; then pour water into the longer arm till the mercury is raised one inch above its former height; the specific gravities of these fluids will be inversely as the heights to which they rise.

PROP. XXXIX. The force with which a body lighter than any fluid endeavours to ascend in that fluid, is as the excess of the specific gravity of the fluid above the solid.

Since ABCD, the fluid in a vessel, will be at rest (Prop. III.) when every part of an imaginary plane SQ, under the surface of the floating body *p t e i*, sustains an equal pressure; if the solid body be of equal specific gravity with the fluid, that is, weighs as much as a quantity of the fluid equal to it in bulk, and whose place it takes up, this imaginary plane being equally pressed by the solid, as if the same space were filled with fluid, the fluid will be at rest, and the solid will neither ascend nor descend. Consequently, if the body be specifically heavier than the fluid, that part of the plane which is directly under the solid being so much more pressed than the other equal parts of the same plane as the solid body is specifically heavier than the fluid, the body must descend with a force equal to that excess; and, on the contrary, if the body be specifically lighter than the fluid, that part of the plane which is directly under the solid being so much less pressed than the other equal parts of the same plane, as the body is specifically lighter than the fluid, it must be buoyed up with a force equivalent to the difference of their specific gravities. Plate 5. Fig. 19.

PROP. XL. Any fluid presses equally against the opposite sides of a solid body immersed in it.

The opposite sides of the solid are at the same depth; and fluids at the same depth press equally. Thus the opposite sides RM, SN, of any body immersed in a vessel of water ABCD, are pressed equally by the surrounding fluid. Plate 1. Fig. 18.

Cor. No motion of the solid will be produced by these opposite lateral pressures.

PROP. XLI. A body immersed in a fluid is pressed more upward than it is downward, and the difference of these two pressures is equal to the weight of as much of the fluid as would fill the space which the body fills.

The body MRNS being immersed in a vessel of water ABCD, its lower part MN must be pressed upward just as much as the water itself, at the same depth MNT, would be if no solid were immersed. Now the water at any depth (by Prop. III.) is pressed as much upward as it is pressed downward. And at the depth MNT, the portion of this stratum MN would, if the solid were away, be pressed downward by a force equal to the weight of the incumbent column of water EMNH. Therefore the force with which MN, that is, the lower part of the solid, is pressed upward, is equal to the weight of as much water as would fill the whole space EHMN. But the solid body RSMN is pressed downward by the weight of the column above it EHRS. Therefore the difference between the two pressures is the Plate 5. Fig. 18.

difference of the weights of the two columns of water EHMN, and EHRS; that is, the upward pressure upon the solid body RSMN exceeds the downward pressure, by a force equal to the weight of as much water as would fill the space RSMN, taken up by the solid body. The case will be the same, whatever is the figure of the body immersed.

PROP. XLII. A body immersed in a fluid, if it be specifically heavier than the fluid, will sink.

Plate 5. Fig. 18. If the body RSMN is specifically heavier than the fluid, it weighs more than a quantity of the fluid of the same bulk with it. Hence the column EHMN, consisting of the column of fluid EHRS and the solid body RSMN, is heavier than the same column would be if it consisted wholly of water. But the upward pressure against MN is (by Prop. III.) equal to the downward pressure of the column of water EHMN, and therefore only sufficient to support the weight of that column. It cannot then support the weight of the heavier column, consisting of a fluid and a solid, EHMN; and that part of this column which is specifically heavier than the fluid, that is, the solid will sink, with a force equal to the difference of the weights of the column of fluid EHMN, and the mixed column EHRS, RSMN.

PROP. XLIII. A body specifically lighter than the fluid, in which it is immersed, will rise to the surface and swim.

Plate 5. Fig. 18. If the solid RSMN be a body specifically lighter than water, the column EHMN will weigh less as it consists of the column of water EHRS, and the solid RSMN, than if it consisted entirely of water. Consequently, the upward pressure upon MN, which is equal to the weight of the column of water EHMN, will be equal to more weight than that of the mixed column EHRS, RSMN; and therefore the lighter part of this column, that is, the solid body, will be carried upward with a force equal to the difference of the weights of the column of fluid EHMN, and the mixed column EHRS, RSMN.

PROP. XLIV. A body which has the same specific gravity with the fluid, in which it is immersed, will remain suspended in any part of the fluid.

Plate 5. Fig. 18. The body RMNS being of the same specific gravity with the fluid, the column EHMN presses downward with the same force, whilst this body makes a part of it, as if the column consisted wholly of water, that is, with a force equal to the upward pressure against MN. Therefore the body RSMN, having its lower surface MN, and in like manner all its parts, pressed by equal forces in opposite directions will remain at rest.

Exp. Let small glass images made hollow, and of specific gravity somewhat less than water, having a small orifice to receive water, swim in a large glass vessel nearly filled with water and covered over closely with a piece of bladder; by pressing the bladder with the hand, the air on the surface is compressed; this pressure is communicated to the air in the images, which consequently receive a larger portion of water, and become in specific gravity as heavy as the water, or heavier, and either float in the water, or sink.

PROP. XLV. A body specifically heavier than the fluid, in which it is immersed, may be supported in it by the upward pressure, if the pressure downward be taken away; and a body specifically lighter than the fluid, in which it is immersed, will not rise in the fluid, if the upward pressure be taken away.

Plate 5. Fig. 18. For, in the first case, the pressure which the solid RSMN sustains from the weight of the fluid being removed, the solid may press downward with a force equal to, or less than, that of the column of fluid EHMN, that is, than that with which it is pressed upward, according to the degree of depth in the fluid at which the solid is placed.

In the second case, as the upward pressure against MN is diminished, the downward pressure of the mixed column EHMN becomes equal to, or greater than, the upward pressure, and the solid will either float in the fluid, or sink.

Exp. For the first part of the proposition, see Prop. III. Exp. 2 and 5. The second part may be thus confirmed. If a plane and smooth piece of hard wood, or of cork, be closely pressed down by the hand upon the plane and smooth bottom of any vessel, whilst mercury is pouring into the vessel; upon removing the pressure of the hand, the downward pressure of the mercury will prevent the wood from rising.

PROP. XLVI. If a body float on the surface of a fluid specifically heavier than itself, it will sink into the fluid till it has displaced a portion of fluid equal in weight to the solid.

Let $p\,t\,e\,i$ be a body, floating on a liquor specifically heavier than itself, it will sink into it till the immersed part, $r\,n\,e\,i$, takes up the place of so much fluid as is equal to it in weight. For, in that case, $e\,i$, that part of the surface of the stratum upon which the body rests, is pressed with the same degree of force, as it would be, were the space $r\,n\,e\,i$ full of the fluid; that is, all the parts of that stratum are pressed alike, and therefore the body, after having sunk so far into the fluid, is in equilibrio with it, and will remain at rest. Plate 5. Fig. 19.

Exp. 1. Place a cube of wood on a small jar, exactly filled with water; a part of its bulk will be immersed, and will displace a quantity of the water; take the cube out of the water, and put it into a scale, with which an empty vessel in the other scale stands balanced; then pour water into that vessel till the equilibrium is restored; that portion of water will fill up the jar in which the cube was placed.

2. Let a glass jar with a weight sufficient to make it sink in water to about two thirds of its length, be placed first in a large vessel of water, and afterwards in one which is very little wider than the jar, and which has in it a small quantity of water, the jar will sink to the same depth in both vessels; that is, till so much of the vessel is under water as is equal in bulk to a quantity of the fluid whose weight is equal to that of the whole vessel.

Cor. Hence arises a rule for estimating the specific gravities of fluids or solids. For, since (by this Prop.) the weights of the water displaced and of the solid are equal, their specific gravities are inversely as their magnitudes; that is, the magnitude of the water displaced is to that of the solid, as the specific gravity of the solid is to the specific gravity of the fluid; or (since the part immersed is equal in magnitude to the fluid displaced) the part immersed is to the whole, as the specific gravity of the solid to the specific gravity of the fluid. Consequently, the greater portion of any given solid is immersed in any fluid, the less is the specific gravity of the fluid; and with respect to solids, inverting the proposition, as the whole is to the part immersed, so is the specific gravity of the fluid to that of the solid; whence, the greater is the portion of any solid immersed in a given fluid, the greater is its specific gravity.

PROP. XLVII. A solid weighs less when immersed in a fluid than in open air, by the weight of a quantity of the fluid equal in bulk to the solid.

If the body immersed were of the same specific gravity with the fluid, (by Prop. XLIV.) it would be supported in the fluid by the upward pressure. The fluid therefore sustains so much of the gravity of the body, or takes away so much of its weight, as is equal to the weight of that quantity of fluid which would fill the place taken up by the body.

Or thus; a body endeavours to descend by its whole weight; but (as was shown, Prop. XLI.) when it is immersed in a fluid, it is supported by a force equal to the weight of an equal bulk of that fluid. And since these two forces act in contrary directions, the weight which the body retains in the fluid will be the difference between them; that is, it weighs as much less in the fluid than in the air, as the weight of a quantity of the fluid equal in bulk to the solid.

Exp. Having provided a solid cylinder of lead which exactly fills a hollow cylinder of brass, place in one scale the hollow cylinder; under the same scale suspend by a string the solid cylinder, and balance the whole by weights; then immerse the solid cylinder in water, and the equilibrium will be restored by filling up the hollow cylinder.

Remark. In strictness both the solid and fluid should be weighed in vacuo. The error, however, arising from the pressure of the air is very small, and may be neglected, unless where the body to be weighed is very light, and also where great precision is required.

Cor. 1. Hence the specific gravities of different fluids may be compared, by observing how much the same solid (specifically heavier than the fluids) loses of its weight in each fluid; that fluid having the greatest specific gravity in which it loses most of its weight.

Exp. Let a cubic inch of wood, made sufficiently heavy to sink in water, be immersed successively in different fluids; it will displace a cubic inch of the fluid in which it is immersed; and since the cube (by Prop. XLVIII.) weighs less in the fluid, by the weight of a quantity of the fluid equal in bulk to the cube, its loss of weight will be the weight of a cubic inch of the fluid.

Cor. 2. The weights which bodies lose in any fluid are proportional to their bulks.

Exp. 1 Two balls of equal bulk, one of ivory, the other of lead, will lose equal weight in water.

2. A piece of copper and a piece of gold being of equal weight in air, the gold outweighs the copper in water.

Cor. 3. If it be known what a cubic inch of any body loses in water, the solid content of any irregular mass of the same kind may be known, by observing how much more or less it loses, than a cubic inch would lose.

Exp. Weigh a cubic inch and any irregular piece of wood of the same kind, and observe the difference of their weights.

Cor. 4. The weight of a solid body of the same specific gravity with the fluid, or of a portion of the fluid itself, suspended in the fluid, is not perceived, because this weight is supported, and not because the gravity of the body is lost or destroyed.

PROP. B. If a and b be the specific gravities of two fluids which are to be mixed together; A and B their magnitudes, and c the specific gravity of the compound; then $A : B :: b - c : c - a$, provided the magnitude of the compound be equal to the sum of the magnitudes of the parts when separate.

Since (by Prop. A.) the weights of bodies are as their magnitudes, and specific gravities, conjointly. the weight of $A = A \times a$, and that of $B = B \times b$; and the weight of the compound $= \overline{A + B} \times c$; but the weight of the compound must be equal to the sum of the weights of the two parts, $\overline{A + B} \times c = A \times a + B \times b$, therefore $A c + B c = A a + B b$, and $A c - A a = B b - B c$; consequently $A : B :: b - c : c - a$.

Exp. Let the specific gravity (to avoid fractions) of gold be 19, of silver 11, and of the compound 14; then the magnitude of the silver in the mixture is to that of the gold, as $19 - 14 : 14 - 11 :: 5 : 3$.

Cor. Hence the solution of that problem which was investigated by Archimedes, in order to detect the fraud of the artist, who, instead of gold, was suspected of having substituted silver, in the crown of Hiero, king of Syracuse. If the proportion of the weights of each body is required, the ratios of their magnitudes, and of the specific gravities, must be taken conjointly; in this case the weights of A and B are as $a \times \overline{b - c} : b \times \overline{c - a}$; that is, the weight of the silver is to that of the gold, as $11 \times 5 : 19 \times 3 :: 55 : 57$.

Schol. We easily deduce from this chapter the methods of obtaining the specific gravities of any bodies, taking rain water as a standard, a cubic foot of which being uniformly found to weigh 1000 avoirdupois ounces.

The weight which a body loses in a fluid is to its whole weight, as the specific gravity of the fluid is to that of the body; where three terms of the proportion being given, the fourth is easily found. Ex. If a guinea weigh in air 129 grains, and on being immersed in water lose $7\frac{1}{4}$ of its weight, the proportion will be $7\frac{1}{4} : 129 :: 1000$ to the specific gravity of a guinea. By this method the specific gravities of all bodies that sink in water may be found.

Cor. 1. Hence if different bodies be weighed in the same fluid, their specific gravities will be as their whole weights directly, and the weights lost inversely.

If a body to be examined consist of small fragments, they may be put into a small bucket and weighed; and then if from the weight of the bucket and body in the fluid, we subtract the weight of the bucket in the fluid, there remains the weight of the body in the fluid.

Cor. 2. If the same body be weighed in different fluids, the specific gravity of the fluids will be as the weights lost.

Exp. The loss of weight sustained by a glass ball in water and milk is respectively 803 and 831 grains; therefore the specific gravity of water is to that of milk as 803 : 831; that is, as 1.000 : 1.034. By the same method, the specific gravity of water is to that of spirits of wine, as 1.000 to .857.

TABLE OF SPECIFIC GRAVITIES.

Platina (pure)	23.000	Marble	2.705
Fine gold	19.640	Green glass	2.600
Standard gold	18.888	Flint	2.570
Mercury	14.019	Ivory	1.825
Lead	11.325	Sulphur	1.810
Fine silver	11.091	Chalk	1.793
Standard silver	10.535	Calculus humanus	1.542
Copper	9.000	Lignum vitæ	1.327
Gun metal	8.784	Coal	1.250
Fine brass	8.350	Mahogany	1.063
Steel	7.850	Milk	1.034
Iron	7.645	Brazil wood	1.031
Pewter	7.471	Box wood	1.030
Cast iron	7.425	Rain water	1.000
Loadstone	4.930	Ice	.908
Diamond	3.517	Living men	.891
White lead	3.160	Ash	.800

Maple	.755	Fir	.550
Beech	.700	Cork	.240
Elm	.600	Common air	$.001\frac{7}{50}$

REMARK 1. The above table shows the specific weights of the various substances contained in it, and the absolute weight of a cubic foot of each body is ascertained in avoirdupois ounces, by multiplying the number opposite to it by 1000; thus the weight of a cubical foot of mercury is 14019 ounces avoirdupois, or 876lb.

REMARK 2. If the weight of a body be known in avoirdupois ounces, its weight in Troy ounces will be found by multiplying it into .91145. And if the weight be given in Troy ounces, it will be found in avoirdupois by multiplying it into 1.0971.

REMARK 3. Mr. Robertson, late librarian to the Royal Society, was the gentleman who investigated the specific gravity of living men, in order to know what quantity of timber would be sufficient to keep a man afloat in water; supposing that most men were specifically heavier than river water; but the contrary appeared to be the case from trials which he made upon ten different persons, whose mean specific gravity was, as expressed in the table, 0.891, or about $\frac{1}{9}$th less than common water.* Phil. Trans. Vol. L.

The scales made use of to determine the specific gravities of bodies, are called the *hydrostatic balance.*

BOOK III. PART II.

OF PNEUMATICS.

CHAPTER. I.

Of the Weight and Pressure of the Air.

DEF. The *Air* or *Atmosphere*, is that fluid which encompasses the earth.

PROP. XLVIII. The air has weight.

This appears from experiment.

EXP. 1. The air being exhausted, by an air-pump, from a glass receiver, the vessel will be held fast by the pressure of the external air.

2. If a small receiver be placed under a larger, and both be exhausted, the larger will be held fast, while the smaller will be easily moved.

3. If the hand be placed upon a small open vessel, in such a manner as to close its upper orifice, it will be held down with great force.

4. The upper orifice of an open receiver being closely covered with a piece of bladder, upon exhausting the receiver, the bladder will burst.

5. In the same situation a thin plate of glass will be broken.

6. Pour mercury into a wooden cup, closely placed upon the upper orifice of an open receiver; when the air beneath is exhausted, the pressure of the external air will force the mercury through the wood, and it will descend in a shower.

7. On a transferrer let the air be exhausted from a long receiver; then let water be admitted through a pipe, by means of a cock; the water will rise in a *jet d'eau.*

8. Fill a glass tube, about 3 feet long and closed at one end, with mercury; then insert the open end in a vessel of mercury; the mercury will remain suspended in the tube by the pressure of the external air upon the surface of the mercury in the vessel; when this pressure is removed, by placing the tube and vessel under a receiver, and exhausting the air, the mercury will sink in the tube, and on re-admitting the air, will rise.

9. If the same immersed tube be suspended from the beam of a balance, the weight necessary to counterpoise it, exclusive of the weight of the tube, is equal to that of the mercury sustained in the barometer by the pressure of the atmosphere; for the weight of the column of air incumbent upon the tube not being counterbalanced by the contrary pressure from below, which is employed in bearing up ths mercury within the tube, must press upon the beam.

* This must have been taken during *inspiration*, and the buoyancy of different individuals is probably very different.

10. Let a barometer tube, instead of being hermetically sealed at the top, be closely covered with a piece of bladder; the mercury will rise to the same height as in a common barometer; and on piercing the bladder with a needle, to admit the air, it will fall.

Schol. Hence the pressure of the atmosphere on or near the surface of the earth is known; the weight of any column of air being equal to the weight of the column of mercury, of the same diameter, supported in the barometer. And, since the height of this column varies with the weight of the atmosphere, the varieties in the weight of the atmosphere are known by the barometer.

11. Let the air be exhausted from a glass vessel, and by means of a cock let the vessel be kept exhausted; weigh the vessel whilst it is exhausted, and when the air is re-admitted; the difference is the weight of so much air as the vessel contains; which difference will be about 324 grains for a thousand cubic inches.

PROP. XLIX. The air presses equally in all directions.

Exp. 1. Let a bladder, filled with air, be placed within a condensing receiver, the condensed air will make the bladder flaccid.

2. In a tall phial let an orifice be made about 3 inches above the bottom; stop this orifice; through a cork in the neck of the phial insert a long tube open at each end; and let its lower end be below the orifice in the side of the phial. The mouth of the phial being closed up about the tube, pour water into the tube till it is full. Upon opening the orifice, the water will be discharged till its surface in the tube is level with the orifice; after which it will cease to flow, because the external lateral pressure of the air balances the perpendicular pressure upon the water in the tube.

3. If a glass vessel be filled with water and covered with a loose piece of paper, on inverting the glass, the water will be kept from falling by the upward pressure of the air.

4. If a vessel be perforated in small holes at the bottom, but closed at the top, the upward pressure of the air will keep the water within the vessel; as will appear by successively stopping and unstopping a small hole in the top of the vessel.

5. Two brass hemispherical cups put close together, when the air between them is exhausted, will be pressed together with considerable force.

6. A syringe being fastened to a plate of lead, and the piston of the syringe being drawn upward with one hand, whilst the lead is held in the other, the air, by its upward pressure, will drive back the syringe upon the piston; whereas if the loaded syringe be hung in a receiver, and the air be exhausted, the syringe and lead will descend; but upon re-admitting the air, they will again be driven upward.

7. If a thin glass vessel, whose aperture is closed, be placed under the receiver of an air pump, and the air exhausted from the receiver; the vessel will be broken by the pressure of air within.

PROP. L. The pressure of the atmosphere varies at different altitudes.

Exp. Put a glass tube, open at both ends, through a cork into a large phial containing a small quantity of coloured water; let the lower end of the tube be in the water; and let the cork and tube be closely cemented to the neck of the bottle. Then, blow through the tube, till the quantity of air within the phial is so increased, that the water will rise above the neck of the phial. Let this phial be placed in a vessel of sand, to keep the air within of the same temperature; the water will stand at different heights in the tube, according to the elevation of the place where it is placed; from whence it appears, that the pressure of the atmosphere varies at different altitudes.

Cor. Hence the proportion of the specific gravity of air to that of water may be determined. If the difference in height of the two places where the above experiment is made be 54 feet, and that difference cause a difference of $\frac{3}{4}$ of an inch in the height of the water; it follows, that a column of water of $\frac{3}{4}$ of an inch, or $\frac{1}{16}$ of a foot, is equiponderant to a column of air of 54 feet having the same base; therefore the gravity of water to that of air, is as 54 to $\frac{1}{16}$, or 864 to 1. In ascending the mountain of Snowdon in Wales, which is 3720 feet perpendicular height, it was found that the barometer sunk $3\frac{8}{10}$ inches. See Art. Barometer, Prop. LVII.

PROP. LI. The force with which the wind strikes upon the sail of a ship, the velocity of the air and the dimension of the sail being given, will be as the square of the sine of the angle of incidence.

Plate 5. Fig. 16. Let AD represent the sail of a ship, with its edge toward the eye; and let a circle be drawn upon the centre K; whence K will be the middle of the sail, and AD its length. If the wind blow perpendicularly against the sail, all the air included within the space FADG will strike upon it. But if the sail is inclined in the position BE, all the air which strikes upon it, is included within the space HBEI.

If it were possible that the sail should be struck with the same quantity of air in the perpendicular position AD as in the oblique position BE, yet the quantity of the oblique stroke would be to the quantity of the direct stroke (by Book II. Prop. XVIII.) as the sine of incidence to the radius; that is, since (supposing LK drawn parallel to BH, the direction of the wind, and BC perpendicular to KL) BKL is the angle of incidence, and BL its sine, as BL to AK.

Again, if it were possible that the oblique stroke of the wind upon the sail BE should be equal to the direct stroke upon AD; yet, the column of air which strikes upon the sail directly, having AD for its base, and the column which strikes obliquely, having BC for its base, the quantity of air which strikes obliquely, is to that which strikes directly, as BC to AD, that is, as BL to AK; but the velocities in either case are supposed to be the same; therefore the momenta, or forces with which the sails are struck, will be as the quantities of matter, that is, as BL the sine of incidence to AK the radius.

Thus, the force with which the wind strikes the sail BC obliquely, is to the force with which it strikes an equal sail AD directly, as BL to AK on two accounts; first, because an oblique stroke is to a direct stroke in this ratio; and secondly, because the quantity of air which strikes the oblique sail is to that which strikes the direct one in the same ratio. Consequently, upon both accounts together, the oblique force is to the direct one as $BL \times BL$ to $AK \times AK$, or as the square of BL the sine of incidence to the square of AK the radius. But, the length of the sail, or AD being given, AK the radius is a given quantity. Therefore the force of the wind, in different obliquities of the sail, will be as BL^2 the square of the sine of incidence.

CHAPTER II.

Of the Elasticity of the Air.

PROP. LII. The air is an elastic fluid, or capable of compression and expansion.

Exp. 1. A blown bladder, pressed with the hand, will return into the form which it had before the pressure.

2. A flaccid bladder, put under a receiver, when the external air is exhausted, becomes extended by the elasticity of the internal air.

3. A bladder suspended within the receiver, with a small weight hanging from it which touches the bottom, when the external air is exhausted, by the expansion of the internal air, will raise the weight.

4. The bladder being put into a box, and a weight laid upon the lid, the weight, on exhausting the air, will be lifted up.

5. If a tube, closed at one end, be inserted at its open end in a vessel of water, the fluid in the tube will not rise to the level of the water in the vessel, being resisted by the elastic force of the air within the tube. On this principle the diving-bell is formed.

6. If a bladder be inclosed in a glass vessel so closely that the air in the vessel without the bladder cannot escape, but the air within the bladder communicates with the external air through the neck of the vessel; the external air being exhausted, the bladder will be closely pressed by the air in the vessel; and when the air is re-admitted, the bladder will be distended.

7. A shrivelled apple, under an exhausted receiver, will have its coat distended by the internal air.

8. In the same situation, the air contained in a fresh egg will expel its contents from an orifice made in its smaller end.

9. On green vegetables, and other substances, placed in a vessel of water under a receiver, whilst the air is exhausting, bubbles will be raised by the expansion of the internal air.

10. Beer, a little warmed, will, from the same cause, whilst the internal air is exhausting, have the appearance of boiling.

11. Let a cylindrical piece of wood (made just specifically heavier than water by fastening to it a small plate of lead) be placed in a vessel of water under a receiver; upon exhausting the air the wood will swim; some particles of air escaping from the wood, and hereby diminishing its specific gravity.

12. Let a glass bulb, having a long neck, be put, with the neck downward, into a vessel of water; put the whole under a receiver, and exhaust the air; on re-admitting the air, that fluid, acting on the surface of the water in the vessel by its elasticity, will cause it to rise in the bulb, or, if the degree of exhaustion be great, nearly fill it. If the air be again exhausted from the receiver, the air remaining in the bulb, by its elasticity will expel the water from the bulb.

13. Place a double transferrer upon the air-pump, with two receivers, exhaust one receiver; then open the pipe between the two receivers; and the air in the unexhausted receiver will, by its elasticity

be in part driven into the exhausted receiver; and both receivers will have equal portions of air; but this air will be rarer in both than the external air; whence both the receivers will be held fast by the external pressure.

PROP. LIII. The elastic spring of the air is equivalent to the force which compresses it.

If the spring with which the air endeavours to expand itself when it is compressed were less than the compressing force, it would yield still farther to that force; if it were greater, it would not have yielded so far. Therefore, when any force has compressed the air so that it remains at rest, the spring of the air arising from its elasticity can neither be greater nor less than this force, that is, must be equal to it.

Exp. Let the air be exhausted from an open tube, whose lower part is inserted in a vessel containing a small quantity of mercury, and let the air within the vessel be prevented from escaping; this air, by its elasticity, will force the mercury up the tube nearly to the height to which it would be raised by the pressure of the atmosphere.

PROP. LIV. The space which any given quantity of air fills is inversely, and its density directly, as the force which compresses it.

Plate 5. Fig. 20. Exp. Let there be a bent tube of the form $n\,k\,g$, open at n and closed at g. Let a small portion of mercury be at the bottom $k\,i$. Then $g\,i$ is filled with air compressed by the weight of the atmosphere, equivalent to the weight of a column of mercury about $29\frac{1}{2}$ inches in height. If more mercury be poured into the orifice n, the weight of this mercury is an addtional compressing force acting upon the air $i\,g$. Since (by Prop. V.) the columns of equal heights $l\,k$, hi, balance each other, the air in the space $g\,i$ is pressed both by the weight of the atmosphere and the column $m\,l$. If therefore $m\,l$ be $29\frac{1}{2}$ inches, the air in gi is pressed with double the weight of the atmosphere, or with two atmospheres; and it will be found that it will be compressed into the space $g\,h$, half the space which the same quantity of air took up when it was pressed only with the weight of the atmosphere; therefore the space is inversely as the compressing force. And its density (Def. V.) is inversely as its bulk, or the space filled by it. Since therefore, both the compressing force and the density of the air are inversely as the space, the density must be directly as the compressing force.

PROP. C. The density of the air being increased, the elasticity is increased in the same ratio.

For (by Prop. LIII.) the elasticity is equivalent to the compressing force; and (by Prop. LIV.) the compressing force is as the density; therefore the elasticity is as the density.

Exp. 1. Condense the air within a globular vessel, having a long neck, by blowing through the neck, the increased elasticity of the air within the vessel will force out water.

2. The glass bulb and vessel, used in Experiment 12, Prop. LII. being placed within a condensing receiver, and the quantity of air in the receiver increased, water will rise into the bulb.

3. The quicksilver in the gauge of the condenser will be forced upward in the tube by increasing the density of the air.

4. Condense the air in different degrees in the condenser, and observe the gauge; and note the different heights at which a column of mercury is supported by air of different degrees of density.

PROP. LV. The air consists of particles which repel each other with forces which are inversely as the distances between their centres.

Plate 5. Fig. 17. An elastic fluid equally compressed in all directions must have all its particles at equal distances from each other; for if the distances are unequal, where it is the least, the repelling force will be the greatest, and the particles will move toward the side where there is less repulsion, till the forces become equal, that is, till the particles are equally distant, or the fluid becomes every where of the same density. Suppose, then, two equal cubes of air, A and B; it is manifest, from the nature of the cube, that the number of particles in the whole mass A is equal to the cube of the number of particles in the line $d\,e$; and, in like manner, that the number of particles in the mass B is equal to the cube of the number of particles in the line $h\,i$. And the density of these two equal cubes of air A and B will be as the number of particles contained in them. Therefore the density of the cube A is to the density of the cube B, as the cube of the number of particles in the line $d\,e$ to the cube of the number of particles in the line $h\,i$. But, since these lines $d\,e$, $h\,i$, are of a given length, the number of particles in each will be greater, as the distances between their centres are less, that is, will be inversely as those distances. Whence, the cube of the number of particles in $d\,e$, $h\,i$, will be inversely as the cube of the distance between their

centres. And it has been shown, that the density of the mass A is to the density of the mass B, as the cube of the number of particles in de to the cube of the number of particles in hi. Therefore the density of A is to the density of B inversely, as the cube of the distance between the centres of the particles.

Also, in compressing any mass, A, every surface, as $defg$, is pressed closer to the surface next beyond it. And the repulsion of the surface $defg$ against the surface next beyond it will be (all other circumstances being equal) as the number of repelling particles in that surface, that is, as the square of the number of particles in the line de. But the number of particles in the line de is inversely as the distance between their centres. Therefore the square of the number of particles in de, that is, the number of repelling particles in the surface $defg$, that is, the repulsion of this surface against the next beyond it, is inversely as the square of the distance between the particles. Again, where the number of particles in each surface is given, if it be supposed that the particles repel each other with a force which is inversely as the distance between their centres, since the surfaces are at the same distance from each other with the particles which compose them, the repulsion of the surfaces must be in the same ratio. Thus, the repulsion in the mass A is to that in the mass B, inversely as the distances of the particles, if only their approach to each other be considered. And it has been shown that the repulsion is inversely as the square of these distances, if only the number of particles be considered. Therefore on both accounts taken together, the repulsion is inversely as the cube of the distance of the particles. And (by Prop. LIII.) the compression is as the repulsion; therefore the compression is inversely as the cube of the distance of the particles.

Now it was shown above, that the density of A is to the density of B inversely as the cube of the distance of the particles. Therefore, when a fluid consists of particles which repel each other with forces inversely as the distances between the centres of the particles, the density of the fluid will be as the compressing force. But it was shown (Prop. LIV.) that the density of the air is as the compressing force. Therefore the air consists of particles which repel each other with forces which are inversely as the distances between their centres.

SCHOL. From the doctrine of the elasticity of the air, the phenomena of sound may be explained.

When the parts of an elastic body are put into a tremulous motion, by percussion, or the like, as long as the tremors continue, so long is the air included in the pores of that body, and likewise that which presses upon its surface, affected with the like tremors and agitations. Now, the particles of air being so far compressed together by the weight of the incumbent atmosphere as their repulsive forces permit, it follows, that those which are immediately agitated by the reciprocal motions of the particles of the elastic body, will, in their approach toward those which lie next them, impel these also toward each other, and hereby cause them to be more condensed than they were by the weight of the incumbent atmosphere, and in their return will suffer them to expand themselves again; hence the like tremors and agitations will be propagated to them; and so on, till having arrived at a certain distance from the body, the vibrations cease, being gradually destroyed by a continual successive propagation of motion to fresh particles of air throughout their progress.

Thus it is that sound is communicated from a tremulous body to the organ of hearing. Each vibration of the particles of the sounding body is successively propagated to the particles of the air, till it reaches those which are contiguous to the *tympanum* of the ear (a fine membrane distended across it), and these particles, in performing their vibrations, impinge upon the tympanum, which agitates the air included within it; which being put into a like tremulous motion, affects the auditory nerve, and thus excites in the mind the sensation or idea of what we call *sound*.

Now, since the repulsive force of each particle of air is equally diffused around it every way, it follows, that when any one approaches a number of others, it not only repels those which lie before it in a right line, but the rest laterally, according to their respective situations; that is, it makes them recede every way from itself as from a centre. And this being true of every particle, the tremors will be propagated from the sounding body in all directions, as from a centre; and further, if they are confined for some time from spreading themselves by passing through a tube, will, when they have passed through it, spread themselves from the end in every direction. In like manner, those which pass through a hole in an obstacle, they meet in their way, will afterward spread themselves from thence, as if that was the place where they began; so that the sound will be heard in any situation whatever, that is not at too great a distance.

The utmost distance, at which sound of any kind has been heard, is about 200 miles, which, is said to have been observed in the war between England and Holland in the year 1672. The watch words *All's well*, given at New Gibraltar, was heard at the Old, a distance of 12 miles. In both these cases. the sound passed over water, which with respect to conducting sound, is of the greatest consequence. By an experiment made on the river Thames, a person was distinctly heard to read at the distance of 140

feet on the water, on land at that of 76; in the latter case no noise intervened, but in the former there was some occasioned by the flowing of the water against the boats. Watermen observe, that when the water is still, the weather calm, and no noise intervenes, a whisper may be heard across the river. After water, stone may be reckoned the best conductor of sound. Brick has nearly the same properties as stone.

Since the repulsive force with which the particles of air act upon each other, is reciprocally as their distances (by Prop. LV.) it follows, that when any particle is removed out of its place by the tremors of a sounding body, or the vibrations of those which are contiguous to it, it will be driven back again by the repulsive force of those toward which it is impelled, with a velocity proportional to the distance from its proper place, because the velocity will be as the repelling force. The consequence of this is, that, let the distance be great or small, it will return to its place in the same time; for the time a body takes up in moving from place to place will always be the same, whilst the velocity it moves with is proportional to the distance between the places. The time therefore in which each vibration of the air is performed, depends on the degree of repulsion in its particles, and so long as that is not altered, will be the same at all distances from the tremulous body; consequently, as the motion of sound is owing to the successive propagation of the tremors of a sounding body through the air, and as that propagation depends on the time each tremor is performed in, it follows, that the velocity of sound varies as the elasticity of the air, but continues the same at all distances from the sounding body.

The velocity of sound, according to Mr. Derham, is at the rate of 1142 feet in a second of time. Hence, with a stop watch, may be easily estimated the distance of thunder, for by multiplying the number of seconds between the flash and clap of thunder by 1142, the distance is given in feet. Or thus, persons in good health have about 75 pulsations at the wrist in a minute, consequently in 75 pulsations, sound flies about 13 miles, that is, one mile in about six pulsations. Example. On seeing the flash of a gun at sea, 54 pulsations at the wrist were counted before the report was heard, consequently the distance of the ship is $\frac{54}{6} = 9$ miles.

Moreover, since the undulatory motion of the air, which constitutes sound, is propagated in all directions from the sounding body; it will frequently happen, that the air, in performing its vibrations, will impinge against various objects, which will reflect it back, and so cause new vibrations the contrary way; now, if the objects are so situated, as to reflect a sufficient number of vibrations back to the same place, the sound will be there repeated, and is called an *echo*. And, the greater the distance of the objects is, the longer will be the time before the repetition is heard. And when the sound in its progress meets with objects, at different distances, sufficient to produce an echo, the same sound will be repeated several times successively, according to the different distances of those objects from the sounding body; which makes what is called a *repeated* echo. Echoes repeat more by night than in the day.

If the vibrations of the tremulous body are propagated through a long tube, they will be continually reverberated from the sides of the tube into its axis, and by that means prevented from spreading, till they get out of it; whereby they will be exceedingly increased, and the sound rendered much louder than it would otherwise be, as in the *speaking trumpet*.

The difference of *musical* tones depends on the different number of vibrations communicated to the air, in a given time, by the tremors of the sounding body; and the quicker the succession of the vibrations is, the acuter is the tone, and the reverse.

PROP. LVI. The elasticity of air is increased by heat.

Exp. To the bottom of a hollow glass ball let an open bended tube be affixed. Let the lower part of the bended tube and part of the ball be filled with mercury; the external surface will be pressed by the weight of the atmosphere; and the internal surface will be equally pressed by the spring of the air inclosed within the vessel. If the ball be immersed in boiling water, the increased elasticity of the included air will raise the mercury in the small tube. The same may be shown by immersing in boiling water a tube, closed at one end, into which a small quantity of mercury has been admitted, inclosing a portion of air within the tube.

Schol. 1. The wind is no other than the motion of the air upon the surface of the globe. The principal cause of the wind is, that the atmosphere is heated over one part of the earth more than over another. For, in this case, the warmer air being rarefied, becomes specifically lighter than the rest; it is therefore overpoised by it and raised upward, the upper parts of it diffusing themselves every way over the top of the atmosphere; while the neighbouring inferior air rushes in from all parts at the bottom; which it continues to do, till the equilibrium is restored. Upon this principle it is, that most of the winds may be accounted for.

Under the *Equator*, the wind is always observed to blow from the east point. For, supposing the sun to continue vertical over some one place, the air will be more rarefied there; and consequently, the neighbouring air will rush in from every quarter with equal force. But, as the sun is continually shift-

ing to the westward, the part where the air is most rarified, is carried the same way; and therefore the tendency of all the lower air, taken together, is greater that way, than any other. Thus the tendency of the air toward the west becomes general, and its parts impelling one another, and continuing to move till the next return of the sun, so much of its motion, as was lost by his absence, is again restored, and therefore the easterly wind becomes *perpetual*.

On each side of the *Equator*, to about the thirtieth degree of latitude, the wind is found to vary from the east point, so as to become north-east on the northern side, and south-east on the southern. The reason of which is, that as the *equatorial* parts are hotter than any other, both the northern and southern air ought to have a tendency that way; the northern current, therefore, meeting in this passage with the eastern, produces a north-east wind on that side; as the southern current, joining with the same, on the other side the *Equator*, forms a south-east wind there.

This is to be understood of open seas, and of such parts of them as are distant from the land; for near the shores where the neighbouring air is much rarefied, by the reflection of the sun's heat from the land, it frequently happens otherwise; particularly on the *Guinea* coast, the wind always sets in upon the land, blowing westerly instead of easterly. This is because the deserts of *Africa* lying near the *Equator*, and being a very sandy soil, reflect a greater degree of heat into the air above them; which being thus rendered lighter than that which is over the sea, the wind continually rushes in upon the land to restore the equilibrium.

That part of the ocean which is called the *Rains*, is attended with perpetual calms, the wind scarcely blowing sensibly either one way or another. For this tract being placed between the westerly wind blowing from the ocean toward the coast of *Guinea*, and the easterly wind blowing from the same coast to the westward thereof, the air stands in equilibrio between both, and its gravity is so much diminished thereby, that it is not able to support the vapour it contains, but lets it fall in continual rain, from whence this part of the ocean has its name.

There is a species of winds, observable in some places within the *Tropics*, called by the sailors *Monsoons*, or Trade Winds, which during six months of the year, blow one way; and the remaining six the contrary. The occasion of them in general is this; when the sun approaches the northern *Tropic*, there are several countries, as *Arabia*, *Persia*, *India*, &c. which become hotter, and reflect more heat than the seas beyond the *Equator*, which the sun has left; the winds therefore, instead of blowing from thence to the parts under the *Equator*, blow the contrary way; and when the sun leaves those countries, and draws near the other *Tropic*, the winds turn about, and blow on the opposite point of the compass.

From the solution of the general trade winds, we may see the reason, why, in the *Atlantic* ocean, a little on this side the thirtieth degree of north latitude, there is generally a west, or south-west wind. For, as the inferior air, within the limits of those winds, is constantly rushing toward the *Equator*, from the north-east point, or nearly so, the superior air moves the contrary way; and therefore, after it has reached these limits, and meets with air, that has little or no tendency to any one point more than to another, it will determine it to move in the same direction with itself.

In our own climate we frequently experience, in calm weather, gentle breezes blowing from the sea to the land, in the heat of the day; which phenomenon is very agreeable to the principle laid down above; for the inferior air over the land being rarefied by the beams of the sun, reflected from its surface, more than that which impends over the water, the latter is constantly moving on to the shore, in order to restore the equilibrium, when not disturbed by stronger winds from another quarter.

From what has been observed, nothing is more easy than to see, why the northern and southern parts of the world, beyond the limits of the trade winds, are subject to such variety of winds. For the air, upon account of the lesser influence of the sun in those parts, being undetermined to move toward any fixed point, is continually shifting from place to place, in order to restore the equilibrium, whenever it is destroyed, by the heat of the sun, the rising of vapours or exhalations, the melting of snow upon the mountains, or other circumstances.

Exp. Fill a large dish with cold water; into the middle of this put a water plate filled with warm water. The first will represent the ocean; and the other an island rarefying the air above it. Blow out a wax candle, and if the place be still, on applying it successively to every side of the dish, the smoke will be seen to move toward the plate. Again, if the ambient water be warmed, and the plate filled with cold water, let the smoking wick of the candle be held over the plate, and the contrary will happen.

Schol. 2. Heat expands all bodies, solid as well as fluid.

Exp. 1, 2. Water may be rarefied into steam, and will become exceedingly elastic, acting with great power, as in the eolipile, and in steam engines. See art. X. Prop. LVII.

3. Metals expand by heat, and the degrees of their expansion are measured by the Pyrometer, which is an instrument invented to render the smallest expansions sensible.

Various machines have been contrived for this purpose, by Ferguson, Dessaguliers, De Luc, &c. but the general principle may be thus illustrated. Let $a\,b\,c$ be a lever, whose fulcrum is b, acting upon Plate 12. Fig. 13.

another lever $c\,d\,e$, whose fulcrum is d; this again acts upon a third lever $e\,f\,g$, whose fulcrum is f, and let x be a metallic rod, one end of which rests against an immoveable obstacle P, and the other end against the lever $a\,b\,c$, at a. If a lamp be put under this rod, the heat will increase its length, and put the levers in motion. Now by the principle of the lever,

$$\text{Vel. of } a : \text{Vel. of } c :: ab : bc$$
$$\text{Vel. of } c : \text{Vel. of } e :: cd : de$$
$$\text{Vel. of } e : \text{Vel. of } g :: ef : fg.$$

Therefore $\text{Vel. of } a : \text{Vel. of } g :: ab \times cd \times ef : bc \times de \times fg.$

Hence, if $a\,b$, $c\,d$, $e\,f$, be small in proportion to $b\,c$, $d\,e$, $f\,g$, a trifling increase in the length x will produce a very considerable motion in the point g, which may be measured upon the graduated arc $y\,z$.

Ex. If $a\,b$, $c\,d$, $e\,f$, be each equal to 1, and $b\,c$, $d\,e$, $f\,g$, each equal to 15, then if the rod increase but the 3375th part of an inch, the point g will describe 1 inch; consequently by dividing each inch in the graduated arc into 20 parts, an expansion in the rod of less than a 60 thousandth part of an inch easily becomes visible.

Mr. Ferguson found the expansion of metals to be in the following proportion; iron and steel 3; copper $4\frac{1}{2}$; brass 5; tin 6; and lead 7. An iron rod 3 feet long, is about $\frac{1}{70}$ of an inch longer in summer than in winter. See Ferguson's first Lecture and Supplement; Dessagulier's Exp. Phil. Chamber's Cyclopædia, by Dr. Rees.

4. Mercury expanding or contracting by an increase or decrease of heat in the air, is made the measure of heat in thermometers. See Art. VIII. Prop. LVII.

Schol. 3. It is found by experiment, that air is necessary to the existence of sound, of animal life, of fire, and of explosion.

Exp. 1. Let a bell ring under an exhausted receiver, and in a condenser.

2. Let a lighted candle be extinguished under a receiver.

3. Let gunpowder fall upon red hot iron placed within an exhausted receiver.

Schol. 4. The elasticity of the air affords a method of determining the depth of the sea where a line cannot be used.

A wine glass immersed in water with its mouth downward will not become filled, because the spring of air will prevent the water from entering beyond a certain point. The *diving bell* is constructed on this principle.

PROP. LVII: To explain the nature and use of sundry Hydraulic and Pneumatic Instruments.

I. The Syphon.

Plate 5. Fig. 23. Let DEC be a bended tube, having one leg longer than the other. This instrument, used for drawing off liquors, is called the syphon. If the shorter leg of the tube be inserted in a vessel of fluid, and if by sucking with the mouth a vacuum be produced in the tube, or if the tube be filled with the fluid before it is used, the fluid will run off from the vessel. The cause of which may be thus explained; the orifice C, of the longer leg, is exposed to the pressure of the atmosphere; also, since the fluid within the shorter leg is supported by the surrounding fluid in the vessel, the pressure upon the orifice D is that of the atmosphere. The two equal orifices are then acted upon by equal pressures; the difference of the lengths of the columns of atmosphere being too small to cause any perceptible difference in their pressure. But these equal pressures are counteracted by the pressures of two unequal columns of fluid ED, EC. If, therefore, the pressures of the columns of atmosphere be more than sufficient to balance those of the columns of fluid, that which acts with the lesser force, that is, the lesser column DE, is more pressed against the column CE, than the column CE is pressed against DE at the vertex E. Consequently, the column EC must yield to the greater pressure, and flow off through the orifice C.

Exp. 1. Draw off water by a syphon.

2. Whilst mercury is passing off from a vessel by a syphon, let the air be exhausted from the vessel, and the fluid will cease to run.

3. Intermitting fountains are natural syphons.

II. The Syringe.

Let a hollow cylindrical tube have a small orifice at one end; at the other end insert a solid cylinder so exactly fitted to the tube, that no air can pass along its sides, and fix a handle to the solid cylinder. If that end of this instrument which has the smaller orifice, be inserted in water, and the solid cylinder or piston be drawn back, a vacuum will be produced within the syringe; and the pressure of

the atmosphere on the surface of the water, meeting with no opposite pressure, will force the water into the tube, from whence it may be forcibly expelled, by pushing down the piston.

III. The Common Pump.

In this useful instrument, a handle, acting upon a pin as a lever of the first kind, draws up a piston AD, fitted to the shaft or barrel of the pump, as described in the syringe. This piston has a hole, over which is a valve of leather, loaded with lead, opening upward. Toward the lower part of the shaft, is inserted a plug C, which also has in it a hole, and a valve which opens upward. When the piston, or sucker, is drawn up from the plug, a vacuum is produced in the shaft between D and C, into which the air contained in the lower part of the pipe expands itself. By repeated strokes the air escapes through the upper valve, and the vacuum becomes so perfect, that the external air, pressing without counteraction upon the surface of the water, in the well or reservoir in which the shaft is supposed to be inserted, forces the water through the valves at C and D, into the space AD; from whence it is prevented from returning downward, by the valves, which are closely pressed down by the incumbent fluid. If therefore the handle be repeatedly lifted up, the column of water will increase upon every stroke, till it rises to the level of the spout, and is discharged. But if the height be more than 34 feet, the water cannot be raised; for such a column is equal to the weight of a column of the atmosphere of the same diameter. Plate 5. Fig. 21

IV. The Forcing Pump.

In this pump, the piston is one entire cylinder, as in the syringe. The water is raised into the pipe between A and D, as in the common pump; from hence it is forced, by the downward pressure of the piston, or forcer, through a tube inserted in the side of the main shaft. In this side-tube a valve is inserted at E to prevent the water from returning, and when a sufficient quantity is raised, it is discharged by the spout. Plate 5. Fig. 22.

The common engine for *extinguishing fires* consists of two such forcing pumps, which convey the water into a reservoir made air tight, into which a pipe is inserted. As this reservoir fills with water, the air within it is proportionally condensed, and therefore forces the water up a cylinder from which it is conveyed, at pleasure, by leathern pipes.

V. The Condenser.

This instrument, which is used to force air into any vessel, is a syringe, having a solid piston, and a valve in the lower part of its barrel which opens downward. By thrusting down the piston, the air is forced through the valve, which is afterward held close by the elasticity of the condensed air. When the piston is lifted up, a vacuum is produced, till it is raised above a small hole in the barrel, when the air rushes in, and is again discharged through the valve.

Artificial Fountains are formed by the help of a condenser, which throws any quantity of air into a vessel in part filled with water; which, by its elasticity, forces the water up into pipes from which it is conveyed at pleasure.

The Air-Gun is an instrument in the form of a gun, by which a quantity of condensed air is suddenly set free, and drives a ball through the barrel with great force.

VI. The Air-Pump.

This instrument, the use of which is to exhaust the air from any vessel, has two strong barrels A, A, which communicate with a cavity in D; within each of which near the bottom, is fixed a valve opening upward, and two pistons, one in each barrel, having a valve which likewise opens upward. These pistons are moved by means of a cog-wheel in the piece TT, to the axis of which the handle B is fixed, and whose teeth catch in the racks of the pistons CC, and move them upward or downward. PQR is a circular brass plate, having at its centre the orifice K of a concealed pipe that communicates with the cavity. In the piece D at V is a screw that closes the orifice of another pipe, for the purpose of admitting the external air when required. Upon W is placed the short barometer guage for the purpose of showing the degrees of exhaustion. When the handle is turned one of the pistons is raised, and a vacuum produced in its barrel. By means of the pipe, which passes from the orifice K in the plate upon which the receiver LM, or vessel to be exhausted, stands, to the part of the barrel beneath the lower valve, the air contained in the receiver, communicating with the barrel, raises the lower valve by its elastic spring, and expands into the vacuum. Thus a part of the air in the receiver is extracted. By turning the handle the contrary way, the same effect is produced in the other barrel; whilst, the first piston being depressed, the air which had passed from the receiver is compressed, and Plate 12. Fig. 12.

escapes through a valve in the piston. This operation is continued till the air is nearly exhausted from the receiver; for it can never be perfectly exhausted, since at each stroke only such a part of the air which remained is taken away, as is to the quantity before the stroke, as the capacity of the barrel, to that of the receiver, pipe, and barrel taken together; which may be easily proved in the following manner.

Let R = the content of the receiver and pipe, B = the content of the barrel.

If L = the quantity of air in R before the stroke, and l = the quantity exhausted by it; and since, the piston being raised, the air is uniformly diffused through R and B, and that in B extracted by the stroke, consequently $L : l :: R + B : B$, or $l : L :: B : R + B$; that is, the quantity of air extracted is to the quantity before the stroke, as the capacity of the barrel is to that of the receiver, pipe, and barrel taken together.

Cor. 1. Let L, M, N, &c. be the quantities of air, before any successive strokes; l, m, n, &c. the quantities exhausted by each stroke; and $L : l :: R + B : B :: M : m :: N : n$, &c. by division $L : L - l$ $(M) :: M : M - m$ $(N) :: N : N - n$ (O), &c. or $L : M :: M : N :: N : O$, &c. therefore L, M, N, &c. are in a decreasing geometric progression, whose common ratio is that of $R + B : R$. If $R = 2B$, then $R + B : R :: 3 : 2$, and $M = \frac{2L}{3}$, $N = \frac{2M}{3}$, &c. and the quantities of air are equal to L, $\frac{2}{3}$ L, $\frac{4}{9}$ L, $\frac{8}{27}$ L, &c.

Cor. 2. Since $L : M :: l : m$; $M : N :: m : n$, &c. l, m, n, &c. are in a decreasing geometric progression, whose common ratio is that of $R + B : R$. If, as in the last Cor. $R + B : R :: 3 : 2$, then $m = \frac{2l}{3}$, $n = \frac{2m}{3}$, &c. and the quantities of air exhausted by the successive strokes are l, $\frac{2}{3}\,l$, $\frac{4}{9}\,l$, $\frac{8}{27}\,l$, &c.

Cor. 3. If R be to B in any finite ratio as 3 : 2, the receiver can never be perfectly exhausted by any finite number of turns; for let the number of turns be n, and Q the last remainder, then $Q = L \times \overline{\tfrac{2}{3}}|^{n}$, supposing L to be the quantity of air in the receiver at first; and this quantity $L \times \overline{\tfrac{2}{3}}|^{n}$ is finite, since n is finite.

VII. The Barometer.

(1) If a glass globe be exhausted of air, and balanced at one end of a beam, upon admitting the air the globe preponderates. This experiment not only, in common with others beforementioned, shows that the air has weight, but also what that weight is. The density of air was found, by Mr. Hauksbee, to be 885 times less than that of water, when the barometer stood at 29½ inches. Hence as a cubic inch of water weighs 253.18 grains Troy, a cubic inch of air weighs 0.286 grains. And if mercury be 14 times heavier than water, the specific gravity of air is to that of mercury as 1 to 885 × 14 = 12390.

Plate 12. Fig. 4. (2) If a glass tube AB, about 32 or 33 inches long, hermetically sealed at one end, be filled with mercury, and then inverted into a bason D of the same fluid, the mercury in the tube will stand at an altitude above the surface of that in the bason between 28 and 31 inches. A tube thus filled, and graduated from 28 to 31 inches, is called a barometer. The height of the mercury in the tube above the surface of the mercury in the bason is called the standard altitude, which, in this country, fluctuates between 28 and 31 inches; and the difference, between the greatest and least altitudes, is called the scale of variation.

Now the mercury in the barometer tube will subside, till the column be equivalent to the weight of the external air upon the surface of the mercury in the bason, and is therefore a true criterion to measure that weight, and chiefly directed to that purpose, in order to foretell the changes in the weather.

Plate 12. Fig. 3. If each inch of the scale of variation AD, (fig. 5, made larger for the sake of perspicuity) be divided into ten equal parts, marked 1, 2, 3, increasing upward, and a *vernier* LM, whose length is $\frac{11}{10}$ths of an inch, be likewise divided into ten equal parts, increasing downward, and so placed as to slide along the graduated scale of the barometer, the altitude of the mercury in the tube, above the surface of that in the bason may be found, in inches and hundredth parts of an inch, by this process. If the surface E of the mercury in the tube do not coincide with a division in the scale of variation, place the index of the vernier M even with this surface, and observing where a division of the *vernier* coincides with one of the scale, the figure in the *vernier* will show what hundredth parts of an inch are to be added to the tenths immediately below the index. If the surface of the mercury be between 6 and 7 tenths above 30 inches, and, the index of the *vernier* being placed even with it, the figure 8 upon the *vernier* coincide with a division upon the scale, the altitude of the barometer will be 30 inches $\frac{6}{10}$ and $\frac{8}{100}$ of an inch. For each division of the *vernier* being greater than that of the scale by $\frac{1}{100}$ of an inch, (for the tenth part of a tenth of an inch is the hundredth part of an inch) and there being eight divisions, the whole must be $\frac{8}{100}$ of an inch above the number 6 in the scale, and the height of the mercury is therefore 30.68 inches.

Cor. 1. Hence, if the atmosphere were homogeneous, its altitude would be easily found. For by the former part of this article, when the mercury stood at $29\frac{1}{2}$ inches, the density of the air was to that of mercury as 1 to 12390; consequently the altitude of a homogeneous atmosphere would be equal to $12390 \times 29\frac{1}{2} = 5.77$ miles. The real height of the atmosphere may be determined from the beginning and end of twilight. See Book VII. Prop. XXXIX.

Cor. 2. The barometer has been applied to the measuring of the heights of towers, mountains, &c. Since 12390 inches of air, near the surface of the earth, is equal to one inch of mercury; 1239 inches, or about 103 feet of air, must be equal to $\frac{1}{10}$ of an inch of mercury. Therefore if a barometer be carried up any great eminence, the mercury will descend $\frac{1}{10}$ of an inch for every 103 feet that the barometer ascends. This corollary supposes that the atmosphere near the surface of the earth is every where of the same density, which is so far from being true, that the conclusions drawn from the supposition deviate from fact even in small altitudes, as appears from the following observations made by Dr. Nettleton.

	Perpen. Altitudes.	Lowest Station.	Highest Station.	Diff.
Town of Halifax	102	29.78	29.66	0.12
Coal mine	236	29.50	29.23	0.27
Halifax-hill	507	30.00	29.45	0.55

See Abr. Phil. Trans. Vol. vi.

M. De Luc, Sir George Shuckburgh, and General Roy, have considered this subject very attentively, and have laid down certain rules, which, with proper corrections, on account of the difference of the temperature of the air, will hold good for all altitudes within our reach. See De Luc on the Modifications of the Atmosphere. Phil. Trans. Vol. lxxvii.

Cor. 3. When the mercury in the barometer stands at the altitude of 30 inches, the pressure of the air upon every square inch is rather more than 15lb. avoirdupois. Now, supposing the surface of a middle-sized man to be $14\frac{1}{2}$ square feet, the pressure upon him, when the air is lightest, will be 13.2 tons, and when heaviest, it will be 14.3 tons, the difference of which is 2464lb. The difference of pressure must affect us in regard to our health and animal spirits, especially when the change takes place suddenly.

For a description of the different kinds of barometers, see Parkinson's Hydrostatics, p.97.

VIII. The Thermometer.

The thermometer is an instrument calculated for measuring the temperature of the air, and other bodies contiguous to it, as to heat and cold, being usually a cylindrical glass tube, containing air, water, oil, spirits of wine, mercury, &c. which fluids are found to occupy different portions of the tube in different temperatures, and these portions being measured, exhibit the different expansions of the included fluid.

AB represents a glass tube, whose end A is blown into a bulb; this bulb and part of the tube being filled with quicksilver, the least change of the bulk of quicksilver, and consequently of the temperature of contiguous bodies, is shown by the rise or fall of the surface in the tube, which is indicated by the scale *a b* affixed to the frame of the instrument. Plate 12. Fig. 2.

The thermometer chiefly used in Great Britain, is that constructed by Fahrenheit; in which there are 180 divisions between the freezing and boiling water points, the freezing point being reckoned 32° above zero, or the commencement of the scale; consequently the boiling water point is 212°.*

A good thermometer must possess the following properties; the capacity of the tube should be very small and regular, and its upper end must be hermetically sealed. The empty space must be as free as possible from air. The scale must be well adjusted, and accurately divided according to the capacity of the tube. Thermometers with small bulbs, and tubes in proportion, are the most to be depended upon, for a large volume of mercury is not sufficiently sensible to the change of temperature.

Since the thermometers of Fahrenheit and Reaumur are those most in use, it will be often found convenient to be able readily to convert the degrees on Fahrenheit's scale into those of Reaumur, and vice versa; and as one degree on Reaumur's scale is equal to 2.25°, or to $\frac{9}{4}$° of Fahrenheit; and as the former scale places the freezing point at zero, and the latter places it at 32; the following canons will reduce the degrees on the one to the corresponding ones on the other.

1. To convert the degrees of Fahrenheit to those of Reaumur; $\frac{\overline{F - 32}}{9} \times 4 = R$; thus the 167° of Fahrenheit answers to the 60° of Reaumur.

* The scale on Reaumur's thermometer, which is principally used on the continent, begins at the freezing point, and proceeds both ways, from 0 or zero. From freezing to boiling water are 80 degrees. For the construction, uses, &c. of this and several other thermometers, see Parkinson's Hydrostatics, p. 154—169.

2. To convert the degrees of Reaumur into those of Fahrenheit; $\frac{R \times 9}{4} + 32 = F.$

Thus the 40° of Reaumur answers to the 122° of Fahrenheit. See No. 4, Appendix to Lavoisier's Chemistry.

It is evident, that the thermometers hitherto described, are limited in their extent. The mercurial thermometer extends no farther than the heat of boiling mercury, which answers to 600° of Fahrenheit's scale; but the heat of solid bodies in the state of ignition exceeds that of boiling mercury. To remedy this defect, Mr. Wedgewood has contrived a thermometer for measuring the higher degrees of heat, by means of a distinguished property of argillaceous bodies, viz. the diminution of their bulk by fire. This diminution commences in a dull red heat, and proceeds regularly as the heat increases, till the clay becomes vitrified. This thermometer, therefore, marks with precision, the different degrees of ignition from the red heat visible only in the dark, to the heat of an air furnace. Its construction is extremely simple. It consists of two rulers fixed upon a flat plane, a little farther asunder at one end than at the other, leaving an open longitudinal space between them. Small pieces of allum and clay, mixed together, are made of such a size as just to enter at the wide end; they are then heated in the fire along with the body whose heat we wish to determine. The fire according to the degree of heat it contains, contracts the earthy body, so that applied to the wide end of the guage, it will slide on toward the narrow end, less or more, according to the degree of heat to which it has been exposed. Each degree of Mr. Wedgewood's thermometer answers to 130 degrees of Fahrenheit; and the scale begins from 1077 of Fahrenheit. Hence the following

TABLE.

	Fahrenheit's scale.	Wedgewood's scale.
Extremity of Wedgewood's scale	32277°	240°
Cast iron melts	21877	160
Least welding heat of iron	12777	90
Fine gold melts	5237	32
Fine silver melts	4717	28
Brass melts	3807	21
Red heat fully visible in day light	1077	0
Red heat fully visible in the dark	947	1
Mercury Boils, also expressed oils	600	
Lead melts	540	Note. If these three metals be mixed together by fusion, in the proportion of 5, 8, and 3, the mixture will melt in a heat below boiling water.
Bismuth melts	460	
Tin melts	408	
Nitrous acid boils	242	
Cow's milk boils	213	
Water Boils	212	
Heat of the human body	92 to 99	
Oil of olives begins to congeal	43	
Water freezes and snow melts	32	
Milk freezes	30	
Urine and vinegar freeze	28	
Strong wine freezes	20	
A mixture of snow and salt freezes	0 to 4	
Mercury freezes	— 39 or 40	
Cold produced at Hudson's Bay, by a mixture of vitriolic acid and snow	— 69	

IX. The Hygrometer.

The hygrometer is an instrument for measuring the degrees of moisture in the air; of which there are various kinds; for whatever contracts and expands by the moisture and dryness of the atmosphere, is capable of being formed into a hygrometer. Such are most kinds of wood; catgut, twisted cord, the beard of wild oat, &c. The following are very simple in their construction, and will serve to explain the principle of the instrument.

Plate 12, Fig. 6. 1. Stretch a catgut or a common cord, ABD, along a wall, passing it over a pulley B; fixing it at one end A, and to the other hanging a weight E, carrying a small index F. Against the same wall, fit a metal plate HI, divided into any number of equal parts, and the hygrometer is complete.

For it is known, that moisture sensibly shortens catgut, cord, &c. and that as the moisture evaporates, they return to their former length. Hence the weight E, with the index, will ascend when the air is moist, and descend when it becomes drier; and the divisions on the scale will show the degrees of moisture or dryness. This hygrometer may be made more sensible and accurate by straining the catgut over several pullies placed in a parallel or any other position.

2. The sponge hygrometer is constructed as follows; BC is the beam of a balance; to the end B is hung a piece of sponge, so cut as to contain as large a superficies as possible, which must be exactly balanced on the other side by another thread of silk D, on which is strung some very small leaden shot at equal distances, so adjusted as to cause an index E to point to G, the middle of the graduated arc FGH, (made large for distinction's sake) when the air is in a middle state between the greatest moisture and the greatest dryness. Under this silk, strung with shot, is placed a shelf I, for that part of the shot to rest upon which is not suspended. When the moisture imbibed by the sponge increases its weight, it will raise the index, and vice versa when the air is dry. Plate 12. Fig. 7.

To prepare the sponge, it may be proper first to wash it in water very clean, and when dry again, dip it in water or vinegar in which there has been dissolved sal ammoniac, or salt of tartar; after which let it dry again. Salt of tartar, or any other salt, or pot-ashes, may be put into the scale of a balance, and used instead of the sponge.

X. Steam Engine.

The steam engine is a machine which derives its moving power from the elasticity and condensation of the steam of boiling water. The high importance of this machine to the mechanical arts of life, especially where immense powers are required, has given birth to many considerable improvements both in its construction and mode of operation.

The following is a description of one of the earliest steam engines, which, as it exhibits the general principles of the machine, will be deemed sufficient in a work only introductory to science. A history of the steam engine, from its first construction by Capt. Savary, down to the present time, in which are included all the great improvements made by the ingenious Mr. Watt of Birmingham, will be found in the Encyclopædia Britannica, Vol. xvii. Part ii.

H represents the boiler on its furnace; E the cylindrical vessel of iron, in which the piston OO moves up and down. The cavity between the piston and bottom of the cylinder is made air tight. F is a cock to admit the steam into the cylinder. IK is a lever, attached to the piston at I, and at K to the piston of the pump which works on that side. PQ is a solid piston moving in the pipe RM, and loaded with a heavy weight at P. ABC is the main pipe that receives the water forced from RM through a valve at C, opening outward. N is an air vessel communicating with the main pipe. At D is a valve opening upward, and at M is the water to be raised. Plate 12. Fig. 11.

The engine is represented at the end of a forcing stroke, which is likewise its position when at rest. Suppose the main pipe ABC to be filled with water, and the water in the boiler H to boil strongly. The cock F being opened, the steam rushes into the cylinder, and being much lighter than air, rises to the top, and expels the air through a valve in the bottom of the cylinder. F is then shut, and the cock G communicating with the main pipe is opened, which, by spouting cold water against the bottom of the piston, condenses the steam. A vacuum being thus obtained, the pressure of the atmosphere forces the piston down to the bottom of the cylinder; the lever IK is moved, and the piston PQ with its weight is raised, and the water ascends in the pipe MR upon the principle of the common pump. The cock G being now shut, and F opened, the steam enters the cylinder, and counteracts the pressure of the atmosphere on the piston OO; consequently, the weight P prevails, and drives down the piston PQ, forcing the water through the valve C into the main pipe and its air vessel. The use of the air vessel is to prevent the main pipe from bursting by the sudden entrance of the water; for the air at N being elastic, gives way to the stroke, and its reaction during the the time of elevating the piston PQ continues the motion of the water, so that its velocity is no more than half what it would have been if it had been impelled by starts, and rested during the raising of the piston. By opening the cock G, and shutting F, (which is done by a single operation) the steam is again condensed, the pressure of the atmosphere again prevails, and thus the work may be continued at pleasure.

The power of some of the steam engines, construced by Messrs. Boulton and Watt, is thus described as taken from actual experiment. An engine, having a cylinder of 31 inches in diameter, and making 17 double strokes per minute, performs the work of 40 horses, working night and day, (for which three relays, or 120 horses must be kept) and burns 11,000 pounds of Staffordshire coal per day. A cylinder of 19 inches, making 25 strokes of 4 feet each per minute, performs the work of 12 horses, working constantly, and burns 3,700 pounds of coal per day. These engines will raise more than 20,000 cubic feet of water, 24 feet high, for every hundred weight of good pit coal consumed by them.

XI. The Hydrometer.

Plate 12. Fig. 3. The Hydrometer, an instrument usually applied to find the specific gravities of liquids, is thus constructed; AB is a hollow cylindrical tube of glass, ivory, copper, &c. joined to a hollow ball D, at the bottom of which is a smaller ball E, containing some quicksilver, or shot, by which the instrument is so poised, that it swims vertically in a liquid. The stem AB is graduated in such a manner, that the figures exhibit the magnitudes of the parts below, and consequently, the specific gravities of the different fluids in which it descends to those figures. Thus if the parts immersed in *water*, and *spirits of wine*, be as 10 to 11.1, then the specific gravity of the water will be to that of the spirits of wine as 11.1 to 10.

To make this instrument of more service, there has been added a little plate, or dish, at the top of the tube, upon which may be placed weights, as convenience may require. For example; if the whole instrument float, immersed in spirits to a certain point, it will require an additional weight to sink it to the same depth in water. Suppose the instrument to weigh 10 dwts. and to be adjusted to rectified spirits of wine, it will then require an additional weight of 1.6 dwt. to sink it to the same point in water. Consequently, the specific gravity of water is to that of rectified spirits of wine as 11.6 to 10, or as 10 to 8.6.

BOOK IV.

OF MAGNETISM.

Def. I. THE earth contains a mineral substance which attracts iron, steel, and all ferruginous substances; this is called a *natural magnet.*

Def. II. The same substance has the power to communicate its properties to all ferruginous bodies; those bodies, after having acquired the magnetical properties, are called *artificial magnets.*

Those magnets are also made without the assistance of the natural magnet, as will hereafter be shown.

Schol. The property of attraction in the magnet was that by which it was first discovered. Every substance that contains iron, is more or less attracted by the magnet; and so universally is this truly important metal disseminated, that there are very few substances which are not in some degree capable of being attracted by the magnet. In this way iron is found to enter into the composition of animals, vegetables, minerals, and even into that of the atmosphere. On this subject, see Cavallo on Magnetism, Chap. vi. Part I.

Def. III. Those points in a magnet, which seem to possess the greatest power, are called the *poles of the magnet.*

Def. IV. The *magnetical meridian* is a vertical circle in the heavens, which intersects the horizon in the points to which the magnetical needle, when at rest, is directed.

Def. V. The *axis of a magnet* is a right line, which passes from one pole to the other.

Def. VI. The *equator of a magnet* is a line perpendicular to the axis, and exactly between the two poles.

Schol. The distinguishing and characteristic properties of a magnet, are, (1.) Its attractive and repulsive powers. (2.) The force by which it places itself, when freely suspended, in a certain direction toward the poles of the earth. (3.) Its dip or inclination toward a point below the horizon. (4.) The property which it possesses of communicating the foregoing powers to iron and steel.

Def. VII. The direction of the *dipping* needle in any place is called the *magnetical line.*

PROP. I. That mineral substance which is called the loadstone, or magnet, has the property of attracting iron; but no other body whatever, unless it has a mixture of iron.

Exp. 1. The action of the magnet on iron may be shown on needles, steel filings, &c.

2. Let a needle be suspended from a loadstone, and a string passing through its eye be fastened to the beam of a balance placed under it; the degree of force with which it is attracted, may be measured.

Schol. Some philosophers have supposed that iron is not the only substance attracted by the magnet. Mr. Kirwan says, that *nickel*, when purified from iron, becomes more, instead of less magnetic, and acquires the properties of a magnet. Mr. Cavallo instituted a number of experiments, with a view of ascertaining whether any other bodies than ferruginous ones, were attracted by the magnet. After all, he does not decide positively on the question.

PROP. II. The action, and reaction of the magnetic power, are mutual and equal.

A piece of iron, or steel, or other ferruginous substance, being brought within a certain distance of one of the poles of a magnet, is attracted by it, so as to adhere to the magnet, and not to suffer itself to be separated without an evident effort. This attraction is also mutual, for the iron attracts the magnet, as much as the magnet attracts the iron; since if they be placed on pieces of wood or cork, so as to float upon the surface of water, it will be found that the iron advances toward the magnet, as well as the magnet advances toward the iron; or, if the iron be kept steady, the magnet will move toward it.

Schol. The strength of magnetic attraction varies according to different circumstances; such as, the strength of the magnet;—the weight and shape of the body presented to it;—the magnetic, or unmagnetic state of that body; the distance between it and the magnet, &c.

The attraction is strongest near the surface of the magnet, and diminishes as it recedes from it; the law of this diminution has not yet been ascertained.

The four following experiments, accurately made by Professor Musschenbroek, will exhibit some of the irregularities respecting magnetic attraction. In these experiments, the magnet was suspended to one scale of an accurate balance, and under it there was successively placed on a table at different distances, another magnet, or piece of iron; and at each distance, the degree of attraction between the iron and the magnet was ascertained by weights put into the other scale. The results were as follow;

Exp. 1. In this experiment a cylindrical magnet, weighing 16 drams, was suspended to the scale; and on the table a piece of iron of the same shape and weight.

Dist. in inches.	Attract. in grains.
6	3
5	$3\frac{1}{2}$
4	$4\frac{1}{2}$
3	6
2	9
1	18
0	57

Exp. 2. A spherical magnet, of the same diameter as the last, but of greater strength, was affixed to the scale, and the cylindrical magnet used in the preceding experiment was placed on the table.

Dist. in inches.	Attract. in grains.
6	21
5	27
4	34
3	44
2	64
1	100
0	260

Exp. 3. Instead of the cylindrical magnet, the cylinder of iron was placed on the table, and under the globular magnet.

Dist. in inches.	Attract. in grains.
6	7
5	$9\frac{1}{2}$
4	15
3	25
2	45
1	92
0	340

Exp. 4. A globe of iron of the same diameter as the magnet, was now placed on the table.

Dist. in inches.	Attract. in grains.
8	1
7	2
6	$3\frac{1}{4}$
5	6
4	9
3	16
2	30
1	64
0	290

Cor. It appears from the second and third experiments, that, when in contact, a magnet attracts another magnet with less force than it does a piece of iron. This has been confirmed by many other experiments. But the attraction between the two magnets begins from a greater distance than between the magnet and the iron; hence it must follow a different law of decrement.*

PROP. III. The attraction and repulsion of magnetism is not sensibly affected by the interposition of bodies of any sort except those which are ferruginous.

Exp. 1. Suppose a magnet placed at an inch distance from a piece of iron requires an ounce of force to remove it, or, which is the same thing, suppose the attraction toward each other is equal to one ounce; it will be found that the same degree of attraction remains constantly unaltered, though a plate of other metal, glass, paper, &c. be interposed between the magnet and the iron, or though they be inclosed in separate boxes of glass or other matter.

* Mr. Coulomb has ascertained that the force of both magnetic and electric influence, like gravitation, is inversely as the square of the distance.

2. Move steel filings placed on a brass plate, in water, &c. by holding a magnet under the vessel.

3. Sprinkle steel dust on a sheet of paper, under which is placed a magnet, or two magnets having their poles opposite to each other, and at the distance of about an inch.

4. A needle under an exhausted receiver will be attracted at the same distance, as in the open air.

Schol. 1. Heat weakens the power of a magnet; and a white heat destroys it entirely. Hence it appears, from this cause alone, besides others which may concur, the power of a magnet must be continually varying.

Schol. 2. The attractive power of a magnet may be increased considerably by gradually adding more weight to it; for it is found that a magnet will keep suspended on one day a little more weight than it did the preceding; which additional weight being added to it, on the following day it will be found that the magnet can keep suspended a weight still greater, and so on, as far as to a certain limit.

On the contrary, by putting a very small weight of iron to it, the magnet may gradually lose much of its strength.

Schol. 3. Among natural magnets, the smallest generally possess a greater attractive power, in proportion to their size, than those which are larger. There have been natural magnets not exceeding 20 or 30 grains, which would lift a piece of iron that weighed 40 or 50 times more than themselves. A small magnet, worn by Sir. I. Newton in a ring, weighing but about 3 grains, is said to have taken up 746 grains, or nearly 250 times its own weight. And Mr. Cavallo has seen one of 6 or 7 grains' weight, which was capable of lifting a weight of 300 grains. But magnets of two pounds and upwards, seldom lift up ten times their own weight of iron.

PROP. IV. The magnetic power may be communicated from the loadstone; and from one piece of iron to another, which then becomes an artificial magnet; and this communication of power is without apparent loss of power in the loadstone.

Exp. 1. Take a bar of soft iron, about three feet long and one inch thick, (some kitchen pokers will answer for this experiment) and place it upright, or rather in the magnetical line. Then present a magnetic needle to the various parts of the bar from top to bottom, and the lower half of the bar will be possessed of the north polarity, capable of repelling the north, and of attracting the south pole of the needle, and the upper half is possessed of the south polarity. The attraction is strongest at the very extremities of the bar, it diminishes as it recedes from them, and vanishes about its middle point.

If the bar is turned upside down, the south pole will become north, and the north will become the south pole. In the southern parts of the globe, the lower part is a south pole; or more generally, the extremities of the bar will acquire the polarities corresponding to the nearest poles of the earth.

If an iron bar be left a long time in the direction of the magnetic line, or even in a perpendicular posture, it will sometimes acquire a great magnetic power. Tongs, pokers, &c. by being often heated, and set to cool again in an erect posture, frequently acquire a considerable magnetic virtue. Magnetism is often communicated to iron and steel by repeated blows of the hammer; by some experiments of Mr. Cavallo, it appears that this effect is often produced on brass, hence it is necessary carefully to examine the brass before it is used in the construction of theodolites, &c.

2. Place two magnets A and B in a right line, so that the north end of the one is opposed to the south end of the other, and at such a distance that the bar to be touched may rest upon them. Take now two other bars, D and E, and apply the north end of D,* and the south end of E to the middle of the untouched bar C, elevating their other ends so as to make an acute angle with the said bar. Plate 13. Fig. 1.

Separate the bars D and E, drawing them different ways along the surface of C, but preserving the same elevation; then removing the bars D and E to the distance of a foot or more from the bar C, and bringing the north and south ends into contact, apply them again to the middle of C. This process being repeated several times to each surface of the bar C, it will be found to have acquired a strong and permanent magnetism.

3. Take twelve bars, six of soft steel, and six of hard, the former to be each three inches long, $\frac{1}{4}$ of an inch broad, $\frac{1}{20}$ of an inch thick; with two pieces of iron, each half the length of one of the bars, but of the same breadth and thickness. The 6 hard bars to be each $5\frac{1}{2}$ inches long, $\frac{1}{2}$ an inch broad, and $\frac{3}{20}$ of an inch thick, with two pieces of iron of half the length but of the same breadth and thickness of one of the hard bars; and let all the bars be marked with a line quite round them at one end; then take an iron poker and tongs, or two bars of iron, the larger they are, and the longer they have been used, the better; and fixing the poker upright, or rather in the magnetical line, between the knees, hold to it near the top, one of the soft bars, having its marked end downward, by a piece of

* The north ends of magnetic bars are generally marked with a cross or straight line, as are also the north ends of the horse-shoe, or any other shaped magnets.

sewing silk, which must be pulled tight by the left hand that the bar may not slide; then grasping the tongs with the right hand, a little below the middle, and holding them nearly in a vertical position, let the bar be stroked by the lower end from the bottom to the top about 10 times on each side, which will give it a magnetic power sufficient to lift a small key at the marked end; which end, if the bar were suspended on a point, would turn toward the north, and is therefore called the north pole; and the unmarked end, for the same reason, is called the south pole. Four of the soft bars being impregnated after this manner, lay the other two parallel to each other, at a quarter of an inch distance, between the two pieces of iron belonging to them, a north and a south pole against each piece of iron; then take two of the four bars already made magnetical, and place them together so as to make a double bar in thickness, the north pole of one even with the south pole of the other; and the remaining two being put to these, one on each side, so as to have two north, and two south poles together, separate the north from the south poles at one end by the interposition of some hard substance I, and place them perpendicularly with that end downward on the middle of one of the parallel bars AC, the two north poles toward its south end, and the two south poles toward its north end. Slide them three or four times backward and forward the whole length of the bar; then removing them from the middle of this bar, place them on the middle of the other bar BD as before directed, and go over that in the same manner; then turn both the bars the other side upward, and repeat the former operation; this being done, take the two from between the pieces of iron; and placing the two outermost of the touching bars in their stead, let the other two be the outermost of the four to touch these with; and this process being repeated till each pair of bars have been touched three or four times over, will give them a considerable magnetic power.

Plate 13. Fig. 2.

Plate 13. Fig. 3.

When the small bars have been thus rendered magnetic, in order to communicate the magnetism to the large bars, lay two of them on the table, between their iron conductors, as before; then form a compound magnet with the six small bars, placing three of them with the north poles downward, and the three others with the south poles downward. Place these two parcels at an angle, as was done with four of them, the north extremity of the one parcel being put contiguous to the south extremity of the other, and, with this compound magnet, stroke four of the large bars, one after another about twenty times on each side, by which means they will acquire some magnetic power.

When the four large bars have been so far rendered magnetic, the small bars are laid aside, and the large ones are strengthened by themselves, in the same manner as was done with the small bars.

To expedite the operation, the bars ought to be fixed in a groove, or between brass pins, otherwise the attraction and friction between the bars will be continually deranging them, when placed between the conductors.

This whole process may be gone through in about ½ an hour, and each of the large bars, if well hardened, will lift about 28 ounces Troy, and they are fitted for all the purposes of magnetism, in navigation and experimental philosophy. The half dozen being put into a case in such a manner that no two poles of the same name may be together, and their irons with them as one bar, they will retain the virtue they have received; but if their power should, by making experiments, be ever so much impaired, it may be restored without any foreign assistance in a few minutes.

This method of communicating magnetism was sent to the Royal Society by Mr. Canton, in the year 1751.

Schol. 1. The magnetic virtue may be readily communicated by the horse-shoe magnet, much in the same way as in the preceding experiment.

Schol. 2. A small compass needle may be touched by being put between the opposite poles of two magnetic bars. Whilst it is receiving the magnetism, it will be violently agitated, moving backward and forward as if it were animated; and when it has received as much magnetism as it can acquire in this way, it becomes quiescent.

Another method of communicating magnetism to a compass needle, is by means of the combined horse-shoe magnet, from the centre of which draw that half of the needle which is to have the contrary pole; from a considerable distance draw the needle over it again. This repeated twenty times or more, and the same for the other half, will sufficiently communicate the power.

PROP. V. Two magnets having a free motion will attract when different poles are directed toward each other, and repel when the adjacent poles are of the same name.

Exp. A needle turning on its centre will be attracted or repelled by another, as different, or the same poles are brought near to each other.

Schol. 1. If the magnetic powers are very unequal; or the two bodies are forcibly brought together, they will attract with the same poles.

Exp. 1. Suspend a magnet by a thread, and let a small needle be brought near it, making poles of the same name contiguous.

2. Bring two very unequal needles into contact at the same poles, suspended in the same manner, they will cohere.

Schol. 2. The following experiments will show the attraction of the magnet on ferruginous bodies which are not magnetic.

Properly speaking, however, the magnet has no action upon unmagnetic bodies, for any ferruginous body becomes magnetic, on being presented to the magnet, and then is attracted by it.

Exp. 1. Place a magnetic needle upon a pin stuck on a table, and when it stands steady, place an iron bar 8 inches long, and $\frac{1}{4}$ an inch thick upon the table, so that one end of it may be on one side of the north pole of the needle, and near enough to draw it a little out of its natural direction. In this situation approach gradually the north pole of a magnet to the other extremity of the bar, and you will see that the needle's north end will recede from the bar in proportion as the magnet is brought nearer to the bar.

The reason of this phenomenon is, that, by the approach of the north pole to the magnet, in the first case, the extremity of the iron bar next to it acquires a south polarity, and consequently, the opposite extremity acquires a north polarity, by which the needle is repelled; but in the second case, when the north pole of the magnet is brought near the bar, the end of the bar next to it acquires a south polarity, and the opposite end, acquiring the north polarity, causes the north end of the needle to recede.

2. Tie two pieces of soft iron wire, AB, AB, each to a separate thread AC, which join at top, and suspend them on a pin so that the wires may hang at some distance from the wall. Then bring the marked end D of a magnetic bar just under them, and it will be seen that the wires repel each other more or less in proportion to the distance of the magnet. The same may be shown by means of the south pole of the magnet. Plate 13. Fig. 5.

If the wires be of soft iron, they will, on removing the magnet, soon collapse; but if steel wires, or two sewing needles be used, they will retain their magnetic virtue, and continue to repel each other.

3. Take four pieces of steel wire, or four common sewing needles, tie threads to them, and join them two and two, as in the last experiment; then bring the same pole of the magnet under both pairs, by which means they will acquire a permanent magnetism, and the wires of each pair will repel each other. After putting the magnet aside, bring one pair of the wires near the other pair, so that their lower extremities may be level, and the four wires will repel each other, and form a kind of square.

4. Strew some iron filings upon a sheet of paper laid on a table, and place a small artificial magnet among them, then give a few gentle knocks to the table with the hand, so as to shake the filings, and they will dispose themselves round the bar in the manner represented by the figure; many particles clinging to one another, and forming themselves into lines, which at the very poles, are in the same direction with the axis of the magnet; a little sideways of the poles they begin to bend, and then they form complete arches, reaching from a point in the north half of the magnet to a point in the other half which is possessed of the south polarity. Plate 13. Fig. 6.

5. Tie a thread to one end of a bit of soft iron wire AB, about four inches long, and suspend it freely; let a bar of soft iron CD be so supported, as to have one of its extremities C about $\frac{3}{4}$ of an inch distant from the lower extremity B of the wire. Bring now either pole of a strong magnet EF under it, and the end B of the wire will recede from C, because they are both possessed of the same polarity; but if the magnet be applied to the upper part of the wire, in the situation GH, then the end B of the wire will be attracted by the extremity C of the iron bar, because, supposing G to be the north pole of the magnet, C acquires a south polarity, and attracts the end B; because B being farthest from the north pole G, acquires also the north polarity. Plate 13. Fig. 7.

Schol. 3. Hence methods are easily devised to ascertain whether a body possesses any magnetism, and in case it does, to find out the poles.

Exp. 1. *To ascertain whether a body has any attraction toward the magnet.*

If the body contain an evident quantity of iron, it will be perceived as soon as it is brought in contact with the magnet, as a certain force will be required to separate them.

If the body be not sensibly attracted by the magnet in this way; let it be placed by means of a piece of cork or wood, upon some water, or mercury, in a common soup plate, in which situation let a magnet be brought sideways to it, and the attraction will be manifest by the body coming toward the magnet.

2. *To ascertain whether a given body has any magnetism.*

The only difference in this experiment from the last is, that instead of a magnet, must be used a piece of soft clean iron, about one inch long, and of half an ounce in weight.

3. *A magnetic body being given, to find out its poles.*

Present the various parts of the surface of the magnetical body successively to one of the poles of a magnetic needle, and the parts possessed of a contrary polarity will be discovered by the needle's standing perpendicularly towards them. Then present the various parts of the surface of the same body to the other pole of the needle.

Def. VIII. There is a point between the two poles where the magnet has no attraction nor repulsion; this point is called the *magnetic centre*, though it is not always exactly between the poles.

PROP. VI. If a magnet be cut through the middle, or any way broken in two, each piece will become a complete magnet, and the parts which were contiguous will become opposite poles.

Plate 13. Fig. 8. Exp. 1. Take a magnetic bar AB, six or eight inches long, and $\frac{1}{4}$ of an inch thick, having only two poles A and B. The magnetical centre of this bar will be in, or very near, its middle C. Now if, by a smart stroke from the hammer, part of the magnet be broken off as FB, it will be found that the part of the fragment contiguous to the fracture has acquired the contrary polarity, and a magnetical centre B will be generated.

At first the magnetic centre of this fragment is nearer to the fracture F, but in time it advances toward the middle of the fragment. The original centre C of AF, after the fracture, will likewise advance nearer to the middle of it.

2. A steel bar, of the same size as that mentioned in the last experiment, being made quite hard, may be broken into two parts, and so pressed together as to appear whole. In this situation it may be rendered magnetic by the application of very powerful magnets to its extremities; and the whole bar will be found to have two poles at its extremities, and one magnetic centre in its middle; but if the parts be separated, each will be found to have two poles and a magnetic centre.

Cor. Hence it is seen that the magnetic centre may be removed;—it may be removed also, by striking a magnetic bar, by heating it, by hard rubbing, &c.

PROP. VII. Magnetism requires some time to penetrate through iron.

Exp. Place a bulky piece of iron weighing 40 or 50 pounds, so near a magnetic needle as to draw it a little out of its direction, apply one of the poles of a strong magnet to the other extremity of the iron, and several seconds will be required before the needle can be affected by it. The interval is greater or less according to the size of the iron and the strength of the magnet.

Def. IX. A magnet is said to be *armed*, when its poles are surrounded with plates of iron or steel.

PROP. VIII. A magnet will take up much more iron when *armed*, than it can alone.

As both magnetic poles together attract a much greater weight than a single one, and as the two poles of a magnet are generally in opposite parts of its surface, in which situation the same piece of iron cannot be adapted to them both at the same time; therefore it has been common, to place two broad pieces of soft iron to the poles of a magnet, and projecting on one side, because in that case, the pieces of iron being rendered magnetic, another piece of iron could be conveniently adapted to their projections, so that both poles may act at the same time. Those pieces of iron, called the *armature*, are generally held fast upon the magnet by means of a silver or brass box. Thus AB represents the magnet CD, CD represents the *armature* or pieces of iron, the projections of which are DD, and to which the piece of iron F is made to adhere. The dotted lines represent the brass box, having a ring E at top by which the armed magnet may be suspended. In this manner the two poles of the magnet, which are at A and B, are made to act at DD. Plate 13. Fig. 4.

For this purpose, and to avoid armature, artificial magnets have been constructed in the shape of a horse-shoe, having their poles in the truncated extremities.

PROP. IX. A magnetical needle, accurately balanced on a pivot or centre, will settle in a certain direction, either duly, or nearly north and south, called the magnetical meridian.

This is known by long experience.

The directive power of the magnet is the most wonderful and useful part of the subject. By it mariners are enabled to conduct their vessels through vast oceans in any given direction; by it miners are guided in their works below the surface of the earth; and travellers conducted through deserts otherwise impassable.

The usual method is to have an artificial magnet suspended, so as to move freely, which will always place itself in or near the plane of the meridian north and south; then by looking upon the direction of this magnet the course is to be directed so as to make any required angle with it. Thus, suppose a vessel setting off from any place in order to go to another which is due west of the former; in that case, the vessel must be so directed that its course may be always at right angles with the situation of the magnetic needle, the north end of which must be to the right hand. A little reflection will show how the vessel may be steered in any other direction. An artificial steel magnet, fitted for this purpose in a proper box, is called the *mariner's compass*, or *sea compass*, or simply the *compass*; which instrument is too well known to need any particular description. The mariner's compass, with the addition of sights, divided circles, &c. for observing azimuths and amplitudes of the heavenly bodies, is called the *azimuth compass.*

Def. X. The deviation of the horizontal needle from the meridian, or the angle which it makes with the meridian, when freely suspended in a horizontal plane, is called the *declination* or *variation of the needle.*

PROP. X. There is generally a small variation in the direction of the magnetic needle, which differs in degree at different places and times.

This is known by observing the different points of the compass at which the sun rises or sets, and comparing them with the true points of the sun's rising or setting, according to astronomical tables. Thus, if the magnetic *amplitude* is 80° eastward of the north, and the true *amplitude* is 82° toward the same side, then the variation of the needle is 2° west. The variation may be estimated from the azimuths in the same way.

Schol. 1. A needle is continually changing the line of its direction, traversing slowly to certain limits toward the east and west. The first good observations on the variations were made by Burrowes about the year 1580, when the variation, at London, was 11° 15′ east, and since that time the needle has been moving to the westward at that place; also by the observations of different persons it has been found to point, at different times, as in the following table.

Years.	Observers.	Variation E. or W. °	′	Years.	Observers.	Variation E. or W. °	′
1580	Burrowes.	11	15 East.	1723	Graham.	14	17 West.
1622	Gunter.	5	56	1747	———	17	40
1634	Gellibrand.	4	3	1774	Royal Soc.	21	16
1640	Bond.	3	7	1775	Royal Soc.	21	43
1657	Bond.	0	0	1776	Royal Soc.	21	47
1665	Bond.	1	23 West.	1777	Royal Soc.	22	12
1666	Bond.	1	36	1778	Royal Soc.	22	20
1672	——	2	30	1779	Royal Soc.	22	28
1683	——	4	30	1780	Royal Soc.	22	41
1692	——	6	00				

By this table it appears, that from the first observation in 1580 till 1657, the change in the variation at London was 11° 15′ in 77 years, which at a mean rate, is nearly 9′ a year. And from 1657 to 1780, it changed 22° 41′, which is at the rate of 11′ a year nearly.

			°	′
At Paris the Variation of the Needle was	in	1550	8	0 East.
		1640	3	0
		1660	0	0
		1681	2	2 West.
		1759	18	10
		1760	18	20

			°	′
At St. Helena the Variation of the Needle was	in	1600	8	0 East.
		1623	6	0
		1677	0	40
		1692	" 1	0 West.

Near the equator, in long. 40° east, the highest variation from the year 1700 to 1756, was 17° 15′ west; and the least 16° 30′ W. In lat. 15° N. and long. 60° W. the variation was constantly 5° E. In lat. 10° S. and long. 60° E. the variation decreased from 17° W. to 7° 15′ W. In lat. 10° S. and long. 5° W. it increased from 2° 15′ to 12° 45′ W. In lat. 15° N. and long. 20° W. it increased from 1° W. to

9° W. In the Indian seas the irregularities were greater, for in 1700, the west variation seems to have decreased regularly from long. 50° E. to long. 100° E; but in 1756 the variation decreased so fast, that there was east variation in long. 80°, 85°, and 90° E. and yet, in long. 95° and 100° E. there was west variation.

In the year 1775, in lat. 58° 17′ S. and long. 348° 16′ E. it was 0° 16′ W. In lat. 2° 24′ N. and long. 32° 12′ W. it was 0° 14′ 45″ W. In lat. 50° 6′ 30″ N. and long. 4° 0′ W. it was 19° 28′ W.

Schol. 2. The variation of the needle is affected by heat and cold. The following is the result of observations made by Mr. Canton at different hours of the day, and also the mean variation for each month in the year.

The Declination observed at different hours of the same day.

June 27, 1759.

	Hours.	Min.	Declin. West.		Degrees of the Thermometer.
Morning	0	18	19°	2′	62°
	6	4	18	58	62
	8	30	18	55	65
	9	2	18	54	67
	10	20	18	57	69
	11	40	19	4	68½
Afternoon	0	50	19	9	70
	1	38	19	8	70
	3	10	19	8	68
	7	20	18	59	61
	9	12	19	6	59
	11	40	18	51	57½

The mean variation for each month in the Year.

	′	″
January - - - - - -	7	8
February - - - - -	8	58
March - - - - - -	11	17
April - - - - -	12	26
May - - - - - -	13	0
June - - - - - -	13	21
July - - - - - .	13	14
August - - - - -	12	19
September - - - -	11	43
October - - - - -	10	36
November - - - -	8	9
December - - - - -	6	58

PROP. XI. A needle which, before it receives the magnetic power, rests on its centre parallel to the horizon, on becoming magnetical will incline toward the earth; this is called the *inclination* or *dip of the needle*.

Exp. Let a small dipping needle be carried from one end of a magnetic bar to the other; when it stands over the south pole, the north end of the needle will be directed perpendicularly to it; as the needle is moved, the dip will grow less, and when it comes to the magnetic centre it will be parallel to the bar; afterward the south end will dip, and the needle will stand perpendicular to the bar when it is directly over the north pole.

Schol. 1. This property of the magnetic needle was first discovered accidentally by Robert Norman, a compass-maker at Radcliffe, about the year 1576. He relates that it being his custom to finish and hang up the needles of his compasses, before he touched them, he found that immediately after the touch, the north point would always dip or incline downward, pointing in a direction under the horizon; so that, to balance the needle again he was forced to put a piece of wax on the south end as a counterpoise. The constancy of the effect led him to measure the angle which the needle would make with the horizon, and he found it at London to be 71° 50′.

It is not yet absolutely ascertained whether the dip varies at the same place; it is now, and has been since the year 1772, about 72°, according to several observations made by Mr. Nairne and the Royal Society. The trifling difference between the first observations of Mr. Norman, and these last of Mr. Nairne, &c. leads us to suppose that the dip is unalterable at the same place.

It is certain, however, that the dip is different in different latitudes, and that it increases in going northward. It appears from a table of observations made with a marine dipping needle of Mr. Nairne, in a voyage toward the north pole in 1773, that

In latitude 60° 18′, the dip was 75° 0′.
In latitude 70 45, the dip was 77 52.
In latitude 80 12, the dip was 81 52.
In latitude 80 27, the dip was 82 2½.

See Phil. Trans. Vol. lxv.

Schol. 2. The phenomena of the compass, and the dipping needle, and of the magnetism acquired by an iron bar in a vertical position, leave no room to doubt but that the *cause* exists in the earth. Dr. Halley supposed that the earth has within it a large magnetic globe, not fixed within to the external parts, having four magnetic poles, two fixed and two moveable, which will account for all the phenome-

na of the compass and dipping needle. This would make the variation subject to a constant law, whereas we find casual changes which cannot be accounted for upon this hypothesis. This the Doctor supposes may arise from an unequal and irregular distribution of the magnetical matter. The irregular distribution also of ferruginous matter in the shell may likewise cause some irregularities.

Mr. Cavallo's opinion is, that the magnetism of the earth arises from the magnetic substances therein contained, and that the magnetic poles may be considered as the centres of the polarities of all the particular aggregates of the magnetic substances; and as these substances are subject to change, the poles will change. Perhaps it may not be easy to conceive how these substances can have changed so materially, as to have caused so great a variation in the poles, the position of the compass having changed from the east toward the west about 33° in 200 years. Also the gradual, though not exactly regular change of variation, shows that it cannot depend upon the accidental changes which may take place in the matter of the earth.

Mr. Churchman of America, says, there are two magnetic poles in the earth, one to the north and the other to the south, at different distances from the poles of the earth, and revolving in different times; and from the combined influence of these two poles, he deduces rules for the position of the needle in all places of the earth, and at all times, past, present, or to come.

The north magnetic pole, he says, makes a complete revolution in 426 years, 77 days, 9 hours, and the south pole in about 5459 years. In the beginning of the year 1777, the north magnetic pole was in 76° 4′ north latitude, and in longitude from Greenwich 140° east; and the south was in 72° south latitude, 140° east from Greenwich.

BOOK V.

OF ELECTRICITY.

Def. I. THE earth, and all bodies with which we are acquainted, are supposed to contain a certain quantity of an exceedingly elastic fluid, which is called the *electric fluid*.

Schol. This certain quantity belonging to all bodies, may be called their natural share; and so long as each body contains neither more nor less than this quantity, it seems to lie dormant, and to produce no effect.

Def. II. When any body becomes possessed of more or less than its natural quantity, it is said to be *electrified*, and is capable of exhibiting appearances which are ascribed to the power of *electricity*.

Schol. This equilibrium could never be disturbed, or, if it was disturbed, would be immediately restored, and therefore be insensible; but that some bodies do not admit the passage of the electric fluid through their pores, and along their surfaces, though others do.

Def. III. When a body has acquired an additional quantity of electric matter; or lost a part of what naturally belonged to it, and it is at the same time surrounded by bodies through which it cannot pass, it must remain in that state, and is said to be *insulated*.

PROPOSITION I.

The Electric Fluid, being excited, becomes perceptible to the senses.

Exp. 1. Let a long glass tube be rubbed with the hand, or with a leathern cushion; the electric fluid being thus excited, will attract light substances, and give a lucid spark to the finger, or any metallic substance, brought near it.

The glass tube is called the *electric*, and all those bodies which are capable, by any means, of producing such effects, are called *electrics*. The hand, or any other body that rubs an electric, is called the *rubber*.

2. As the exciting of a tube is very laborious for the operator, and the electricity procured by that means is small in quantity; globes and cylinders are used for this purpose. These, by a proper apparatus, are made to revolve on their axes, and a rubber of leather is applied to the equatorial parts of the revolving glass, which become electrical by the friction. The electricity of the globe, or cylinder, is received by a metallic conductor insulated on a glass supporter.

Plate 13. Fig. 9. A cylinder or globe thus fitted up is called an electrical machine. C represents a glass cylinder about 1 foot in diameter and 20 inches long, which is turned by means of a wheel; the rubber or cushion is supported behind the cylinder by two upright springs that appear beneath, and are fastened to two cross bars of glass. B is a metallic conductor, supported on two pillars of glass; from the end nearest to the cylinder issue several points, and at the other end the ball E projects by means of a wire. Sparks given by the conductor of a machine of this construction and magnitude are from 12 to 14 inches long. A chain D must connect the rubber with the earth.

Schol. 1. In all experiments in electricity the greatest care should be taken to keep every part of the apparatus clean, and as free as possible from dust and moisture. When the weather is clear, and the air dry, especially in clear frosty weather, the electrical machine will always work well. But in very hot, or damp weather, the machine is not so powerful.

Before the machine is used, the cylinder should be first wiped very clean with a soft linen cloth; and afterward with a clean hot flannel, or old silk handkerchief.

Sometimes it will be necessary to apply to the rubber a very small quantity of amalgam made with one part of zinc, and four or five of mercury.

Schol. 2. Respecting the theory of electricity, there are two different hypotheses, one that there is only one fluid, and the other that there are two. Dr. Franklin's hypothesis is the former, and it depends on the following principles. (1.) That all terrestrial bodies are full of the electric fluid. (2.) That the electric fluid violently repels itself, and attracts all other matter. (3.) By exciting an electric the equilibrium of the electric fluid contained in it is destroyed, and one part contains more than its natural quantity, and the other less. (4.) Conducting bodies, connected with that part which contains more electric fluid than its natural quantity, receive it, and are charged with more than their natural quantity; this is called *positive* electricity; if they be connected with that part which has less than its natural quantity, they part with some of their own, and contain less than their natural quantity; this is called *negative* electricity. (5.) When one body positively and another negatively electrified are connected by any conducting substance, the fluid in the body which is positively electrified rushes to that which is negatively electrified, and the equilibrium is restored. These are the principles of *positive* and *negative* electricity. The other hypothesis is, that there are two distinct fluids, which was suggested by M. Du Faye, upon his discovery of the different properties of excited glass, and excited resins, sealing-wax, &c. The following are the principles of this theory. (1.) That the two powers arise from two different fluids which exist together in all bodies. (2.) That these fluids are separated in non-electrics, by the excitation of electrics, and from thence they become evident to the senses, they destroying each other's effects when united. (3.) When separated they rush together again with great violence, in consequence of their strong mutual attraction, as soon as they are connected by any conducting substance. These are the principles of *vitreous* and *resinous* electricity.

PROP. II. The electric fluid passes easily along the surfaces of some bodies; whilst other bodies do not convey it; the former are called *Conductors*, the latter *Non-conductors*, or *Electrics*.

Exp. The metallic cylinder being fixed upon glass supporters, and placed near the electric machine, will, by means of the pointed wires, receive the electric fluid from the glass cylinder, and the fluid will be diffused over the whole surface of the metallic cylinder, from whence it cannot pass through the glass supporters which are electric, but may be conveyed away by any metallic or other conducting substances, brought near, or into contact with it. This metallic cylinder is called the *Prime Conductor*, or the *Conductor*.

PROP. III. Some conductors are more perfect than others; and the electric fluid passes through that which is most perfect.

Exp. The fluid will pass through a wire held in the hand.

Schol. 1. The following bodies are conductors and electrics, disposed in the order of their degrees of perfection. Conductors; gold, silver, copper, brass, iron, tin, quicksilver, lead, the semi-metals, ores, charcoals, water, ice, snow, salts, soft stones, smoke, steam. Non-conductors, or Electrics; glass, and all vitrifications, even those of metals; precious stones, resins, gums, amber, sulphur, baked wood, bituminous substances, wax, silk, cotton, feathers, wool, hair, paper, air, oil, hard stones. Many electrics become conductors, when heated, and all when moistened.

Schol. 2. Glass vessels, made for electrical purposes, are often rendered very good electrics by use and time, though they might be very bad ones when new. And some glass vessels, which had been long used for excitation, have sometimes lost their power almost entirely. Dr. Priestley mentions several instances of very long tubes, which, when first made, answered the purposes of electricity admirably, but after a few months they have become almost useless.

Schol. 3. An exhausted glass vessel on being rubbed shows no signs of electricity upon its external surface. But the electric power of a glass cylinder is the strongest when the air within is a little rarefied. If the air be condensed, or the cylinder be filled with some conducting substance, it is incapable of being excited. Nevertheless, a solid stick of glass, sealing-wax, sulphur, &c. may be excited.

Schol. 4. The same substance, by different preparations, is sometimes a conductor, and at others an electric. A piece of wood just cut from a tree is a good conductor;—let it be baked, and it becomes an electric; burn it to charcoal, and it is a good conductor again;—lastly, let this coal be reduced to ashes, and these will be impervious to electricity. Such changes are also observable in many other bodies; and very likely in all substances there is a gradation from the best conductors to the best non-conductors of electricity.

PROP. IV. Non-conductors retain the fluid on a small part of their surface where the friction has acted; conductors diffuse it over all their surface, and therefore cannot confine it, unless they be surrounded entirely by non-conductors, or be *insulated.*

Exp. Observe the partial distribution of the fluid on an excited electric, and its universal diffusion over a conductor. If a finger, or any other conductor, be presented to an excited glass, cylinder, tube, &c. it will receive a spark, and in that spark, a small part only of the electricity of the electric; because the excited electric being a non-conductor, cannot convey the electricity of all its surface to that point to which the conductor has been presented. But if any conducting substance be brought to a charged metallic conductor, it will receive in one spark nearly the whole of the electricity accumulated upon it. The small part which remains is very trifling in comparison of the first spark, and is called the *residuum.*

Def. IV. A body is said to be *positively electrified,* when it has thrown upon it a greater quantity of the electric fluid than its natural share.

Def. V. A body is said to be *negatively electrified,* when it has a less quantity of the electric matter than is natural to it.

PROP. V. The electric fluid may be excited by rubbing, by pouring a melted electric into another substance, by heating and cooling, and by evaporation.

Exp. 1. In working the electrical machine, the fluid is excited by friction. Rubbing is the general mean by which all electric substances that are at all excitable may be excited. Whether they be rubbed with electrics of a different sort, or conductors, they always show signs of electricity, and in general stronger when rubbed with conductors, and weaker when rubbed with electrics.

2. When sulphur is melted into an earthen vessel, if the vessel be supported by a conducting substance, the sulphur, when cold and separated from the vessel, is strongly electrical, and will attract light bodies.

3. If sulphur be melted into glass vessels, when cold, the glass, whether supported by electrics or not, will be positively electrified, and the sulphur negatively.

4. Melted sealing-wax, when poured into sulphur, becomes positively electrified, and the sulphur negatively.

5. Melted sealing-wax poured into glass cups acquires a negative electricity; upon being separated, the glass is positive.

6. Sulphur, melted into metallic cups, shows no signs of electricity till it is separated from the cup, when the cup is negative and the sulphur is positive.

7. If a stick of sealing-wax be broken into two pieces, the extremities that were contiguous will be found electrified, one positively, and the other negatively.

8. The tourmalin, a stone which is generally of a deep red, or purple colour, about the size of a walnut, and found in the East Indies, while kept in the same degree of heat, shows no signs of electricity, but will become electrical by increasing or diminishing its heat, and stronger in the latter than in the former case. (1.) Its electricity does not appear all over its surface, but only on two opposite sides, which may be called its poles, and they are always in one right line with the centre of the stone, and in the direction of the strata; in which direction the stone is absolutely opaque, though on the other side it is semitransparent. (2.) Whilst the tourmalin is heating, one of its sides (call it A) is electrified plus; the other (call it B) minus. But when it is cooling, A is minus, and B is plus. (3.) If this stone be excited by friction, then both its sides at once may be made positive. (4.) If a tourmalin be cut into several parts, each piece will have its positive and negative poles, corresponding to the positive and negative sides of the stone from which it was cut.

Schol. These properties are now found to belong to several hard and precious stones, as well as to the tourmalin.

9. Electricity may be produced by the evaporation of water in this manner:—Upon an insulat-

ing stand, as a wine glass, place an earthen vessel, as a crucible, a basin, &c. and put into it three or four lighted coals. Let a wire be put with one end among the coals, and with the other let it touch a very sensible electrometer. Then pour in a spoonful of water at once upon the coals, which will occasion a quick evaporation; and at the same time the electrometer will diverge. For a description of the electrometer, see Prop. XII. Schol.

PROP. VI. **The electric fluid may be lodged in electrics, or in insulated conductors, in a greater quantity than naturally belongs to them, or they may be *positively* electrified.**

Exp. In working the machine, the cylinder acquires more than its natural quantity of fluid by excitation, the conductor, by communication; for, while there is a free conveyance of fluid from the earth to the rubber, by means of a conducting supporter, the conductor will be highly electrified.

Schol. The electric matter with which the prime conductor is loaded, is not *produced* by the friction of the cylinder against the rubber. It is only *collected* by that operation from the rubber, and all the bodies that are contiguous to it. If, therefore, the rubber be well insulated, the friction of the cylinder will produce but little electricity; for in that case the rubber can only part with its own share, which is very inconsiderable. In this situation, if the finger be presented to the rubber, sparks will be seen to dart from it to the rubber, to supply the place of that electric matter which had passed from it to the cylinder; if the conductor be also insulated, these sparks will cease as soon as it is fully loaded.

PROP. VII. **The electric fluid being accumulated on any body will pass to any conductor brought near to the body; if it pass from, or be received by, pointed wires, it will be conveyed in a continued stream; if it pass from, or be received by, a surface which has no sharp points, it will be discharged with an instantaneous explosion or spark.**

Exp. 1. Receive the fluid from the conductor upon a pointed wire, and upon a brass ball.

2. The fluid will be diffused through the surrounding atmosphere, by wires placed upon the conductor.

Cor. Hence arises the necessity of keeping the whole surface of the conductor free from points.

Schol. When a conductor is electrified by communication, its whole electric power is discharged at once, on the near approach of a conductor communicating with the earth; whereas an excited electric, in the same circumstances, loses its electric power only in the parts near to the conductor.

PROP. VIII. **If conductors be insulated, they will retain a greater or less quantity of the electric fluid (the power of the machine being given) proportional to the extent of surface in the conductor.**

Exp. Observe the difference in the magnitude and distance of sparks taken from a small conductor, and of those taken from a large one.

There is a limit, beyond which this proposition will not hold true, but which experiment has not yet ascertained. For it is certain, that if the conductor be very long, it will discharge itself over the cylinder back to the rubber long before it is fully charged. The late Mr. G. C. Morgan, whose memory will be ever dear to the editor of this work, asserts, that by the most powerful excitation of a cylinder, 14 inches in diameter, the spark afforded by a conductor 8 inches in diameter, and 12 feet long, did not equal half the length of that procured from the same cylinder with a conductor of equal diameter, but shortened to 6 feet. And he thinks that a conductor of half that length even, and about 16 inches in diameter, would have yielded a longer spark than either of the preceding. See Morgan's Lect. on Elect. Vol. I. p. 54, &c.

PROP. IX. **A body may be deprived of part of its natural portion of electric fluid, or be *negatively* electrified.**

Exp. If the rubber which communicates the fluid to the glass cylinder, and from thence to the conductor, be insulated, because by working the machine a quantity of its fluid is conveyed away, and it cannot receive a fresh supply through its supporter, it will be in an exhausted or negative state.

Schol. If negative electricity be required, then the chain which connects the rubber with surrounding objects, and consequently with the earth, the great reservoir of the electric fluid, must be removed from the insulated rubber, and hung to the prime conductor; for in this case the electricity of the conductor will be communicated to the ground, and the rubber will appear strongly negative. Another conductor may be connected with the insulated rubber, and then as strong negative electricity may be obtained from this as positive can be in the case before mentioned.

The patent machine of Mr. Nairne is admirably adapted for the purposes both of positive and negative electricity.

PROP. X. When bodies are negatively electrified, they receive the fluid from other bodies brought near them.

Exp. 1. Let two insulated conductors, one of which is connected with the glass cylinder, the other with the rubber, be electrified; whilst they are in this state let them be brought near each other; a spark will pass from that which (by Prop. VI.) is positively, to that which (by Prop. IX.) is negatively electrified.

2. Let two persons standing on glass feet be electrified, first, both positively, or both negatively, they will not, on contact, communicate the fluid to each other; but let them be electrified, the one positively and the other negatively, by making a communication from one to the conductor, and from the other to the rubber; on contact, the former will give, and the latter receive a spark.

PROP. XI. From a pointed body positively electrified the fluid will be seen to stream out, toward any unelectrified body brought near it, in a conical pencil of rays; whereas, in passing from the unelectrified body to a pointed body negatively electrified, it will form a globular flame, or star, about its point.

Exp. 1. Observe in a dark room, the different appearances of the electric fluid at the extremity of a pointed wire, when the point is presented to an insulated conductor positively, and when it is presented to one negatively, electrified; or when such a wire is fixed upon a conductor positively or negatively electrified.

2. Within a *luminous conductor* electrified positively, (viewed in a dark room) the fluid will be seen passing in the form of a pencil from one wire, and received in the form of a star upon the other; and the reverse if it be electrified negatively.

PROP. XII. If two bodies be electrified, both positively, or both negatively, they repel each other; but if one be electrified positively, and the other be negatively or not at all electrified, they attract each other.

Exp. 1. Light feathers, or hair, connected with the conductor, appear repellent, but are attracted by bringing any non-electrified body near them.

2. The hair of a person electrified becomes repellent.

3. In the graduated electrometer the ball is repelled according to the degree in which the conductor is electrified.

4. Downy feathers, paper figures, threads of flax, thistle down, gold leaf, brass dust, or other light bodies, brought near to the conductor, are alternately attracted and repelled. This will not take place if the bodies be laid on a plate of glass.

5. Two bells being suspended by wires from a brass rod connected with the conductor, and a third by a silk cord, and two small balls of brass suspended by a silken thread between the bells, the fluid will be communicated from the conductor to the outer bells, and by the balls to the middle bell, and from thence conveyed by a chain to the earth; the balls in receiving and communicating the fluid are attracted and repelled succesively, and produce ringing.

6. Let water flow from a capillary tube, from which, before it is electrified, it passes in drops; upon being electrified, the particles of fluid will be separated, and their motion accelerated.

These appearances will be presented, whether the conductor be positively or negatively electrified.

7. Mr. Symmer, in the year 1759, presented to the Royal Society some papers upon the electricity of silk stockings. He had been accustomed to wear two pairs of silk stockings, a white pair under black. When these were pulled off together, no signs of electricity appeared, but on pulling off the black from the white, he heard a snapping noise, and in the dark perceived sparks of fire. On this subject he has related a number of very curious experiments on the attraction and repulsion of the stockings, and upon their different states of electricity.

Cor. Since it is found that rubbed glass electrifies any insulated conductor positively, it may be determined whether any body is electrified positively or negatively, by bringing it near to a pith-ball, or down-feather, positively electrified, and observing whether the ball or feather be attracted or repelled by the body.

Exp. Bring a pith-ball or down-feather, suspended by a silken thread and positively electrified by any rubbed glass surface, near to another pith-ball or feather suspended by a flaxen thread from a conductor connected with the cylinder; then bring the same near to a conductor connected with the rubber.

Schol. The electrometer is an instrument invented to measure the degree of electrification of any body. Small degrees of electricity are shown by the divergence of two very small pith-balls, *a*, *b*, suspended upon parallel threads, straws, &c. These balls presented to a body in its natural state will not be affected; but if the body be electrified, they will be attracted by it and diverge. Plate 13. Fig. 10.

Another very useful and common electrometer consists of an upright stick, AB, to which is affixed a graduated semicircle; D is a pith-ball stuck upon the end of a fine straw, which by means of an axis at C, is moveable in a plane parallel to that of the semicircle. This electrometer is fixed upright on a prime conductor; and when it is not electrified, the radius will hang down, and according to the intensity of the electric state given to the conductor, the repulsion must cause the ball to ascend. The ascent will be marked by the graduations. Plate 13. Fig. 11.

Mr. Cavallo has invented a very sensible electrometer, well adapted for the observation of the presence and quality of natural and artificial electricity. ABC is the brass case containing the instrument. When the part AB is unscrewed, and the electrometer taken out, it appears as represented in ABDC. A glass tube, CDNM, is cemented into the piece AB. The upper part of the tube is shaped tapering to a small extremity, which is entirely covered with sealing-wax. Into this tapering part a small tube of glass is cemented; the lower extremity being also covered with sealing-wax projects a small way within the tube CDNM. Into this smaller tube, a wire is cemented, which, with its under extremity, touches a flat piece of ivory H, fastened to the tube by means of a cork. The upper extremity of the wire projects about a quarter of an inch above the tube, and screws into the brass cap EF, which cap is open at the bottom, and serves to defend the waxed part of the instrument from the rain. From H are hung two fine silver wires, having very small corks at the lower ends, which, by their repulsion, show the electricity. IM, and KN, are two slips of tin-foil stuck to the inside of the glass, and communicating with the brass bottom AB. They serve to convey away that electricity, which when the corks touch the glass, is communicated to it, and might disturb their free motion. Plate 13. Fig. 12.

When this instrument is used to observe artificial electricity, it is set on a table, and electrified by touching the brass cap EF with an electrified body; in this state, if any electrified substance is brought near the cap, the corks of the electrometer, by their converging, or diverging more, will show the species of electricity.

When it is to be used to try the electricity of fogs, &c. it must be unscrewed from its case, and held a little above the head by the bottom AB, so that the observer may conveniently see the corks, which will immediately diverge if there is any sufficient quantity of electricity in the air, the nature of which may be ascertained by bringing an excited piece of sealing-wax toward the brass cap EF.

PROP. XIII. From the sharp points of electrified bodies there proceeds a current of air.

Exp. 1. A wire with sharp points bended in opposite directions, and suspended on the point of a perpendicular wire inserted in the conductor, will be carried round by the current proceeding from the points.

2. Let several pieces of gilt paper be stuck like vanes into the sides of a cork, through the centre of which a needle passes; suspend the whole by a magnet, and present one of the vanes to the point of a wire inserted in the conductor; they will be put into motion.

PROP. XIV. Some bodies, upon being rubbed, are electrified positively, and others negatively; and the same bodies are capable of being electrified positively, or negatively, as they are rubbed with different substances.

Exp. Smooth glass becomes positively electrified by being rubbed with any substance hitherto tried, except the back of a living cat; rough glass becomes positively electrified by being rubbed with dry oiled silk, sulphur, and metals; negatively, with woollen cloth, sealing-wax, paper, the human hand. White silk becomes positively electrified by being rubbed with black silk, metals, black cloth; negatively, with paper, hairs, the hand. Black silk will be positively electrified with red sealing-wax; negatively, with hare's skin, metals, the hand. Sealing-wax will be negatively electrified with the hand, leather, woollen cloth, paper, hare's skin. Baked wood will be positively electrified with silk; negatively with flannel. If these and other substances, being electrified, be brought near to a pith-ball or down-feather, as described Prop. XII. Cor. Exp. it will appear whether they are electrified positively or negatively.

PROP. XV. Bodies insulated, if placed within the influence of an electrified body, will be electrified, at the part adjacent to that body, in the manner contrary to that of the electrified body.

Exp. 1. Bring a conductor (without pointed wires) near to the glass cylinder, whilst the machine is working; if the conductor be not insulated, it will be negatively electrified till it is brought so near as to receive sparks from the cylinder; if the conductor be insulated, it will, in the same situation, be electrified negatively, in the parts nearest the cylinder, and positively in the parts more remote; as may be seen by bringing an excited glass tube (which is positively electrified) near to a ball suspended from the conductor. Compare Prop. XII. Cor.

2. Let two pith-balls be so suspended by flaxen threads as to be in contact when unelectrified; on being brought near to a body electrified positively, they will repel each other, being electrified negatively: if the balls be suspended in the same manner by silken threads, they will, in the same situation, be positively electrified.

Plate 13. Fig. 13. 3. Let PC be an electrified prime conductor, and AB a metallic body placed within its atmosphere, but beyond the striking distance. Now from the principles already explained, it is evident that the electrical atmosphere of the prime conductor must be *positive* or *negative*. (1.) If it be *positive*, then the adjacent part A of the metallic body AB, will be found to be electrified negatively; the remote part B, will be electrified positively; and there will be a certain point D, in its natural state, or not electrified at all. (2.) If the prime conductor be charged with negative electricity, then A will be positive, B, negative, and still some point, as D, will be found unelectrified, which is called the *neutral* point.

Earl Stanhope has demonstrated, by a considerable number of experiments, that the neutral point D is the fourth point of a harmonical division of the line CAB. Consequently the points C, A, and B being given, the neutral point D may be always found. For by the proportion assumed by his lordship, as the whole line BC is to the part CA, so is the remote part BD to the middle term DA; therefore by composition,

$$BC + CA\ (BA + 2AC) : CA :: BD + DA\ (BA) : AD.$$

Thus, if BA be 40 inches, and CA 36, then AD is equal to $12\frac{6}{7}$ inches.

Cor. 1. From the nature of this proposition, it is evident that the neutral point D can never be farther from A than half the distance between A and B, supposing the electrified conductor PC to be removed to an infinite distance.

Cor. 2. It is likewise evident, that the evanescent position of the neutral point D must be A, when the end A of the metallic body AB comes into contact with the charged body PC.

Schol. From the above considerations, Lord Stanhope has, with great ingenuity, proved by an elaborate mathematical demonstration, illustrated and confirmed by a great variety of experiments, that the density of an electrical atmosphere superinduced upon any body must be inversely as the square of the distance from the charged body.

4. Let a circular plate composed of resin and sulphur, or of sealing-wax, be negatively electrified by rubbing it with flannel; whilst it is in this state, let a metallic plate of the same form and size, having a glass handle fastened to its centre, be placed, by means of the handle, on the electrified plate; then receive a spark from the metallic plate with the finger; after which the metallic plate, being removed by the glass handle, will be found to be positively electrified. This instrument is called an electrophorus.

5. Let one side of a plate of glass be electrified positively, the other side will attract light bodies, being negatively electrified.

6. Let a plate of glass be placed between two metallic plates about two inches in diameter smaller than the plate of glass, and let the plates be supported by a conductor; upon positively electrifying the upper metallic plate, by means of a wire connected with the prime conductor, the fluid not being able to pass along the glass, will be accumulated upon the part contiguous to the upper metallic plate; whilst the lower metallic plate, being within the electric influence of the upper, will be negatively electrified.

PROP. XVI. When any electric substance is electrified, it will continue in that state till some conductor conveys away the accumulated or restores the deficient fluid; which will be done more or less rapidly, according to the degree of conducting power in the conductor, and the number of points in which it touches the electric.

Exp. 1. When the metallic plate in the electrophorus is electrified (as described Prop. XV. Exp. 4.) by setting it upon the electric plate, touching it with the finger, and separating it successively, many sparks may be obtained, without again exciting the electric plate; for this plate being negatively electrified, the metallic plate on being touched with the hand, becomes positively electrified (by Prop. XV.) and the electric plate remains long in its negative state, because not being a conductor, its deficiency will be slowly supplied from the air where its surface is not covered.

2. If a glass vessel, a common drinking-glass, for instance, held in the hand, receive the electric fluid on the inside from a wire, or chain, fixed on the conductor, pith-balls, placed under the vessel upon a conducting supporter, will continue long in motion.

3. Let a plate of glass be electrified in the manner described in Prop. XV. Exp. 6. Because one side of the plate is positively electrified, and the other negatively, if a communication be made from one metallic plate to the other by means of some conductor, part of the accumulated fluid will suddenly pass to the side which is deficient; upon a second application of the plates of metal to the glass, there will be a second explosion.

Schol. AB is an electric jar, coated with tin-foil on the inside and outside, within three inches of the top, having a wire with a round brass knob K, at its extremity. This wire passes through the cork D, that stops the mouth of the jar, and, at its lower end, is bended or branched so as to touch the inside coating in several places. Coated jars may be made of any form and size, and are called *Leyden Phials*, or *Leyden Jars.* Plate 13. Fig. 14.

A number of jars combined, make what is termed an electrical *battery;* they all stand in a box, the the bottom of which is covered with tin, thus all their outsides are connected; and by means of wires and brass rods, their insides are also connected.

The discharging rod consists of a glass handle A, and two curved wires BB, which move by a joint C, fixed to the brass cap of the glass handle A. The wires BB are pointed, and the points enter the knobs DD, to which they are screwed, and may be unscrewed from them at pleasure. By this construction, the balls or points may be used as occasion requires. The wires being moveable at the joint C, may be adapted to smaller or larger jars at pleasure. Plate 13. Fig. 15.

PROP. XVII. If a glass plane, or cylindrical vessel, *coated* on both sides with tin-foil, or any other conducting substance, be *charged,* that is, positively electrified on one side, and consequently negatively electrified on the other; a communication being made from one side to the other by some conductor, the plane, or vessel, will be suddenly *discharged,* with an explosion.

There is a strong attraction (compare Prop. XII. and XV.) between the fluids on opposite sides of the glass, or the fluid which is accumulated on one side makes a powerful effort toward the other side where the fluid is deficient; but the substance of the glass itself being impervious to the electric fluid, the accumulated fluid cannot pass to the deficient side till a communication is made between them by some conducting substance. When such a communication is made, because the metallic coating touches the whole surface of the electrified glass, the whole quantity of redundant fluid easily passes from the side which was positively electrified to the other.

Exp. 1. Let a plate of glass coated with tin-foil (except about $1\frac{1}{2}$ inch from the edge) be charged, as described in Prop. XV. Exp. 6. Upon making a communication from one side to the other by the discharging rod, there will be a sudden discharge.

2. Let the same be done with the *Leyden Phial.*

3. Charge a jar, coated on the inside, with water, shot, or brass dust, and held on the outside by the hand; then discharge it in a dark room.

4. If two equal circular brass plates, one of which is suspended by a long metallic rod from the conductor parallel to the horizon, and the other, supported by a conductor, is placed parallel and opposite to the first, be electrified; the plate of air between them will be charged by the brass plates.

5. Let one coated jar be suspended by a wire under another; let the upper jar be charged by taking sparks from the conductor; the lower uninsulated jar will be charged with the fluid which passes from the side negatively electrified of the upper jar.

6. Discharge, in a dark room, a jar imperfectly coated.

Cor. 1. A coated jar cannot be charged unless its outer surface be connected with some conductor. For without such a conductor, the fluid cannot pass from or to the outer surface, which is necessary in order to charge the jar.

Cor. 2. When a coated glass vessel is charged, the charge of electric fluid is in the glass and not in the coating.

Exp. Lay a plate of glass between two metallic plates, as described Prop. XV. Exp. 6. Having charged the plate of glass, remove the upper plate of metal by a glass handle, with some non-conducting subtance, as silk; remove the electrified glass plate, and place it between two other plates of metal unelectrified and insulated; the plate of glass thus coated afresh will still be charged.

Schol. The discharge of a plate of glass, Leyden Phial, &c. is made by restoring the equilibrium which was destroyed by the charging; and it is effected by forming a communication between the overloaded and the exhausted side; and if the communication be made by metal, or other good conductors, the equilibrum will be restored with violence, the redundant electricity on one side will rush with great rapidity through the metallic communication to the exhausted side, and a large explosion will be made, that is, the flash of electric light will be very visible, and the report will be loud.

PROP. XVIII. If the conductor be electrified positively, that side of the jar with which it has a communication will be electrified positively, the other negatively.

Exp. 1. Charge one jar on the inside positively, and another negatively, and observe, in a dark room the different appearances of the fluid, upon the point of a wire brought near to the ball which is connected with the inner side of each jar: when the point is presented to the jar positively electrified on the inner side, it will exhibit the appearance of a star; when presented to the other, that of a pencil.

2. Observe the different appearances, in a dark room, when with the same charged jar the point is presented toward the side positively, and toward the side negatively, electrified.

3. Between two jars, charged one negatively and the other positively, suspend by a silken string a cork ball, from which short threads hang freely; the ball will pass with a rapid motion from one to the other, and, being first attracted toward the jar positively electrified, then toward the other, it will receive the fluid from the former, and communicate it to the latter, till both are discharged. If both be charged in the same manner, the cork will remain at rest.

4. If, after a jar is charged, the uncoated part of the jar be moistened by the breath, or by steam, the jar placed upon a conductor will be gradually discharged, and the fluid will be seen, in a dark room, to flash strongly from one side to the other; if the jar be insulated, the flashes will be greatest on the side positively electrified.

5. Let a discharging rod be applied without its balls to a charged jar, in such a manner as to discharge the jar gradually; the point which approaches toward the side positively electrified, will, in a dark room, exhibit a star; the other point, a pencil.

6. Within the receiver of an air-pump place two well polished brass balls, the lower supported on a brass stem by the plate of the pump, the other fixed on a stem which is moveable in the neck of the receiver; let the balls be brought within the distance of four or five inches from one another; then let the upper ball be connected with the conductor, and electrified positively; a lucid atmosphere will, in a dark room, appear on the lower surface of the upper ball; whereas if the upper ball be negatively electrified, the lucid atmosphere will be seen on the lower ball.

PROP. XIX. The electric fluid can be conveyed through an insulated conductor of any length, and its passage from one side of a charged jar to the other, is apparently instantaneous, through whatever length of a metallic, or other good conductor, it is conveyed.

Exp. 1. Let a long wire, passing round a room, suspended by silk cords, be a part of the circuit of communication from one side of a charged jar to the other; the discharge will be apparently at the same instant in which the communication from one side to the other is completed.

2. Let any number of persons make a part of the circuit of communication; the fluid will pass instantaneously through the whole circuit.

Schol. The shock of the Leyden jar has been transmitted through wires of several miles in length, without taking any sensible space of time. Dr. Priestley relates several curious experiments made with a view of ascertaining this point soon after the invention of the Leyden Phial. See Priestley's Hist. of Elect.

PROP. XX. The sudden discharge of a charged jar gives a painful sensation to any animal, placed in the circuit of communication, called the *electric shock*.

The discovery of the effects of electricity, as exhibited by the Leyden jar, immediately drew the attention of all the philosophers in Europe. The account which some of them gave of the experiments to their friends, border very much on the ludicrous. M. Musschenbroeck, who tried the experiment with a glass bowl, told M. Reaumur, in a letter written soon after the experiment, that he felt himself struck in his arms, shoulder, and breast, so that he lost his breath; and it was two days before he recovered from the effects of the blow and the terror. He added, that he would not take a second shock for the whole kingdom of France.

M. Allamand, who made the experiment with a common beer glass, said, that he lost his breath for some moments, and then felt such an intense pain all along his right arm, that he was apprehensive of bad consequences; but it soon went off without any inconvenience.

Notwithstanding the parade made by these philosophers, the shock was probably, not by any means stronger than what many children 6 or 7 years old would bear without the smallest hesitation. Their descriptions must have arisen from terror, or love of the marvellous.

Cor. The force of the electric shock may be increased, by increasing the surface of the coated glass.

Exp. 1. A battery being charged, a fine metallic wire brought into the circuit will be melted.

2. If a plane piece of metal be placed upon one of the rods of the discharger, and upon the other a needle with the point opposite to the surface of the metal, upon discharging the battery, the surface of the piece of metal will be marked with coloured circles, occasioned by thin laminæ of metal raised in the explosion.

3. If a piece of gold-leaf be put between two pieces of glass, and the whole fast bound together, the metal will be melted, and a metallic stain will be seen in both glasses.

4. If a shock be sent through a needle, it will give it magnetic polarity.

5. An animal or plant may be killed by being placed in the circuit of a battery.

Schol. Persons, not thoroughly conversant in electricity, should be very cautious in using large batteries; they should be sure that they are perfect masters of a small force, before they meddle with a greater. Such a force of electricity as may be accumulated in batteries is not to be trifled with, since the consequences, if not fatal, may be great and lasting. A large shock, taken through the arms and breast, which an operator is most in danger of receiving, might possibly injure the lungs, or some other vital part; and if the shock were taken through the head, which may easily happen when a person is stooping over the apparatus in order to adjust it, it might affect his intellects for the remainder of life.

PROP. XXI. If the circuit be interrupted, the fluid will become visible, and where it passes, it will leave an impression upon any intermediate body.

Exp. 1. Let the fluid pass through a chain, or through any metallic bodies placed at small distances from each other; the fluid, in a dark room, will be visible between the links of the chain, or between the metallic bodies.

2. If the circuit be interrupted by several folds of paper, a perforation will be made through them, and each of the leaves will be protruded by the stroke from the middle toward the outward leaves.

3. Let a card be placed under wires which form the circuit, where the circuit is interrupted for the space of an inch; the card will be discoloured. If one of the wires be placed under the card, and the other above it, the direction of the fluid may be seen.

4. Spirits of wine, or gunpowder, being made part of the circuit, may be fired.

5. Inflammable air may be fired by an electric gun.

PROP. XXII. The atmosphere is electrified, sometimes positively, and sometimes negatively.

Exp. Let a kite be sent up into the air with cord, consisting of copper thread twisted with twine; let the lower end of the cord be insulated by a silk line; a metallic conductor suspended from the lower end of the cord will be positively or negatively electrified. The air at some distance from houses, trees, masts of ships, &c. is generally electrified positively; particularly in frosty, clear, or foggy weather. For the particular construction of the electrical kite, and other instruments used with it, see Cavallo's Elect. Vol. ii. Chap. 1.

Schol. The following general laws have been deduced by Mr. Cavallo, from a great number of experiments made during two years in almost every degree of the atmosphere from 15° to 80° of Fahrenheit's thermometer.

1. The air appears to be electrified at all times; its electricity is constantly positive, and much stronger in frosty than in warm weather; but it is by no means less in the night than in the day time.

2. The presence of clouds generally lessens the electricity of the kite.

3. When it rains, the electricity of the kite is generally negative, and very seldom positive.

4. The aurora borealis seems not to affect the electricity of the kite.

5. The electrical spark, taken from the string of the kite, or from any insulated conductor connected with it, especially if it does not rain, is very seldom longer than $\frac{1}{4}$ of an inch, but it is exceedingly pungent. When the index of the electrometer is not higher than 20°, the person that takes the spark will feel the effect of it in his legs; it appearing more like the discharge of an electric jar, than the spark taken from a prime conductor.

6. The electricity of the kite is in general stronger or weaker, according as the string is longer or shorter; but it does not keep any exact proportion to it. The electricity, for instance, brought down by a string of an hundred yards, may raise the index of the electrometer to 20°, when with double that length of string the index of the electrometer will not go higher than 25°.

7. When the weather is damp, and the electricity is pretty strong, the index of the electrometer, after taking a spark from the string, or presenting the knob of a coated phial to it, rises surprisingly quick to its usual place, but in dry and warm weather it rises exceedingly slow.

PROP. XXIII. The electric fluid and lightning are the same substance.

Their properties and effects are the same. Flashes of lightning are generally seen to form irregular lines in the air; the electric spark, when strong, has the same appearance. Lightning strikes the highest and most pointed objects; takes in its course the best conductors; sets fire to bodies; sometimes dissolves metals; rends to pieces some bodies; destroys animal life; in all of which it agrees (as has been shown) with the phenomena of electric fluid. Both causes have the same power of making iron magnetic. Lightning has been known to strike men with blindness. Dr. Franklin produced a similar effect on a pigeon by the electrical fluid. Lastly, the lightning being brought from the clouds to an electrical apparatus, by a kite or wire, will exhibit all the appearances of the electric fluid.

Exp. Take a Leyden phial, 5 inches in diameter, and 13 inches in height; on the inside let the coating rise till its upper edge be $2\frac{1}{4}$ inches from the rim of the vessel; on the outside let the coating rise no higher than one inch from the bottom. When the phial is thus coated, let it be charged, and a spark will pass from the tin-foil on the outside to that on the inside; but its form will resemble that of a tree, whose trunk will increase in magnitude and brilliancy, and consequently in power, as it approaches the edge, owing to ramifications which it collects from all parts of the glass. Within two inches of the edge, it becomes one body, or stream, and along that interval its greatest force acts.

When two clouds, or the two correspondent parts of a cloud, have their equilibrium restored by a discharge, the appearances are exactly similar to those of the preceding experiment. Each extremity of the flash is formed by a multitude of little streams, which gather into one body, whose power is undivided in that interval only which separates the positive from the negative.

PROP. XXIV. Buildings may be secured from the effects of lightning, by fixing a pointed iron rod higher than any part of the building, and continuing it, without interruption, to the ground, or the nearest water.

The electric fluid will, by means of the pointed rod, be gradually conveyed from the cloud to the earth by a continued stream, and thus prevent the effects of a sudden and violent explosion.

Exp. Let a board, shaped like the gable end of a house, be fixed perpendicularly upon a horizontal board; in the perpendicular board let a hole be made, about an inch square and $\frac{1}{4}$ inch deep; in this hole let a piece of wood nearly of the same dimensions be so inserted as to fall easily out of its place, and let a wire be fastened diagonally to this square piece of wood; let another wire, terminated by a brass ball, be fastened to the perpendicular board, with its ball above the board, and its lower end in contact with the diagonal wire in the square piece of wood; let the communication be continued by a wire to the bottom of the perpendicular board. If the wires in this state be made part of a circuit of communication, on discharging the jar the square piece of wood will not be displaced; but if the communication be interrupted by changing the direction of the diagonal wire, the square piece of wood will, upon the discharge, be driven out of its place.

If instead of the upper brass ball, a pointed wire be placed above the perpendicular board, the discharge may be drawn off without an explosion.

Schol. The following directions are given by Earl Stanhope, to persons erecting conductors of lightning.

(1.) The rods must be made of such substances as are, in their nature, the best conductors of electricity.

(2.) The rods must be uninterrupted, and perfectly continuous.

(3.) They must be of sufficient* thickness.

(4.) They must be perfectly connected with the common stock, that is, the earth, or the nearest water.

(5.) The upper extremity of the rods must be finely tapered, and as accurately pointed as possible.

(6.) The rods must be very prominent, and several feet above the chimneys.

(7.) Each rod must be carried in the shortest convenient direction from its upper end to the common stock.

(8.) There should be no prominent bodies of metal on the top of the building proposed to be secured, but such as are connected with the conductor by some proper metallic communication.

(9.) There should be a sufficient† number of substantially erected, high, and pointed rods. See

* Perhaps $\frac{3}{4}$ of an inch.

† So many that no part of the building may be more than 30 or 40 feet from one.

"Principles of Electricity," by Charles Viscount Mahon, now Earl Stanhope. To the same work, the reader must be referred for an account of a discovery made by his lordship in the science of electricity, which he denominated the "*returning stroke*," by which, he asserts, that persons may be killed, and other vast mischief ensue by lightnings at the distance of several miles from the flash. It is proper also to observe that several respectable electricians, though willing to admit the fact as discovered by Earl Stanhope, yet do not seem to think that the danger attending the returning stroke can ever be great or formidable. See Cavallo's Elect. Vol. ii. and iii. Morgan's Lectures on Elect. Vol. ii. Dr. Hutton's Dict. Art. *Returning Stroke.*

PROP. XXV. The electric fluid passes easily through a *vacuum.*

The air being a non-conductor, in proportion as it is removed, the effort of the electric fluid on the surface of the body positively electrified to pass to the next conductor, meets with less resistance, and therefore is diffused over a greater space.

Exp. 1. Let a jar be charged in *vacuo.*

2. Let a *luminous conductor* be placed in the circuit, and observe the fluid passing through it.

3. Let a *vacuum* be made a part of the circuit in discharging a phial.

4. Make a *vacuum* in a double barometer, and let the fluid pass from one leg to the other by connecting one of the vessels of mercury with the conductor.

5. The electric fluid may be made to pass through a large tube three feet in length, and four or five inches in diameter, if, being well exhausted, one end of it be connected with a large conductor.—The preceding experiments are to be performed in a dark room.

Schol. 1. From the resemblance between these electrical appearances, and the atmospherical phenomena of the *Aurora Borealis*, meteors, &c. it is inferred, that these phenomena are produced by the electric fluid.

Schol. 2. The success of the foregoing experimeuts depends, it is highly probable, upon the air in the jar, tube, &c. being rarefied in a high degree; for Mr. W. Morgan, a gentleman deeply skilled in calculations and political arithmetic, has shewn that a *perfect vacuum* is absolutely impermeable to the electric fluid. See Phil. Trans. vol. lxxv.

PROP. XXVI. Some fishes have the property of giving shocks analogous to those of artificial electricity; namely, the *Torpedo*, the *Gymnotus electricus*, and the *Silurus electricus.*

If the torpedo, whilst standing in water, or out of water, but not insulated, be touched with one hand, it generally communicates a trembling motion or slight shock to the hand. If the torpedo be touched with both hands at the same time, one hand being applied to its under, and the other to its upper surface, a shock will be received exactly like that occasioned by the Leyden Phial. When the hands touch the fish on the opposite surfaces, and just over the electric organs, then the shock is the strongest; but no shock is felt, if both hands are placed upon the electric organs of the same surface; which shows that the upper and lower surfaces of the electric organs are in opposite states of electricity, answering to the *plus* and *minus* sides of a Leyden Phial.

The shock given by the torpedo, when in air, is about four times as strong as when in water; and when the animal is touched on both surfaces by the same hand, the thumb being applied to one surface, and the middle finger to the opposite, the shock is felt much stronger than when the circuit is formed by both hands.

This power of the torpedo is conducted by the same substances which conduct electricity, and is interrupted by those substances which are non-conductors of electricity. A circuit may be made of several persons joining hands, and the shock will be felt by them all at the same time; but the shock will not pass through the least interruption of continuity, not even the distance of the two hundreth part of an inch.

No electric attraction or repulsion could be ever observed to be produced by the torpedo, nor indeed by any of the electric fishes. The shocks of the torpedo seem to depend on the will of the animal.

The gymnotus electricus, or electric eel, possesses all the electrical properties of the torpedo, but in a superior degree. When small fish are put into the water wherein the gymnotus is kept, they are generally stunned or killed by the shock, and then they are swallowed, if the animal be hungry.

The strongest shock of the gymnotus will pass a very short interruption of continuity in the circuit. When the interruption is formed by the incision made by a penknife on a slip of tin-foil that is pasted on glass, and that slip is put into the circuit, the shock, in passing through that interruption, will show a small but vivid spark, plainly to be seen in a dark room.

The gymnotus seems also to be possessed of a sort of *new sense*, by which he knows whether the bodies presented to him are conductors or not. This fact was ascertained by a great number of experiments, made by Mr. Walsh.

The silurus electricus is known to have the power of giving the shock, but we have a very imperfect account of its properties.

A fourth electrical fish was found on the coast of Johanna, one of the Comora islands, in lat. 12° 13′ south, by William Patterson; and an account of it was published in the 76th vol. of the Phil. Trans.

Schol. 1. When electricity is strongly communicated to insulated animal bodies, the pulse is quickened, and perspiration increased; and if they receive, or impart electricity on a sudden, a painful sensation is felt at the place of communication. But what is more extraordinary is, that the influence of the brain and nerves upon the muscles seems to be of an electric nature.

We are indebted for this discovery to M. Galvani, a learned Italian, who has denominated that part of science, Animal Electricity. We shall, without pretending to enter at large on the subject, give the result of the principal observations hitherto made, together with three or four illustrative experiments.

1. The nerve of the limb of an animal being laid bare, and surrounded with a piece of tin-foil, if a communication be formed between the nerve thus armed, and any of the neighbouring muscles, by means of a piece of zinc, strong contractions will be produced in the limb.

2. If a portion of the nerve which has been laid bare be armed as above, contractions will be produced as powerfully, by forming the communication between the armed and bare part of the nerve, as between the armed part and muscle.

3. A similar effect is produced by arming a nerve, and simply touching the armed part of it with the metallic conductor.

4. Contractions will take place if a muscle be armed, and a communication be formed by means of the conductor between it and a neighbouring nerve. The same effect will be produced if the communication be formed between the armed muscle and another muscle which is contiguous to it.

5. Contractions may be produced in the limb of an animal by bringing the pieces of metal into contact with each other at some distance from the limb, provided the latter make part of a line of communication between the two metallic conductors.

6. Contractions can be produced in the amputated leg of a frog, by putting it into water, and bringing the two metals into contact with each other at a small distance from the limb.

7. The influence which has passed through, and excited contractions in one limb, may be made to pass through and excite contractions in another limb.

8. The heart is the only involuntary muscle, in which contractions can be excited by these experiments.

9. Contractions are produced more strongly, the farther the coating is placed from the origin of the nerve.

10. Animals which were almost dead have been found to be considerably revived by exciting this influence.

11. When these experiments are repeated upon an animal that has been killed by opium, or by the electric shock, very slight contractions are produced; and no contractions whatever will take place in an animal that has been killed by corrosive sublimate, or that has been starved to death.

12. Zinc appears to be the best exciter when applied to gold, silver, molybdena, steel, or copper. The latter metals, however, excite but feeble contractions when applied to each other. Next to zinc, in contact with these metals, tin and lead, and silver and lead, appear to be the most powerful exciters.

Exp. 1. Place the limb of an animal, a frog for instance, upon a table; hold with one hand the principal nerve previously laid bare, and in the other hold a piece of zinc; let a small plate of lead or silver be then laid upon the table, at some distance from the limb, and a communication be formed, by means of water, between the limb and the part of the table where the metal is lying. If now, the silver be touched with zinc, contractions will be produced in the limb the moment that the metals come into contact with each other. The same effect will be produced, if the two pieces of metal be previously placed in contact, and the operator touch one of them with his finger.

2. Let two amputated limbs of a frog be taken; let one of them be laid upon a table, and its foot be folded in a piece of silver; let a person lift up the nerve of this limb with a silver probe, and another person hold in his hand a piece of zinc, with which he is to touch the silver including the foot; let the person holding the zinc in one hand, catch with the other the nerve of the second limb, and he who touches the nerve of the first limb is to hold in the other hand the foot of the second; let the zinc now be applied to the silver, including the foot of the first, and contractions will be immediately excited in both limbs.

3. Take a living flounder, lay it flat in a pewter plate, or upon a sheet of tin-foil, and put a piece of silver, as a shilling or half a crown, upon the fish. Then by means of a piece of metal, complete

the communication between the pewter plate, or tin-foil, and the silver piece, on doing which the animal will give evident tokens of being affected.

4. Let a person lay a piece of zinc upon his tongue, and a half crown, or other silver, under it; on forming a communication between those two metals, by bringing their two edges into contact, he will perceive a peculiar sensation, a kind of cool, sub-acid taste, not exactly like, and yet not much different from that produced by artificial electricity. See Cavallo's Elect. Vol. iii.

Schol. 2. Electricity has been administered for various diseases. Mr. Cavallo has taken great pains in ascertaining the cases in which electricity has been successfully applied. We are informed by that gentleman, that *rheumatic disorders*, even of long standing, are relieved, and generally quite cured. Deafness, the toothach, swellings in general, inflammations of every sort, palsies, ulcers, cutaneous eruptions, the St. Vitus' dance, scrofulous tumours, cancers, abscesses, nervous headach, the dropsy, gout, agues, and obstructions, have all been considerably relieved, and in many instances perfectly cured, by the application of electricity. A full account of the method of administering electricity in the cases above mentioned, with an accurate description of the instruments used, may be seen in the 2d Vol. of Cavallo's Complete Treatise of Electricity.

PROP. XXVII. There is a considerable analogy and difference between magnetism and electricity.

The power of electricity is of two sorts, positive and negative; bodies possessed of the same sort of electricity, repel each other, and those possessed of different sorts attract each other. In magnetism, every magnet has two poles; poles of the same name repel each other, and the contrary poles attract each other.

In electricity, when a body in its natural state is brought near to one electrified, it acquires a contrary electricity, and becomes attracted by it. In magnetism, when a ferruginous substance is brought near to one pole of a magnet, it acquires a contrary polarity, and becomes attracted by it.

One sort of electricity cannot be produced by itself. In like manner, no body can have one magnetic pole without the other.

The electric virtue may be retained by electrics, but it easily pervades non-electrics. The magnetic virtue is retained by ferruginous bodies, but it easily pervades other bodies.

On the contrary, the magnetic power differs from the electric, in that it does not affect the senses with light, smell, taste, or noise, as the electric does.

Magnets attract only iron, whereas the electric power attracts bodies of every sort.

The electric virtue resides on the surface of electrified bodies, but the magnetic is internal.

A magnet loses nothing of its power by magnetising other bodies, but an electrified body loses part of its electricity by electrifying other bodies. See Cavallo's Magnetism, Part ii. Chap. 2.

BOOK VI.

OF OPTICS; OR, THE LAWS OF LIGHT AND VISION.

CHAPTER. I.

Of Light.

DEF. I. LIGHT is that which, proceeding from any body to the eye, produces the perception of *seeing*.

DEF. II. A *Ray of Light* is any exceedingly small portion of light as it comes from a luminous body.

DEF. III. A body, which is transparent, or affords a passage for the rays of light, is called a *Medium*.

DEF. IV. Rays of light which, coming from a point, continually separate as they proceed, are called *Diverging Rays*.

DEF. V. Rays which tend to a common point, are called *Converging Rays*. The divergency, or convergency, of rays, is measured by the angle contained between the lines which the rays describe.

DEF. VI. Rays of light are *parallel*, when the lines which they describe are parallel.

DEF. VII. A *Beam* of light is a body of parallel rays; a *Pencil* of rays, is a body of diverging or converging rays.

DEF. VIII. The point, from which diverging rays proceed, is called the *radiant point*; that, to which converging rays are directed, is called the *focus*.

Plate 6, Fig. 1. If the rays proceed from B; BD, BA, BC, BE, are diverging rays, and B is the radiant; if the rays tend toward B; DB, AB, &c. are converging rays, and B is the focus.

Fig. 2. If the rays AC, BC, converge to the focus C, passing on from thence in a right line, they become diverging, and C becomes a radiant.

DEF. IX. A ray of light, bent from a straight course in the same medium, is said to be *inflected*.

PROPOSITION I.

Rays of light consist of particles of matter.

For, like all matter with which we are acquainted, they are capable of being inflected out of their course by attraction.

EXP. 1. If a beam of light be admitted into a dark room through a small hole, and the edge of a knife be brought near the beam, the rays, which would otherwise have been in a straight line, will be inflected toward the knife. The edge of any other thin plate of metal &c. produces the same effect.

2. The shadow of a small body, as a hair, a thread, &c. placed in a beam of the sun's light, will be much broader than it ought to be if the rays of light passed by these bodies in right lines.

3. A beam of light passing through an exceedingly narrow slit, not above $\frac{1}{400}$ part of an inch broad, will be split into two, and leave a dark space in the middle.

PROP. II. Every visible body emits particles of light from its surface in all directions, which, passing without obstruction, move in right lines.

Wherever a spectator is placed with respect to a luminous body, every point of that part of the surface which is turned toward him is visible to him; the particles of light are, therefore, emitted in all directions, and those rays only are intercepted in their passage by an interposed object, which would be intercepted upon the supposition that the rays move in right lines.

Exp. 1. Let a portion of a beam of light be intercepted by any body, the shadow of that body will be bounded by right lines passing from the luminous body, and meeting the lines which terminate the opaque body.

2. A ray of light, passing through a small orifice into a dark room, proceeds in a straight line.

3. Rays will not pass through a bended tube.

Schol. Rays of light are properly represented by right lines.

PROP. III. The rays of light move with great velocity.

The velocity of light is much greater than that of sound; for the flash of a gun, fired at a considerable distance, is seen some time before the report is heard. The clap of thunder is not heard till some time after the lightning has been seen.

This proposition is proved by observations made on the satellites of the planet Jupiter, and on the aberration of the rays of light from the fixed stars, as will be shown in treating upon Astronomy; from whence it will be seen, that the velocity is at the rate of 200,000 miles in one second of time.

PROP. IV. The particles of light are exceedingly small.

Otherwise their velocity would render their momentum too great to be endured by the eye without pain.

Exp. 1. If a candle be lighted, and there be no obstacle to obstruct the progress of its rays, it will fill all the space within two miles every way, before it has lost the least sensible part of its substance.

2. Rays of light will pass without confusion through a small puncture in a piece of paper, from several candles in a line parallel to the paper, and form distinct images on a sheet of pasteboard placed behind the paper.

PROP. V. The quantities of light, received from a luminous body upon a given surface, are inversely as the squares of the distances of the surface from the luminous body.

Let ABD, EFG, be two concentric spherical surfaces; of which let ELFI, AHBK, be two similar portions. Let the rays CE and CF, with the rest proceeding from the centre C, fall upon the portion ELFI, and cover it; it is evident from inspection, that the same rays at the distance CH will cover the portion AHBK only; now these rays being the same in number at each place, will be as much thinner in the former, than they are in the latter, as ELFI is larger than AHBK; but these spaces being similar portions of the surfaces of spheres, have the same ratio to each other, that the surfaces themselves have; that is, they are to each other as the squares of their radii CL, CH; the density of the rays is therefore inversely as the squares of these radii, or of their distances from the luminous point C. Plate 6. Fig. 3.

Exp. The light, passing from a candle through a square orifice, will diverge as it proceeds, and will illuminate surfaces which will be to each other as the squares of their distances from the candle. Thus at the distance AF the candle will illuminate the square BF; at the distance AO it will illuminate the surface CO equal to four times BF, and at the distance AS it will illuminate the surface DS equal to nine times BF; but AF, AO, and AS, are as 1, 2, and 3; consequently the illuminated surfaces are as the squares of the distances. Plate 12. Fig. 8.

PROP. VI. If the distance between rays diverging from different radiant points be the same, the distances of the radiant points are inversely as the divergency of the rays.

Let D and E be two different radiants; and let the rays diverging from D describe the lines DA, DB, and the rays diverging from E describe the lines EA, EB; so that, at the points A and B, the distance between the former rays shall be the same with the distance between the latter, and let EC, DC, be the perpendicular distances of the radiants E, D. At the point E make the angle ZEC equal to ADC, which is half ADB; whence ZEC and ADC (El. V. 7.) have the same ratio to AEC. But if these angles are small, they are very nearly in the proportion of their tangents ZC, AC. And because the angle ADC is equal to the angle ZEC (El. I. 28.) AD is parallel to ZE; and because these lines are Plate 6. Fig. 4.

parallel, (El. I. 29.) the angles CAD, CZE, are equal; whence the two triangles ZEC, ADC, are equiangular, and (El. VI. 4.) EC is to DC, as ZC to AC, or (from what was shown above) as ADC to AEC; that is, the distance of the radiant E is to the distance of the radiant D, as half the angle of divergency of the rays which proceed from D is to half the divergency of the rays which proceed from E, or as the whole angle of divergency ADB to the whole angle of divergency AEB; that is, the distances of the radiants are inversely as the divergency of the rays.

PROP. VII. If the distance between converging rays tending to different foci be the same, the distances of the foci are inversely as the convergency of the rays.

Plate 6. Fig. 4. Let AD, BD, be lines described by rays converging to the focus D, and AE, BE, lines described by other rays converging to E, and let the distance AB, at the points A and B, be the same between the former and the latter rays. The angles ADB, AEB, are in this case the angles of convergency; and EC, DC, are distances of the foci to which they respectively tend. Now it was proved in the last Prop. that EC is to DC as ADB is to AEB. Therefore the distances of the foci are inversely as the convergency of the rays.

PROP. VIII. If rays proceed from a radiant at an infinite distance, their divergency is considered as nothing, and the rays are considered as parallel.

Since (by Prop. VI.) the divergency of rays is inversely as the distance of the radiant, when the distance of the radiant is infinitely great the angle of divergency is infinitely small, and the rays may be considered as parallel.

Cor. Hence all the rays which come from the centre, or any other given point, of the sun's surface, are considered as parallel.

PROP. IX. If rays tend to a focus at an infinite distance, their convergency is considered as nothing, and the rays are considered as parallel.

Since (by Prop. VII.) the convergency is inversely as the distance of the focus, when that distance is infinitely great, the angle of convergency is infinitely small.

CHAPTER II.

Of Refraction.

SECT. I.

OF THE LAWS OF REFRACTION.

Def. X. A ray of light bent from a straight course by passing out of one medium into another, is said to be *refracted.*

Def. XI. The *Angle of Incidence* is that, which is contained between the line described by the incident ray, and a line perpendicular to the surface on which the ray strikes, raised from the point of incidence.

Def. XII. The *Angle of Refraction* is that, which is contained between the line described by the refracted ray, and a line perpendicular to the refracting surface at the point in which the ray passes through that surface.

Def. XIII. The *Angle of Deviation* is that which is contained between the line of direction of an incident ray, and the direction of the same ray after it is refracted.

Plate 6. Fig. 6. AC is a ray of light; HK the surface of the refracting medium; CF the refracted ray; OP the perpendicular; ACO the angle of incidence; PCF the angle of refraction; and FCL the angle of deviation.

Plate 6. Fig. 5. Schol. The radiant point and focus may be either real or imaginary. If the rays *r n*, *r o*, diverging from the radiant *r*, suffer refraction and move on in the directions of the lines *n* A, *o* B, which produced in the contrary direction would meet in R, this radiant point is imaginary.

If the rays I *p*, L *q*, tending toward the point F, be refracted at *p* and *q*, and acquire a direction toward *f*, the focus F is imaginary.

PROP. X. The attracting force of any medium, acting upon a ray of light, is every where perpendicular to the refracting surface.

If the medium be uniform in all its parts, its immediate power upon the ray of light will be equally strong in every point of a plane drawn parallel to the refracting surface; though its strength may be different in the next parallel plane, and so onward as far as that power is extended on each side of the surface of the medium. The extent of this power will therefore be terminated by two planes, parallel to each other and to the refracting surface. Let R be a particle of light, acted upon by the refractive power of the medium whose refracting surface is DC. It is evident that the refractive power at O will move the particle R in the direction RO; and taking any two points D, C, at equal distances on each side of O, the powers at D and C being equal, and acting at equal distances, RD, RC, equally inclined to RO, cannot move R in any direction but that of RO. The same may be shown of the powers at every point of the line DC, and in every line parallel to DC, that is, of the whole power of the medium. Plate 6. Fig. 8.

PROP. XI. A ray of light, in passing out of a rarer into a denser medium, is refracted toward a perpendicular to the surface of the denser, raised from the point in which the ray meets the medium; in passing out of a denser into a rarer medium, it is refracted from the same perpendicular.

Let a ray of light, AC, pass obliquely out of a rarer medium X, into a denser medium Z; let HK be the plane surface of the denser medium; from the point C, in which the ray AC passes into the denser medium, raise the perpendicular OCP; the ray will be refracted out of the direction ACL, toward the perpendicular OCP. Plate 6. Fig. 6.

Because the ray is more attracted by the denser medium than by the rarer, it will be accelerated on entering the medium Z; for whilst the ray is so near the surface of the medium Z as to be within its attraction, and more attracted toward the denser than toward the rarer, this attraction conspires with the motion of the ray, and, consequently, increases its velocity. And, since the action of the attracting force of the medium Z, must (by Prop. X.) be in the direction of a line OCP perpendicular to its surface, if the oblique motion of the ray in the direction AC be resolved into two others, AD parallel to the surface HK, and AB, or DC, perpendicular to it, the parallel motion AD cannot be accelerated or retarded by the attraction which acts in the direction OC; the change of velocity, therefore, which the ray receives from the attracting force, must be made in the perpendicular part of its motion DC. Take CG greater than DC representing the perpendicular motion of the ray after passing into the denser medium; and take CE equal to AD representing the parallel part of the motion of the ray, which, because it is parallel to AB, remains the same when the ray enters the denser medium. The ray, therefore, at its entering the medium Z, may be considered as acted upon by two forces CE, CG, and consequently (Book II. Prop. XIV.) will describe CF the diagonal of a parallelogram, the sides of which are CE, CG. Now, of these sides, CE remaining the same, whilst CG becomes greater than CD, the angle GCF (from the nature of the parallelogram) will be less than the angle NCL, equal (El. I. 15.) to ACD. Therefore the ray, after it has passed into the denser medium, makes a less angle with the perpendicular OCP than AC, the ray before it passes into the denser medium; that is, the ray, in passing out of the rarer into the denser medium is refracted toward the perpendicular. On the contrary, whilst the ray of light FC is passing out of the denser medium Z into the rarer medium X, it is more attracted by the denser than by the rarer medium, and is therefore more drawn toward the former than toward the latter; whence the attraction opposes the motion of the ray, and will retard it as much, as in passing out of the rarer into the denser medium it was accelerated; and, consequently, the effect will be the reverse of that which was shown in the former case.

SCHOL. 1. Although there is no doubt that refraction is performed gradually, and in time, during which the light really describes a curve line extending quite through the refracting space, and connecting the refracted with the incident ray, which are tangents to this curve at its respective extremities, yet both the time and space are so small, that experiment has never been able to render even the space perceptible, so that the incident and refracted rays are commonly considered as forming a perfect angle precisely at the surface separating the two mediums.

SCHOL. 2. The principles of optics are demonstrated upon the supposition that light is a homogeneal substance; and though light will appear to be compounded of several kinds of rays, yet the principles of refraction, reflection, &c. are mathematically true when applied to rays of any one sort.

EXP. 1. Let a perpendicular cylindrical vessel be so placed that the sun, shining upon its side NA, may cast a shadow of the side to a point L in the bottom of the vessel. This shadow is terminated by SNL, a ray which passes, in a right line, by the edge of the vessel. If the vessel be filled with water, the shadow will recede, as the water is poured into the vessel, from the point L, which terminated it Plate 6. Fig. 7.

when the vessel was empty, toward the side NA, on which the sun shines, and will be terminated by the the ray ONC; that is, the ray SNL, which first terminated the shadow, by passing out of the air into the water, is refracted toward AN, a line drawn perpendicular to the surface of the water at the point in which the ray enters the water; or the angle of refraction is less than the angle of incidence.

2. Let a small bright object be laid upon the bottom of a cylindrical vessel NBAL at C. Let the spectator's eye be so placed at S, as just to lose sight of the object at C; that is, so that a ray passing in a right line from the remote edge of the object toward the eye at S will be intercepted by the edge of the vessel, or that the first ray which is not intercepted will pass in the direction ONC above the eye. Whilst the eye continues in the same situation, if the vessel be filled with water, the object will become visible; that is, the ray which passed from the remote edge of the object, in a right line CNO, by the vessel, in entering the air is refracted into the direction NS, toward the eye, or from the perpendicular PNA.

PROP. XII. All refraction is reciprocal.

Plate 6. Fig. 6. The ray AC, in passing out of the medium X into Z, is refracted into CF, because it is accelerated at its entrance into Z by the the greater attraction of the denser medium; and the ray FC in passing out of Z into X is refracted into CA, because it is retarded by the same attraction. Since, then, the acceleration and retardation are produced by the same degree of attraction in opposite directions they will be equal to one another, and the refractions produced by them will be equal, but in opposite directions; that is, if the refracted ray becomes the incident ray, the incident ray will become the refracted one; or, the refractions are reciprocal.

PROP. XIII. In any two given mediums, the sine of any one angle of incidence has the same ratio to the sine of the corresponding angle of refraction, as the sine of any other angle of incidence has to the sine of its corresponding angle of refraction.

Plate 16. Fig. 1. Let AC represent the velocity of a ray of light obliquely incident on the plane surface FG of a denser medium W at the point C. Resolve this motion, which is constant and invariable in the same medium, into AB perpendicular to the refracting surface and BC parallel to it. Of these (as was shown in Prop. XI.) only the perpendicular motion AB is accelerated, the parallel motion BC continuing unaffected by the attraction. Next (since most probably the intensity of the refractive power is greatest precisely at the refracting surface, and gradually diminishes on each side to the limits of its action, thus producing a variable acceleration,) suppose the refracting space to be divided into strata by planes parallel to the refracting surface at the point through which the ray passes, and so near to each other that the refractive power, and, of course, the acceleration may be considered uniform between them. Now (B. II. Prop. XXVI. Cor. 4.) the space passed through by a motion uniformly accelerated is proportional to the difference of the squares of the initial and final velocities. But the space passed through by the perpendicular part of the motion of any ray while passing through the same stratum, is always the same, (that is, the thickness of the stratum) whatever the obliquity of the ray may be; therefore the difference between the square of the perpendicular velocity of *any* ray at its *entering* any one stratum and the square of its perpendicular velocity at its *leaving* the same stratum is the same as the difference between the squares of the perpendicular velocities of any *other* ray at its entering and leaving the same stratum, however different the obliquity, and, of course, the actual perpendicular velocities of the rays may be. And as the number of these differences, with reference to any single ray, is equal to the number of strata assumed, the sums of an equal number of equal differences must be equal. Therefore the whole differences produced in passing through all the strata or whole refracting space, or the difference between the squares of the perpendicular velocities in one medium until the rays enter the refracting space, and in the other after emerging from it, is always the same, however the actual perpendicular velocities may vary with the obliquity of incidence, and however the refracting force may vary during refraction. Take $CD = BC$ to represent the parallel motion of the ray after refraction, as that continues unaltered, and draw DE in the denser medium perpendicular to the surface to represent the perpendicular velocity of the refracted ray. The square of DE must exceed the square of AB by a certain quantity which continues constant in all positions of the ray AC. Now since $CD^2 = BC^2$, and DE^2 exceeds AB^2 by a constant quantity, $CD^2 + DE^2$ or CE^2 must exceed $BC^2 + AB^2$ or AC^2 by the same constant quantity. Therefore as AC and AC^2 are constant quantities, so CE^2 and CE are also constant; and as AC expresses the direct velocity of the incident ray which never varies with the angle of incidence, so CE expresses the direct velocity of the refracted ray, which is the same at all angles. On the centre C, and with CB or CD, representing the parallel velocity, as a radius, describe a circle; it is plain that AC is the secant of the angle ACB or cosecant of the angle of incidence, and CE the secant of the angle DCE or cosecant of the angle of refraction; and as these are constant quantities, however the radius or parallel motion may vary, the coescants of incidence and refraction

in the same two mediums have a constant ratio to each other; but sines are inversely as the cosecants of the same angles, therefore the sines also have always the same ratio to each other. When light passes from a denser medium into a rarer, the same reasoning may easily be adapted to the retardation —the differences of the squares both of the perpendicular and direct velocities of the rays before and after refraction being the same at all angles of inclination, and the direct velocity of the refracted ray being less than that of the incident ray by a certain constant quantity, &c. the terms of the ratio being reversed.

If a, b, d, e, be read instead of A, B, D, E, respectively, the figure exhibits an example of a much greater angle of incidence.

Cor. 1. Hence, when the angle of incidence is increased, the corresponding angle of refraction will also be increased; because the ratio of their sines cannot continue the same, unless they be both increased; and if two angles of incidence be equal, the angles of refraction will also be equal.

Cor. 2. Hence the angle of deviation varies with the angle of incidence.

Schol. 1. If a ray of light, AC, pass obliquely out of air into water, AD, the sine of the angle of incidence ACD is to NS, the sine of the angle of refraction NCF, nearly as 4 to 3; therefore, supposing the sines proportional to the angles, the sine of FCL, the angle of deviation, is as the difference between AD and NS, that is, as 4 — 3, or 1; whence the sine of incidence is to the sine of the angle of deviation as 4 to 1. In like manner it may be shown, that, when the ray passes obliquely out of water into air, the sine of the angle of incidence will be to that of deviation, as NS to AD — NS, that is, as 3 to 1. In passing out of air into glass, the sine of the angle of incidence is to that of refraction, as 3 to 2, and to that of deviation, as 3 to 3 — 2, or 1; and in passing out of glass into air, the sine of the angle of incidence is to that of refraction, as 2 to 3, and to that of deviation as 2 to 1. Plate 6. Fig. 6.

Cor. 3. Hence a ray of light cannot pass out of water into air at a greater angle of incidence than 48° 36′, the sine of which is to radius as 3 to 4. Out of glass into air the angle must not exceed 40° 11′, because the sine of 40° 11′ is to radius as 2 to 3 nearly; consequently, when the sine has a greater proportion to the radius than that mentioned, the ray will not be refracted.

Schol 2. It must be observed, that when the angle is within the limit, for light to be refracted, some of the rays will be reflected. For the surfaces of all bodies are for the most part uneven, which occasions the dissipation of much light by the most transparent bodies; some being reflected, and some refracted, by the inequalities on the surfaces. Hence a person can see through water, and his image reflected by it at the same time. Hence also, in the dusk, the furniture in a room may be seen by the reflection of a window, while objects that are without are seen through it.

Exp. Upon a smooth board draw a right line BCD, and on its extremities erect the perpendiculars BA and DE in opposite directions; on the middle C, of line BD, as a centre, and with any extent greater than BC, intersect the line BA suppose in A, and with an extent greater than CA in the ratio of 4 to 3, intersect the line DE, perhaps in E. Then if pins be stuck perpendicularly at A, C, and E, and the board be dipped in the water as far as the line BD, the pin at E will appear in the same line with the pins at A and C. This shows, that the ray which comes from the pin E is so refracted at C, as to come to the eye along the line CA; whence the sine of incidence is to the sine of refraction as 4 to 3. If other pins were fixed along CE, they would all appear in AC produced; which shows that the ray is bent at the surface only. The same may be shown, at different inclinations of the incident ray, by means of two moveable rods turning upon the centre C, which always keep the ratio of the sines, as 4 to 3. Also the sun's rays, coinciding with AC, may be shown to be refracted in the same manner. Plate 16 Fig. 1.

PROP. XIV. Rays of light, which pass perpendicularly out of one medium into another, suffer no refraction.

When AC, the incident ray, concides with OC, the perpendicular, the action of the medium Z or X to accelerate or retard the motion of the ray, being perpendicular to its surface, cannot turn the ray out of its perpendicular path. Plate 6. Fig. 6.

PROP. XV. When parallel rays pass obliquely out of one medium into another through a plane surface, they will continue parallel after refraction.

Let AB, CD, be parallel rays, falling on the plane surface RBD of a medium of different density; because they make equal angles of incidence with their respective perpendiculars OP, ST, they will suffer an equal degree of refraction; that is, the angles of refraction EBP, FDT, will be equal; whence the refracted rays BE, DF, will be parallel. Plate 6. Fig. 9.

PROP. XVI. Through a plane surface, if diverging rays pass out of a rarer into a denser medium, they are made to diverge less; and if they pass out of a denser into

a rarer medium, they are made to diverge more: If converging rays pass out of a rarer into a denser medium, they will be made to converge less; it out of a denser into a rarer, to converge more.

Plate 4. Fig. 10. Let the diverging rays AB, AE, AF, pass out of a rarer into a denser medium, through the plane surface GH, and let the ray AB be perpendicular to that surface; the rest being refracted toward their respective perpendiculars IK, LM, and that the most which falls the farthest from B, they will proceed in the directions EN and FO, diverging in a less degree from the ray AP, than they did before refraction; whereas, had they proceeded out of a denser into a rarer medium they would have been refracted from their perpendiculars EK, FM, and those the most which were the most oblique, and therefore would have diverged more than before. Fig. 11. Again, let the converging rays AB, CD, EF, pass out of a rarer into a denser medium, through the plane surface GH, and let the ray AB be perpedicular to that surface: the other rays being refracted toward their respective perpendiculars IK, LM, and EF being refracted more than CD, they will proceed in the directions DN, FN, converging in a less degree toward the ray AN, than they did before; whereas, had the first medium been the denser, they would have been refracted the other way, and therefore have converged more.

Def. XIV. A *Lens* is a round piece of polished glass, which has both its sides spherical, or one spherical and the other plane.

A lens may either be convex on both sides, plano-convex, concave, plano-concave, or convex on one side and concave on the other; which last is called a *meniscus*.

In plate 6. fig. 12. sections of these, formed by a plane passing perpendicularly through their centres, are represented.

Def. XV. The *Axis of a Lens* is a right line passing through its centre, perpendicular to both its surfaces, and the extremities of the axis are its poles.

Each kind of lens is generated by the revolution of a section of the lens about this line. Thus, in the first lens, if *a c b*, *a d b*, revolve about *c d*, the convex lens will be formed.

Def. XVI. In every beam of light, the middle ray is called the *Axis*.

Def. XVII. Rays are said to fall *directly* upon a lens, if their axis coincides with the axis of the lens; otherwise, they are said to fall *obliquely*.

Def. XVIII. The point, in which parallel rays are collected by passing through a lens, is called the *Focus of parallel rays* of that lens.

PROP. XVII. Through a convex surface of the denser medium, parallel rays, passing out of a rarer into a denser medium, will become converging;—diverging rays will be made to diverge less, to become parallel, or to converge, according to the degree of divergency before refraction, or of the convexity of the surface;—rays converging toward the centre of convexity will suffer no refraction;—rays converging to a point beyond the centre of convexity will be made more converging;—and rays converging toward a point nearer the surface than the centre of convexity, will be made less converging by refraction;—and when the rays proceed out a denser into a rarer medium, through a concave surface of the denser, the contrary occurs in each case.

Plate 6. Fig. 13. Let AB, ID, be parallel rays entering a denser medium through the convex surface CDE, whose centre of convexity is L; and let one of these, ID, be perpendicular to the surface. This will pass on through the centre without suffering any refraction; but the other, being oblique to the surface, will be refracted toward the perpendicular LB, and will therefore be made to proceed in some line, as PK, converging toward the other ray, and meeting it in K, the focus. Had one ray diverged from the other, suppose in the line MP, it would, by being refracted toward its perpendicular LB, have been made either to diverge less, be parallel, or to converge. Let the line ID be produced to K; and if the ray had converged, so as to have described the line NBL, it would then have been coincident with its perpendicular, and have suffered no refraction. If it had proceeded with less convergency toward any point beyond L in the line IK, it would have been made to converge more by being refracted toward the perpendicular LB, which converges more than it; and had it proceeded with more convergency than BL,

that is, toward any point between D and L, being refracted toward the perpendicular, it would have been made to converge less.

And the contrary happens, when rays proceed out of a denser into a rarer medium, through a concave surface of the denser. For being now refracted from their respective perpendiculars, as they were before toward them, if they are parallel before refraction, they diverge afterward; if they diverge, their divergency is increased; if they converge in the direction of their perpendiculars, they suffer no refraction; if they converge less than their respective perpendiculars, they are made to converge still less, to be parallel, or to diverge; if they converge more, their convergency is increased. All which may clearly be seen by the figure, imagining the rays AB, ID, &c. bent the contrary way in their refractions to what they were in the former cases.

Exp. Let parallel, diverging, and converging rays pass through a convex lens; the several cases of this proposition will be confirmed.

If CDEH be a convex lens, whose axis is IK, let L be the centre of the first convexity CDE, and M that of the other CHE; and let the ray AB be parallel to the axis; through B draw the line LN, which will be perpendicular to the surface CDE at that point. The ray AB in entering the denser substance of the lens will be refracted toward the perpendicular, and therefore proceed after it has entered the surface at B in some direction inclined toward the axis, as BP. Through M the centre of convexity of this surface and the point P draw the line MR, which passing through the centre will be perpendicular to the surface at P, and the ray now entering a rarer medium will be refracted from the perpendicular into some direction as PK. In like manner, and for the same reasons, the parallel ray ST on the other side the axis, and also all the intermediate ones, as XZ, &c. will meet it in the same point, unless the rays AB and ST enter the surface of the lens at too great a distance from the axis IK, the reason of which will be afterward explained.

The point K where the parallel rays AB, ST, &c. are supposed to be collected by passing through the lens CE, is called the focus of parallel rays of that lens.

If the rays come diverging from a point equally distant from the surface as the focus of parallel rays, they will be rendered parallel; if from a point farther from the surface than L, they will be brought to a point beyond L; if from a point nearer than L, they will diverge less; as may be inferred from Prop. XII.

If the rays come converging toward L, they will suffer no refraction; if toward a point beyond L, they will become more converging; if toward a point nearer the surface than L, they will become less converging; as is sufficiently explained in the proof of this proposition.

Schol. If the rays AB, CD, EF, be parallel to each other, but oblique to GH, the axis of the lens IK, or if the diverging rays CB, CF, proceed as from some point C, which is not situated in the axis of the lens, they will be collected into some point as L, not directly opposite to the radiant C, but nearly so; for the ray CD, which passes through the middle of the lens, and falls upon the surface of it with some obliquity, will itself suffer a refraction at D and N; but it will be refracted the contrary way in one place from that in the other; and these refractions will be equal in degree, if the surfaces are parallel, as we may easily perceive if we imagine ND to be a ray passing out of the lens both at N and D, for it is evident the line ND has an equal inclination to each surface at both its extremities. Upon which account the difference between the situation of the point L, and one directly opposite to C, is so small, that it is generally neglected; and the focus is supposed to be in that line, in which a ray that would pass through the middle point of the lens, were it to suffer no refraction, would proceed. Plate 6. Fig. 14.

PROP. XVIII. When rays pass out of a rarer into a denser medium, through a concave surface of the denser, if the rays are parallel before refraction, they are made to diverge;—if they are divergent, they are made to diverge more, to suffer no refraction, or to diverge less, according as they proceed from some point beyond the centre, from the centre, or from some point between the centre and the surface;—if they are convergent, they are either made less converging, parallel, or diverging, according to their degree of convergency before refraction;—and the reverse, in passing out of a denser into a rarer medium through a convex surface of the denser.

Let MF, OI, be two parallel rays entering a concave and denser medium, the centre of whose convexity is H, and the perpendicular to the refracting surface at the point F is LH; the ray OI, if we suppose it perpendicular to the surface, will proceed on directly without refraction, but the oblique ray MF, being refracted toward the perpendicular HL, will recede from the other ray OI. If the ray MF had proceeded from a point in OI farther from the surface than H, it would have been bent nearer to the Plate 6. Fig. 15.

perpendicular, and therefore have diverged more; if it had diverged from the centre H, it would have fallen in with the perpendicular HL, and not have been refracted at all; and had it proceeded from a point nearer the surface than the centre H, it would, by being refracted toward the perpendicular HL, have proceeded in some line nearer it than it otherwise would have done, and so would diverge less than before refraction. Lastly, if it had converged, it would have been rendered less converging, parallel, or diverging, according to the degree of convergency, which it had before it entered into the refracting surface.

If the same rays proceed out of a denser into a rarer medium through a convex surface of the denser, the contrary happens in each supposition; the parallel rays are made to converge; those which diverge less than their respective perpendiculars, that is, those which proceed from a point beyond the centre, are made less diverging, parallel, or converging, according to the degree in which they diverge before refraction; those which diverge more than their respective perpendiculars, that is, those which proceed from a point between the centre and the refracting surface, are made to diverge still more. And those which converge, are made to converge more. All which may easily be seen by considering the situation of the rays with respect to the perpendicular HL.

Exp. Let parallel, diverging, and converging rays pass through a concave lens; the several cases of this proposition will be confirmed; thus, let ABCD represent a concave lens, EO its axis, FH the radius of the first concavity, IK that of the second; produce HF to L, and let MF be a ray of light entering the lens at the point F. This ray being refracted toward the perpendicular FL, will pass on to some point, as K, in the othersurface, more distant from the axis than F, and being there refracted from the perpendicular IK, will be diverted farther still from the axis, and proceed in the direction KN, as from some point O, on the first side of the lens. In like manner other rays, as PQ, parallel to the former, will proceed after refraction at both surfaces as from the same point O; which upon that account will be the imaginary radiant of parallel rays of this lens.

Plate 6. Fig. 15.

If the rays diverge before they enter the lens, their imaginary radiant is then nearer the lens than that of the parallel rays. If they converge before they enter the lens proceeding toward some distant point in the axis, as E, they are then rendered less converging: if they converge to a point at the same distance from the lens with the focus of parallel rays, they then go out parallel; if to a point at a less distance, they remain converging, but in a less degree than before they entered the lens.

Schol. If the lens is plane on one side, and convex or concave on the other, the refraction is similar, but in a less degree. In a meniscus, if he convexity on one side be equal to the concavity on the other, the two sides will produce equal and contrary effects, and the inclination of the rays to each other will be the same after refraction as before. If the convexity be greater than the concavity, the meniscus will have the effect of a lens which has its convexity equal to the excess of the convexity of the meniscus above its concavity; and the reverse, if its concavity exceed its convexity.

PROP. XIX. When diverging rays are made to converge by passing through a convex lens, as the radiant approaches toward the lens on one side, the focus departs from it on the other; and the reverse.

For, the nearer the radiant point is to the lens, the more the rays which fall upon the lens diverge before refraction; whence (the power of the refracting medium being given) they will converge the less after refraction, and have their focal point at the greater distance from the surface: on the contrary, the more remote the radiant point is from the lens, the less the incident rays will diverge, and consequently the more will the refracted rays converge, and the nearer will the focus be to the surface, till, at an infinite distance of the radiant, the rays are collected in the focus of parallel rays.

PROP. XX. When the radiant point is at that distance from the surface, at which parallel rays coming through it from the other side would be collected, rays flowing from that point become parallel on the other side.

Plate 6. Fig. 13.

It manifestly follows from Prop. XII. that if the parallel rays AB, ID, ST, in passing through CDE, are brought to a focus in K, rays from K as a radiant point will, after refraction, proceed in the parallel lines BA, DI, TS.

PROP. XXI. When rays pass out of one medium into another of different density through a plane surface, if they diverge, the distance of the imaginary radiant will be to that of the real radiant;—if they converge, the distance of the real focus will be to that of the imaginary focus of the incident rays, as the sine of the angle of incidence is to that of the angle of refraction.

This proposition admits of four cases.

Case 1. Of diverging rays passing out of a rarer into a denser medium.

Let X represent a rarer, and Z a denser medium, separated from each other by the plane surface AB; suppose CD and CE to be two diverging rays proceeding from the point C, the one perpendicular to the surface, the other oblique; through E draw the perpendicular PK. The ray CD being perpendicular to the surface, will proceed on in the right line CQ, but the other falling upon it obliquely at E, and there entering a denser medium, will suffer a refraction toward the perpendicular EK. Let then EG be the refracted ray, and produce it back till it intersects DC produced also in F; this will be the imaginary radiant. On the centre E with the radius EF, describe the circle AFBQ, and produce EC to H; draw HI the sine of the angle of incidence, and GK that of refraction; equal to this is FP or CM, which let be drawn. Now if we suppose the points D and E contiguous, or nearly so, then will the line HE be almost coincident with FD, and therefore FD will be to CD, as HE to CE or (El. VI. 4.) as HI to CM; that is, the distance of the imaginary radiant of the ray EG, is to the distance of the real radiant, as the sine of the angle of incidence is to that of the angle of refraction.* Plate 6. Fig. 16.

Case 2. Of diverging rays proceeding out of a denser into a rarer medium.

Let X be the denser, Z the rarer medium, FD and FE two diverging rays proceeding from the point F; and supposing the perpendicular PK drawn as before, FP will be the sine of the angle of incidence of the oblique ray FE, which in this case being refracted from the perpendicular, will pass on in some line as ER, which being produced back to the circumference of the circle will cut the ray FD somewhere, suppose in C; this therefore will be the imaginary focus of the refracted ray ER; draw RO, the sine of the angle of refraction, to which HI will be equal; but here also FP or its equal CM, is to HI, as EC to EH, or (if the points D and E be considered as contiguous) as DC to DF; that is, the sine of the angle of incidence is to the sine of the angle of refraction, as the distance of the imaginary radiant to that of the real radiant.

COR. The image R of a small object G, placed under water, is one fourth nearer the surface than the object. And hence the bottom of a pond, river, &c. is one third deeper than it appears to a spectator. If a river appear to be only $4\frac{1}{2}$ feet deep, it will be 6 feet; a person not apprised of this, might venture into the water at the hazard of his life. And hence also we have the reason of the common phenomenon of a shilling, or other object, placed in an empty vessel, appearing to be elevated higher and higher as the vessel is filled with water. Suppose the vessel empty, CQ its side opaque, G the object; if the eye be at H, the object will be hidden by the side CQ; but by filling the vessel with water up to DB, it will become visible, and seen at R, the ray EG being refracted into HE. And if the eye be so placed as to see the object at G when the vessel was empty, while it is filling the object will appear to rise gradually. Hence appears the reason why a straight stick HER, (Fig. 16.) immersed obliquely into water, appears bent at the surface DB; the image of the part within lying above the object.

Case 3. Of converging rays passing out of a denser medium into a rarer.

Next; let Z be the denser, X the rarer medium, and GE the incident ray; this will be refracted from the perpendicular into a line as EH; then all things remaining as before, GK, or its equal FP, or CM, will be the sine of the angle of incidence, and HI that of refraction; but these lines, as before, are to each other, as DC to DF; that is, the focal distance is to the distance of the imaginary focus, as the sine of the angle of incidence to that of the angle of refraction. Plate 6. Fig. 16.

Case 4. Of converging rays passing out of a rarer into a denser medium.

Let Z be the rarer, X the denser medium, and RE the incident ray; this will be refracted toward the perpendicular into a line, as EF; C will be the imaginary focus, and F the real one; HI, which (El. I. 26.) is equal to RO, will be the sine of the angle of incidence, and FP that of the angle of re-

* Whereas IE is to ME, or ND to CD, as HI to CM, that is, as the sine of the angle of incidence to that of the angle of refraction, which (Prop. XIII.) is always the same, the line IN is in all inclinations of the ray CE, at the same distance from CM. Consequently, had CE been coincident with CD, the point H had fallen upon N; and because the circle passes through both H and F, F would also have fallen upon N; upon which account the imaginary radiant of the ray EG would have been there. But the ray CE being oblique to the surface DB, the point H is at some distance from N; and therefore the point F is necessarily so too, and the more so, the greater that distance is. Hence it is manifest, that no two rays flowing from the radiant point C, and falling with different obliquities on the surface BD, will, after refraction there, proceed as from the same point; therefore, strictly speaking, there is no one point in the line D produced, that can more properly be called the focus of rays flowing from C, than another; for those which enter the refracting surface near D, will after refraction proceed, as has been observed, from the parts about N; those which enter near E, will flow as from the parts about F; those which enter about T, as from some points in the line DF produced, &c. And it is farther to be observed, that when the angle DCE becomes large, the line NF increases apace; whence those rays which fall near T proceed after refraction, as from a more diffused space, than those which fall at the same distance from each other near the point D. Upon which account it is usual, with optical writers, to suppose the distance between the points where the rays enter the plane surface of a refracting medium, to be inconsiderable with regard to the distance of the radiant point, if they diverge; or to that of their imaginary focus, if they converge; and unless there be some particular reason to the contrary, they consider them as entering the refracting medium in a direction as nearly perpendicular to its surfaces as may be.

fraction; but these are to each other (El. VI. 2.) as DF to DC; and therefore the focal distance, is to that of the imaginary focus, as the sine of the angle of incidence is to that of the angle of refraction.

PROP. XXII. When parallel rays fall upon a spherical surface of different density, the focal distance will be to the distance of the centre of convexity, as the sine of the angle of incidence is to the difference between that sine and the sine of the angle of refraction.

Case 1. Of parallel rays passing out of a rarer into a denser medium through a convex surface of the denser.

Plate 6. Fig. 17. Let AB represent a convex surface; C its centre of convexity; HA and DB two parallel rays, passing out of a rarer medium into a denser, the one perpendicular to the refracting surface, the other oblique; draw CB; this being a radius, will be perpendicular to the surface at the point B; and the oblique ray DB being in this case refracted toward the perpendicular, will proceed in some line, as BF, meeting the other ray in F, which will therefore be the focal point; produce CB to N, then will DBN, or (El. I. 29.) its equal BCA be the angle of incidence, and FBC that of refraction. Now, since any angle has the same sine with its supplement to two right ones, the angle of FCB being the supplement of ACB, which is equal to the angle of incidence, may here be taken for that angle; and therefore, as the sides of a triangle have the same relation to each other as the sines of their opposite angles have, FB being opposite to this angle, and FC being opposite to the angle of refraction, they may here be considered as the sines of the angles of incidence and of refraction; and for the same reason CB may be considered as the sine of the angle CFB, which angle being, together with the angle FBC, equal to the external one ACB (El. I. 32.) it is itself equal to the difference between those two last angles; and therefore the line FB is to CB, as the sine of the angle of incidence is to the sine of an angle which is equal to the difference between the angle of incidence and of refraction. Now because in very small angles as these are, for we suppose in this case also the distance AB to vanish (the reason of which will be shown in the note), their sines will have nearly the same ratio to each other that they themselves have, the distance FB will be to CB as the sine of the angle of incidence is to the difference between that sine and the sine of the angle of refraction; but because BA vanishes, FB and FA are equal, and therefore FA is to CA in that ratio. *

Case 2. Of parallel rays passing out of a denser into a rarer medium through a concave surface of the denser.

Plate 6. Fig. 17. Let AB be the concave surface of the denser medium, C the centre of convexity, and HA and DB two parallel rays. Through B, the point where the oblique ray DB enters the rarer medium, draw the perpendicular CN; and let the ray DB, being in this case refracted from the perpendicular, proceed in the direction BM; produce BM back to H; this will be the imaginary radiant, and DBN, or its equal ACB, will be the angle of incidence, and CBM, or its equal (El. I. 15.) HBN, that of refraction; then because NBD and DBH together are equal to NBH, the angle of refraction, therefore BCA, which is equal to the first, and AHB, which is equal to the second, are together equal to the angle of refraction; and therefore, since one of them, BCA, is equal to the angle of incidence, the other, AHB, is the difference between that angle and the angle of refraction. Now the sine of the angle BCA, the angle of incidence, is to the sine of the angle AHB, as BH to BC; but the distance AB vanishing, HB is to CB, as HA to CA; that is, the sine of the angle of incidence is to the sine of an angle which is the difference between the sine of the angle of incidence and that of refraction, as the distance of the focus from the surface is to that of the centre from the same.

Case 3. Of parallel rays passing out of a rarer into a denser medium through a concave surface of the denser.

Let AB be the concave surface of the denser medium, and let LB and FA be the incident rays. Now whereas, when DB was the incident ray, and passed out of a rarer into a denser medium, as in Case the first, it was refracted into a line BF, this ray LB having the same inclination to the perpendicular, will also suffer the same degree of refraction, and will therefore pass on afterward in the line FB pro-

* It appears from the above proposition, that the focal distance of the oblique ray DB is such, that the line BF shall be to the line CB or CA as the sine of the angle of incidence to the sine of an angle equal to the difference between the angle of incidence and of refraction; therefore so long as the angles BCA, &c. are small, so long the line FB is nearly of the same length, because small angles have nearly the same ratio to each other that their sines have. But when the point B is removed far from A, so that the ray DB enters the surface, suppose about O, the angles BCA, &c. becoming large, the sine of the angle of incidence begins to bear a considerably less ratio to the sine of an angle which is equal to the difference between the angle of incidence and of refraction than before, and therefore the line BF begins to bear a much less ratio to BC; wherefore its length decreases apace; upon which account those rays which enter the surface about O, not only meet nearer the centre of convexity than those which enter at A, but are collected into a more diffused space. Hence it is, that the point where those only which enter near A, are collected, is reckoned the true focus; and the distance AB, in all demonstrations relating to the foci of parallel rays entering a spherical surface, whether convex or concave, is supposed to vanish.

duced, toward P. So that, whereas in that case the point F was the real focus of the incident ray DB, the same point will, in this case, be the imaginary radiant of the incident ray LB; but it was there demonstrated, that the distance FA is to CA, as the sine of the angle of incidence is to the difference between that and the sine of the angle of refraction, therefore the radiant distance of the refracted ray BP is to the distance of the centre of convexity in that ratio.

Case 4. Of parallel rays passing out of a denser into a rarer medium through a convex surface of the denser.

Let AB be the convex surface of the denser medium, and let LB and FA be the incident rays, as before. Now whereas, when DB was the incident ray passing out of a denser into a rarer medium, it was refracted into BM, as in Case the second, having a point as H in the line MB produced for its imaginary radiant; therefore LB, for the like reason as was given in the last case, will in this be refracted into BH, having the same point H for its real focus. So that here also the focal distance will be to that of the centre of convexity, as the sine of the angle of incidence is to the difference between that and the sine of the angle of refraction.

COR. 1. Hence, the sines of the angles of incidence and of refraction of parallel rays being given, and also the distance of the centre of convexity from the surface, the focus of any lens may be easily found.

COR. 2. The distance of the centre of a glass sphere, from the principal focus of rays refracted by it, is nearly equal to a radius and a half of the sphere. For in this case it will be $FA : CA :: I : 2I-2R$, (where I and R signify the sines of incidence and refraction); therefore $FA=\frac{3CA}{6-4}=\frac{3CA}{2}$. And if the sphere be of water instead of glass, $FA=\frac{4CA}{2}=2CA=$ the diameter of the sphere.

PROP. XXIII. When diverging or converging rays enter into a medium of different density through a spherical surface, the ratio compounded of that which the focal distance bears to the distance of the radiant point (or of the imaginary focus of the incident rays, if they converge) and of that, which the distance between the same radiant point (or imaginary focus) and the centre, bears to the distance between the centre and the focus, is equal to the ratio, which the sine of the angle of incidence bears to the sine of the angle of refraction; that is, $FD \times CA : DA \times CF :: FB : BG$.

Case 1. Of diverging rays passing out of a rarer into a denser medium through a convex surface of the denser, with such a degree of divergency, that they shall converge after refraction.

Let BD represent a spherical surface, and C its centre of convexity; and let there be two diverging rays, AB and AD, proceeding from the radiant point A, the one perpendicular to the surface, the other oblique. Through the centre C produce the perpendicular ray AD to F, and draw the radius CB and produce it to K, and let BF be the refracted ray; then will F be the focal point; produce AB to H, and through the point F draw the line FG parallel to CB. AB being the incident ray, and CK perpendicular to the surface at the point B, the angle ABK, or which is equal to it, because of the parallel lines CB and FG, FGH is the angle of incidence. Now since the supplement of any angle to two right ones has the same sine with the angle itself, the sine of the angle FGB, which is the supplement of FGH to two right ones, may be considered as the sine of the angle of incidence; for which sine the line FB, as the sides of a triangle have the same ratio to each other, that the sines of their opposite angles have, may be taken. Again, the angle FBC is the angle of refraction, or its equal, because alternate to it, BFG, to which BG being an opposite side, may be taken as the sine. But FB is to BG in a ratio compounded of FB to BA, and of BA to BG; for the ratio that any two quantities bear to each other is compounded of the ratio which the first bears to any other, and of the ratio which that other bears to the second. Now FB is to BA, supposing BD to vanish, as FD to DA; and BA is to BG, because of the parallel lines CB and FG, as AC to CF. That is, the ratio compounded of FD, the focal distance, to DA, the distance of the radiant point, and of AC, the distance between the radiant point and the centre, to CF, the distance between the centre and the focus, is equal to that which the sine of the angle of incidence bears to the sine of the angle of refraction.* Plate 6. Fig. 19.

* Since the focal distance of the oblique ray AB is such that the compound ratio of FB to BA and of AC to CF shall be the same, whatever be the distance between B and D; it is evident that, AC being always of the same length, the more the line AB lengthens, the more FB must lengthen too, or else FC must shorten; but if BF lengthens, CF will do so too, and in a greater ratio with respect to its own length than BF will, therefore the lengthening of BF will conduce nothing toward preserving the equality of the ratio; but as AB lengthens, BF and CF must both shorten, which is the only possible way in which the ratio can be continued the same. And it is also apparent, that the farther B moves from D toward O, the faster

Case 2. Of converging rays passing out of a rarer into a denser medium through a concave surface of the denser with such a degree of convergency, that they shall diverge after refraction.

Plate 6. Fig. 19. Let the incident rays be HB and FD passing out of a rarer into a denser medium through the concave surface BD, and tending toward the point A, from whence the diverging rays flowed in the other case; then the oblique ray HB having its angle of incidence HBC equal to ABK, the angle of incidence in the former case, will be refracted into the line BL, such that its angle of refraction KBL will be equal to FBC, the angle of refraction in the former case; that is, it will proceed after refraction in the line FB produced, having the distance of the imaginary radiant the same as the focal distance FD with the diverging rays AB, AD, in the other case. But, by what has been already demonstrated, the ratio compounded of FD, the distance of the imaginary radiant, to DA, in this case, the distance of the imaginary focus of the incident rays, and of AC, the distance between the same imaginary focus and the centre, to CF, the distance between the centre and the imaginary radiant, is equal to that which the sine of the angle of incidence bears to the sine of the angle of refraction.

Case 3. Of diverging rays passing out of a rarer into a denser medium through a convex surface of the denser, with such a degree of divergency as to continue diverging.

Plate 6. Fig. 18. Let AB, AD, be the diverging rays, and let their divergency be so great, that the refracted ray BL shall also diverge from the other; produce LB back to F, which will be the imaginary radiant; draw the radius CB, and produce it to K; produce BA likewise toward G, and draw FG parallel to BC. Then will ABK be the angle of incidence, whose sine BF may be taken for, as being opposite to the angle BGF, which is the supplement of the other to two right ones. And LBC is the angle of refraction, or its equal KBF, or which is equal to this, BFG, as being alternate; therefore BG, the opposite side to this may be taken for the sine of the angle of refraction. But BF is to BG, for the like reason as was given in Case the first, in a ratio compounded of BF to BA, and of BA to BG. Now BF is to BA, (DB vanishing) as DF to DA, and because of the parallel lines FG and BC, the triangles CBA and AGF are similar, therefore BA is to AG, as CA to AF; consequently, BA is to BA together with AG, that is, to BG, as CA is to CA together with AF, that is, CF. Therefore the ratio compounded of DF, the distance of the imaginary radiant, to DA, the distance of the real radiant point, and of CA, the distance between the real radiant point and the centre, to CF, the distance between the centre and the imaginary radiant, is equal to that which the sine of the angle of incidence bears to the sine of the angle of refraction.

Schol. By making HB and CD the incident rays, the proposition may be proved of converging rays passing out of a rarer into a denser medium, through a concave surface of a denser, with such convergency, that they shall continue to converge. Also, by the same method of reasoning as in the preceding cases, the proposition may be proved, of diverging rays passing out of a denser into a rarer medium through a concave surface of the denser, and of converging rays passing out of a denser into a rarer medium through a convex surface of the denser. And all those cases which are the converse of the preceding, admit of a similar proof; for, when rays pass out of one medium into another, the sine of the angle of incidence has the same ratio to the sine of the angle of refraction, as the sine of the angle of refraction has to the sine of the angle of incidence, when they pass through the same lines of direction the contrary way.

Case 4. Of rays passing out of a denser into a rarer medium, from a point between the centre of convexity and the surface.

Plate 6. Fig. 20. Let AB, AD, be two rays passing out of a denser into a rarer medium, from the point A, which is taken between C, the centre of convexity, and the refracting surface BD; through B draw CK, and let BL be the refracted ray; produce BL back to F, and draw FG parallel to BC. Then will ABC be the angle of incidence, of which BF, being opposite to its alternate and equal angle BGF, is the sine. LBK will be the angle of refraction, or its equal FBC, of which BG, being opposite to its supplement to two right angles BFG, is the sine. But, BF is to BG in the compound ratio of BF to BA, and of BA to BG; and (BD vanishing) BF is to BA as DF to DA, and because the lines CB and FG are parallel, BA is to BG as CA to CF.

Case 5. Of rays passing out of a rarer into a denser medium from a point between the centre of concavity and the surface.

AB lengthens, and therefore the farther the rays enter from D, the nearer to the refracting surface is the place where they meet, but the space they are collected in is the more diffused; and therefore in this case, as well as those taken notice of in the two preceding notes, different rays, though flowing from the same point, will constitute different foci; and none are so effectual as those which enter at or very near the point D. And since the same is observable of converging as well as of diverging rays, none, except those which enter very near that point, are usually taken into consideration; upon which account it is, that the distance DB, in determining the focal distances of diverging or converging rays entering a convex or concave surface, is supposed to vanish.

Let AB, AD, be two diverging rays passing out of a rarer into a denser medium through the refracting surface BD, whose centre of concavity is C, a point beyond that from whence the rays flow. Plate 6. Fig. 21. Through B draw CK, and let BL be the refracted ray; produce it back to F, and draw FG parallel to BC, meeting BA in G. ABC will be the angle of incidence, of which BF, being opposite to its supplement to two right angles BGF, is the sine. The angle of refraction is LBK, or its equal FBC, of which BG, being opposite to its alternate and equal angle BFG, is the sine. But BF is to BG in the compound ratio of BF to BA and of BA to BG; and (BD vanishing) BF is to BA as DF to DA; and because of the parallel lines CB and GF, the triangles AFG and ABC are similar. BA therefore is to AG, as CA to AF; consequently, BA is to BA — AG, that is, to BG, as CA is to CA — AF, that is, to CF.

In like manner the Proposition may be proved of rays passing out of a denser into a rarer medium toward a point between the centre of convexity and the surface, and in all other supposable cases.

COR. Hence the distance of the radiant point, or of the imaginary focus, being given, and also the radius of convexity, and the sines of the angles of incidence and refraction in the two mediums, the focus of any lens may be thus found.

Let it be required to determine the focal distance of diverging rays passing out of air into glass through a convex surface, and let the distance of the radiant point be 20, and the radius of convexity be 5; let the focal distance be expressed by x; then, because by the preceding Proposition the ratio compounded of that which the focal distance bears to the distance of the radiant point, (that is, in this supposition, of x to 20,) and of the ratio which the distance of the same radiant point from the centre bears to the distance between the centre and the focus (in this case, of 25 to $x - 5$,) is equal to the ratio which the sine of the angle of incidence bears to the sine of the angle of refraction (that is, of 17 to 11), we shall have in the instance before us the following proportion, $\left.\begin{matrix} x : 20 \\ 25 : x - 5 \end{matrix}\right\} :: 17 : 11$, and compounding them in one, which is done by multiplying the two first parts together, we have $25x : 20x - 100 :: 17 : 11$, $x = \frac{1700}{65}$.

SECT. II.

Of Images produced by Refraction.

PROP. XXIV. Rays of light flowing from the several points of any object, farther from a convex lens than its principal focus, by passing through the lens, will be made to converge to points corresponding to those from which they proceeded, and will form an image.

Let ABC be a luminous or illuminated object. From every point, as A, B, C, rays diverge in all directions. Plate 6. Fig. 22. Let some of these rays fall upon a convex lens GHK placed in a hole GK, in the window shutter of a dark room ML, at a greater distance from the object than the principal focal distance of the lens. BH being the axis, will (by Prop. XIV.) pass through the lens without refraction in the direction BHE. But the collateral rays BG, BK, made equally convergent by the lens, will cross the axis at E; that is, all the rays which come from the point B in the object, will be united behind the lens in the focus E. In like manner, among the rays AG, AH, AK, which diverge from the point A, whilst AH the axis (as was shown Prop. XVII. Schol.) may be considered as if it went straight through the lens, the other rays will be made to converge, and will be united in a focus at F; and also, the rays from C will be united in D. The same may be shown concerning every other point in the object. Consequently, there will be as many correspondent foci in the image as there are radiant points in the object; and these foci will be disposed in the same manner with respect to one another as the radiants, and will therefore form an image. The object must be farther from the lens than its principal focus, else the rays from the several radiants would not converge, but either become parallel or diverging (by Prop. XVII.), whence no image would be formed.

EXP. 1. Let the rays of the sun pass through a convex lens into a dark room, and fall upon a sheet of white paper placed at the distance of the principal focus from the lens.

2. The rays of a candle, in a room from which all external light is excluded, passing through a convex lens, will form an image on white paper.

PROP. XXV. The image produced by rays of light passing through a convex lens is inverted.

Plate 6. Fig. 22. The focus in which the rays, that come from any point A, or B, are united, is in the axis AHF, or BHE, of the beam, whether it fall directly or obliquely upon the lens. But the axis (by Prop. XVII.) is the middle ray of a cone of rays whose base is the surface of the lens, and vertex the radiant point. Every axis, therefore, as AHF, BHE, must pass through H, the middle point of the lens, and consequently must cross one another in that point; from which it is manifest, that the rays from the lowest point C of the object will become the highest point of the image D; and that the image will be, with respect to the original object, inverted.

PROP. XXVI. The image will not be distinct, unless the plane surface, on which it is received, be placed at the distance of the principal focus of the lens.

For otherwise the rays which come from a single point in the object, will not have its corresponding point in the image, but will be spread over a larger surface.

PROP. XXVII. As an object approaches a convex lens, its image departs from it, and as the object departs, the image approaches.

Plate 6. Fig. 22. As the object ABC approaches the lens, the several radiants approach it; and consequently (by Prop. XIX.) the several foci which form the image FED recede; and the reverse. But the image can never be nearer the lens than its focus of parallel rays, since this is the place of the image, when the object is infinitely distant.

PROP. XXVIII. When the object is placed parallel to the image, the diameter of the object is to the diameter of the distinct image, as the distance of the object from the lens, is to the distance of the image from the lens.

Plate 6. Fig. 22. The radiant A (as appears from Prop. XXV.) is represented by its focus in the point F, where the line AH, produced behind the lens, cuts a plane passing through the focus of parallel rays parallel to the lens. In like manner the radiant C is represented by its corresponding focus in the point D, in which the same plane is cut by the line CH produced. If therefore the distance of the extremities of the object, or its length, be AC, the length of its image will be DF. Since therefore AC is parallel to DF, the alternate angles ACD, CDF, (El. I. 29.) are equal; and also the alternate angles CAF, AFD; whence (El. VI. 4.) AC is to FD, as CH to DH, as BH is to EH, that is, the height of the object is to that of the image, as the distance of the object from the lens, to the distance of the image from the lens. Any diameter or line drawn across the object may be proved, in like manner, to have the same ratio to any corresponding diameter or line drawn across the image.

PROP. XXIX. When the image appears confused, it is larger than when it is distinct.

For the rays, in this case, are not received upon the white surface exactly at the distance from the lens at which they are brought to a focal point, but either at a distance greater or less; and in either case the rays which come from any radiant points at the extremities A and C, will not be collected into points on the plane at F and D, but be spread over a small circular space round these points; whence the confused image will be larger than the distinct image.

PROP. XXX. The object and distinct image are similar surfaces.

Though the side of any object which is toward the lens be not a plane surface, yet the light is reflected from it in the same manner as if the figure of the object were drawn upon the plane surface of a piece of canvass, and differently shaded. Therefore the side of the object next to the lens may be considered as a plane figure. And since (by Prop. XXVIII.) the height of the object is to that of the picture, as the distance of the object from the lens, to the distance of the image from the lens, and also the breadth of the object in any part, to the breadth of the image in the corresponding part, in the ratio of these distances; it follows (El. V. 11.) that the height of the object is to the height of the image, as the breadth of the object in any part is to the breadth of the image in its corresponding part; that is, the object and image are similar surfaces.

PROP. XXXI. The diameter of an image formed by rays passing from a given object through a convex lens, increases as the object approaches the lens, and decreases as the object recedes from the lens.

The diameter of the image (by Prop. XXVIII.) increases as its distance increases, and decreases

as its distance decreases. And (by Prop. XXXVII.) the distance of the image increases as the distance of the object decreases, and the reverse. Whence the diameter of the image increases as the distance of the object decreases, and decreases as the distance of the object increases.

PROP. XXXII. When the distance of the object is given, the diameter of the image is as the diameter of the object.

If the object AC remain at the distance BH from the lens, the image (by Prop. XXVII.) will remain at the distance EH; whence the ratio of BH to EH, and consequently (by Prop. XXVIII.) the ratio of the diameter AC to its correspondent diameter DF, is given, or is invariable. Consequently, if AC increases or decreases, DF must proportionally increase or decrease; that is, the diameter of the image is directly as the diameter of the object. Plate 6. Fig. 22.

PROP. XXXIII. When the diameter and distance of the object are given, the diameter of the image will be as its distance from the lens.

If the diameter and distance of the object are given, it is manifest that the diameter of the image cannot be varied without changing the lens. But if, instead of the lens GHK, one less convex, or more convex, be used, the rays will be brought to a focus, and the image (by Prop. XXIV.) will be formed at a greater or less distance from the lens. And since (by Prop. XXVIII). the distance of the object is always to the distance of the image, as the diameter of the object to the diameter of the image; the first and third terms remaining invariable, the second and fourth must be increased or diminished proportionally; that is, the diameter of an image will be directly as its distance from the lens.

PROP. XXXIV. When the diameter and distance of the object are given, the area of the image is as the square of its distance from the lens.

Because the surface of the image (by Prop. XXX.) is similar to the surface of the object, whilst the surface of the object remains the same, the image, in every variation of its magnitude, must be similar to itself. But the areas of similar surfaces (El. VI. 20. Cor. 1.) are as the squares of their homologous sides, that is, as the squares of their heights or breadths. Therefore the area of the image, is always as the square of its diameter. And the diameter of the image, when the diameter and distance of the object are given, is (by Prop XXXIII.) as its distance from the lens; therefore the square of its diameter, or its area, is as the square of its distance from the lens.

PROP. XXXV. Though the distance of the object from the lens be varied, the image may be preserved distinct without varying the distance of the plane surface which receives it.

This will be the case, if as much as the image is brought forward by the removal of the object, it is thrown backward by diminishing the convexity of the lens, and the reverse; or the image may be preserved distinct without changing the lens, by increasing or diminishing the distance of the lens from the plane surface which receives the image, in the same ratio as the distance of the object from the lens is increased or diminished; which may be done either by moving the lens or the plane surface.

PROP. XXXVI. The distances of the object and image, and the diameter of the object being given, the diameter of the image will not be altered by changing the area of the lens.

The height of the image DF is the distance between the two extreme foci F and D; the former of which is always in the axis AHF of the cone which has A for its vertex, and the latter in the axis CHD of the cone whose vertex is C, which axes cross each other in H. Since therefore DF, the the height of the image, is the distance between these two lines AHF, CHD, where they meet the plane surface, the height of the image will be the same, whether the whole area, GHK is open, or only a small part of it at H. Plate 6. Fig. 22.

PROP. XXXVII. When the object is near the lens, but not so near as the principal focus, in order to make the image distinct, the area of the lens must be small.

If the object was as near to the lens as the principal focus, or nearer, no image (by Prop. XVII.) could be formed. But let the object A be at a distance from the lens NP, very little greater than that of the principal focus; then the extreme rays AN, AP, of the cone NAP, diverging more than the rays AD, AE, if the plane surface, which is to receive the rays, is placed where these rays are collected to a focus, the extreme rays AN, AP, diverging more, will not be collected, and the image on Plate 6. Fig. 23.

the plane surface will be confused. To prevent this, the extreme rays must be excluded by diminishing the area of the lens, or of the hole where it is placed. If the radiant A were at a greater distance, this would be unnecessary. Supposing the lens at SR, the extreme rays AN, AP, would pass above or below the lens, and only the middle rays, which are brought to a focus on the plane surface, would pass through the lens.

PROP. XXXVIII. The brightness of an image, when its distance from the lens is given, is as the area of the lens.

Plate 6. Fig. 22. When the whole area GHK is open, the entire cone of rays AGK passes through the lens from the point A, and is brought to a focus at F; but when the area is diminished to a small surface at H, the greatest part of the cone is excluded, and no rays, but the axis AH and those which are near it can pass through the lens; whence it is manifest, that the focal point F must be more illuminated by the rays from A when the area of the lens is GHK, than when the area is diminished. The same may be said of every other cone of rays, and of every other point in the image. Therefore the whole image, although (by Prop. XXXVII.) made more distinct by diminishing the area, will be made fainter or less bright.

PROP. XXXIX. The brightness of the image, when the area of the lens and the distance of the object are given, is inversely as the square of its distance from the lens.

The area of the lens and distance of the object being given, the number of rays which pass through the lens and form the image, is given. Now the same number of rays spread over a larger surface will not illuminate it so strongly as they would a smaller surface; that is, the brightness will be inversely as the illuminated area; and the area of the image is (by Prop. XXXIV.) as the square of its distance from the lens; whence its brightness is inversely as the square of its distance.

PROP. XL. The heat at the focus of a burning glass, when the area of the glass is given, is inversely as the square of the focal distance.

The distance of the burning spot, that is, the image of the sun, from the lens, is the focal distance, because the sun's rays are parallel. And because the heat and the brightness at the focus are as the number of rays collected, the heat is as the brightness. But the brightness (by Prop. XXXIX.) is inversely as the square of the distance of the image from the lens; therefore the heat is in the same ratio, that is, in this case, inversely as the square of the focal distance of the glass.

PROP. XLI. The heat at the focus of a burning glass, when the focal distance is given, is as the area of the glass.

The brightness is (by Prop. XXXVIII.) as the area of the lens, and the heat is as the brightness; therefore the heat is also as the area of the lens.

PROP. XLII. The heat at the focus of a burning glass is to the common heat of the sun, inversely as the area of the focus to the area of the glass.

The brightness, or the heat, must be inversely as the space or area over which the rays which cause them are spread, that is, inversely as the area of the focus to the area of the glass.

Schol. This proposition supposes all the rays which fall upon the lens to pass through it to the focus.

Plate 12. Fig. 9. Exp. Most of the preceding Propositions from Prop. XIX. to XXXIX. may be confirmed, in a room from which all external light is excluded, by placing a convex lens x, fixed in a frame which moves perpendicularly upon an oblong bar of wood, or table BD, at distances such as the Propositions require from a lighted candle Q placed perpendicularly on the same bar of wood, and receiving the images upon white paper q. Upon this bar of wood, on one side of a line over which the convex lens is placed, let a line perpendicular to the last mentioned line be divided into parts 1, 2, 3, 4, &c. each equal to the distance of the focus of parallel rays; and on the other side of the lens let a line be divided in the same manner, and let the first division which is farther from the lens than the focus, be subdivided into parts respectively equal to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. of the distance of the focus of parallel rays; if a candle be placed over the division 2, it will form a distinct image on a paper held over the division $\frac{1}{2}$; if the candle be over 3, the image will be at $\frac{1}{3}$, &c. whence it appears, that the distances of the correspondent foci vary reciprocally. Prop. XL. XLI. XLII. may be confirmed by holding a large double convex lens, or burning glass, in the sun's rays, and receiving the image on white paper, or other substances.

CHAPTER. III.

Of Reflection.

SECTION I.

OF THE LAWS OF REFLECTION.

DEF. XIX. A *Ray of Light*, turned back into the same medium in which it moved before its return, is said to be *reflected*.

DEF. XX. The *Angle of reflection* is that, which is contained between the line described by a reflected ray and a line perpendicular to the reflecting surface at the point of reflection.

Let AC be the incident ray falling upon the reflecting surface DE, CB will be the reflected ray, OC the perpendicular, ACO the angle of incidence, and OCB the angle of reflection. Plate 6. Fig. 25.

PROP. XLIII. The reflection of light from transparent bodies is either partial or total; the partial reflection happens either at the first or second surface, the total, at the second surface only.

When a beam of light falls upon a thick piece of polished glass, all the rays will not pass through it; but some of them will be reflected from the first surface of the glass, where the beam enters. At the second surface, some of the rays will also be reflected. These partial reflections happen, whatever is the obliquity of the rays. The total reflection happens when the angle of incidence, or the obliquity, is greater than 41 degrees. All the light which then comes to the second surface will be reflected. See Cor. 3. Prop. XIII.

SCHOL. The rays of light are not reflected by striking upon the solid parts of bodies.

At least as many rays are reflected from the second surface when the light passes out of glass into air, as from the first when it passes out of air into glass; but if the reflection were caused by the striking of the rays upon solid parts of bodies, since glass is denser than air, that is, has more solid parts in a given space, a greater quantity of rays would be reflected from the first surface than from the second. Besides, it seems improbable that, at the second surface, with one degree of obliquity, the rays should meet with nothing but pores or interstices in the air, and pass freely into it, and that with a greater degree of obliquity, it should meet with nothing but solid parts, and be totally reflected. Again, since water is denser than air, if the reflection were owing to the striking of the rays upon the solid body, it might be expected that the light would be more perfectly reflected in passing out of glass into water than into air; whereas, it is found, that if water be placed behind the glass instead of air, rays will not be reflected at the second surface, though their obliquity be greater than 41 degrees; hence also it is manifest, that the reflection is not owing to the striking of the rays upon the second surface of the glass; for then it would be the same, whatever were the medium beyond it.

PROP. XLIV. Reflection is caused by the powers of attraction and repulsion in the reflecting bodies.

Supposing that bodies attract those rays which are very near them, and repel those a little farther from them; and calling the space contiguous to the surface of the glass, where the rays are attracted, the attracting surface; and the space next to this, the repelling surface; the proposition may be thus proved. Plate 6. Fig. 24.

Let GG, MM, be the repelling surface of a piece of glass, and R a a ray falling upon it. This ray when it enters the surface will be retarded by the repulsion, and consequently, refracted from the perpendicular at a. And this repulsion increasing till the ray gets into the surface of attraction, the ray will be constantly retarded, that is, turned out of its straight course at b, c, d, &c. till it becomes parallel to MM at f, the limit of the repelling surface. And in this situation of the ray, the repelling force, which had retarded, will now constantly accelerate it, and consequently it will be continually refracted toward the perpendicular, at g, h, i, &c. till it emerge from the surface at l; when, the repelling force ceasing to act, the ray will proceed in a right line. Thus the ray, by reflection, is made to describe the curve $a f l$.

PROP. XLV. The angle of reflection is equal to the angle of incidence, and their complements are also equal.

In all cases of reflection, the rays (by Prop. XLIV.) describe such a curve at *a f l*. And since they describe one half of this curve by being retarded, and one half by being similarly accelerated, one half will be similar to the other; whence one half will make the same angle with a perpendicular at *f*, as the other half makes with it. And the bending of the rays is made within so very small a compass, that is, the curve *a f l* is so small, that it may be neglected, as in fig. 25, where the angle ACO is equal to the angle BCO, and consequently, the angle ACD equal to BCE.

Exp. 1. Having described a semicircle on a smooth board, and from the circumference let fall a perpendicular bisecting the diameter, on each side of the perpendicular cut off equal parts of the circumference; draw lines from the points in which those equal parts are cut off to the centre; place three pins perpendicular to the board, one at each point of section in the circumference, and one at the centre; and place the board perpendicular to a plane mirror. Then look along one of the pins in the circumference to that in the centre, and the other pin in the circumference will appear in the same line produced with the first; which shows that the ray which comes from the second pin, is reflected from the mirror at the centre to the eye, in the same angle in which it fell on the mirror.

2. Let a ray of light passing through a small hole into a dark room, be reflected from a plane mirror; at equal distances from the point of reflection, the incident and the reflected ray will be at the same height from the surface.

PROP. XLVI. All reflection is reciprocal.

Plate 6. Fig. 25. If the ray AC, after it has been reflected in the line CB, is turned back again in the direction CB, it will be reflected (by Prop. XLV.) into AC. Therefore if ACO be the angle of incidence, OCB will be the angle of reflection; and if OCB be the angle of incidence, ACO will be the angle of reflection.

Schol. Sir Isaac Newton explains the cause of reflection by supposing, that light in its passage from the luminous body is disposed to be alternately reflected by, and transmitted through any refracting surface it may meet with; that these dispositions which he calls Fits of easy reflection and easy transmission, return successively at equal intervals; and that they are communicated to it, at its first emission out of the luminous body it proceeds from, probably by some very subtle and elastic substance diffused through the universe, in the following manner. As bodies falling into water, or passing through the air, cause undulations in each, so the rays of light may excite vibrations in this elastic substance. The quickness of these vibrations depending on the elasticity of the medium (as the quickness of the vibrations in the air, which propagate sound, depend solely on the elasticity of the air, and not upon the quickness of those in the sounding body), the motion of the particles of it may be quicker than that of the rays; and therefore when a ray, at the instant it impinges upon any surface, is in that part of a vibration of this elastic substance which conspires with its motion, it may be easily transmitted, and when it is in that part of a vibration which is contrary to its motion, it may be reflected. He farther supposes, that when light falls upon the first surface of a body, none is reflected there, but all that happens to it there is, that every ray that is not in a fit of easy transmission, is there put into one, so that when they come at the other side (for this elastic substance easily pervading the pores of bodies, is capable of the same vibrations within the body as without it,) the rays of one kind shall be in a fit of easy transmission, and those of another in a fit of easy reflection, according to the thickness of the body, the intervals of the fits being different in rays of a different kind.

PROP. XLVII. Rays of light reflected from a plane surface, have the same degree of inclination to each other that their respective incident ones have.

Plate 6. Fig. 26. The angles of reflection of the rays AC, AI, AK, being equal to that of their respective incident ones, it is evident that each reflected ray will have the same degree of inclination to the surface DE, from whence it is reflected, that its incident one has; but it is here supposed that all those portions of surface, from whence the rays are reflected, are situated in the same plane; consequently, the reflected rays FC, LI, MK, will have the same degree of inclination to each other that their incident ones have, from whatever part of the surface they are reflected.

Cor. Parallel rays falling on a reflecting plane surface are reflected parallel.

PROP. B. If a plane mirror revolve upon an axis, the angular velocity of the reflected ray is double that of the mirror.

Plate 6. Fig. 25. Let DE be a mirror, AC an incident ray, and CB a reflected ray. If the mirror turn upon an axis at C so as to come into the situation FG, then the incident ray AC will be reflected into the line CH.

Now the angle BCH, which expresses the angular velocity of the reflected ray BC, is double of the angle DCF, which is the angular velocity of the mirror. For the angle ACD = BCE = BCH + HCE. (Prop. XLV.) For the same reason ACF = HCG = HCE + (ECG) FCD. Therefore the angle DCF (ACD — ACF) = BCH — DCF, consequently the angle BCH = 2DCF; that is, the angular velocity of the reflected ray is double that of the mirror.

Schol. Upon this Proposition depend the construction and use of Hadley's quadrant.

PROP. XLVIII. Parallel rays reflected from a concave surface are made converging.

Let AF, CD, EB, represent three parallel rays falling upon the concave surface FB, whose centre is C. To the points F and B draw the lines CF, CB; these being drawn from the centre will be perpendicular to the surface at those points. The incident ray CD also passing through the centre will be perpendicular to the surface, and therefore will return after reflection in the same line; but the oblique rays AF and EB will be reflected into the lines FM, BM, situated on the contrary side of their respective perpendiculars CF and CB. They will therefore proceed converging after reflection toward some point, as M, in the line CD. Plate 6. Fig. 27.

PROP. XLIX. Converging rays falling upon a concave surface are made to converge more.

Let GB, HF, be the incident rays. Now, because these rays have larger angles of incidence than the parallel ones, AF, EB, in the foregoing case, their angles of reflection will also be larger than theirs; they will therefore converge after reflection, suppose in the lines FN and BN, having their point of concourse N farther from C than the point M, to which the parallel rays AF and EB converged in the foregoing case. Plate 6. Fig. 27.

PROP. L. Diverging rays, falling upon a concave surface, if they diverge from the focus of parallel rays, become parallel;—if from a point nearer the surface than that focus, will diverge less than before reflection;—if from a point between that focus and the centre, will converge after reflection to some point, on the contrary side of the centre, and farther from the centre than the point from which they diverged;—if from a point beyond the centre, the reflected rays will converge to a point on the contrary side, but nearer to it than the point from which they diverged;—if from the centre, they will be reflected thither again.

Let the incident diverging rays be MF, MB, proceeding from M, the focus of parallel rays; then as the parallel rays AF and EB were reflected into the lines FM and BM, these rays will now on the contrary be reflected into them. (By Prop. XLVI.) Plate 6. Fig. 27.

Let them diverge from N, a point nearer to the surface than the focus of parallel rays; they will then be reflected into the diverging lines HF and BG, which the incident rays GB and HF described, which were shown to be reflected into them in the foregoing proposition; but the degree wherein they diverge, will be less than that wherein they diverged before reflection.

Let them proceed diverging from X, a point between the focus of parallel rays and the centre; they then make less angles of incidence than the rays MF and MB, which became parallel by reflection; they will consequently have less angles of reflection, and proceed therefore converging toward some point, as Y; which point will always fall on the contrary side of the centre, because a reflected ray always falls on the contrary side of the perpendicular with respect to that on which its incident one falls; and therefore will be farther distant from the centre than X.

If the incident ones diverge from Y, they will after reflection converge to X, those which were the incident rays in the former case being the reflected ones in this.

Lastly, if the incident rays proceed from the centre, they fall in with their respective perpendiculars, and for that reason are reflected thither again.

Exp. Place a concave mirror at proper distances from an open orifice, or a convex or a concave lens, through which a beam of solar rays passes, according to the three preceding propositions.

PROP. LI. Parallel rays reflected from a convex surface are made diverging.

Let AB, GD, EF, be three parallel rays falling upon the convex surface BF, whose centre of convexity is C, and let one of them, GD, be perpendicular to the surface. Through B, D, and F, the Plate 6. Fig. 28.

points of reflection, draw the lines CV, CG, and CT, which because they pass through the centre will be perpendicular to the surface at those points. The incident ray GD, being perpendicular to the surface, will return after reflection in the same line, but the oblique ones AB and EF in the lines BK and FL situated on the contrary side of their respective perpendiculars BV and FT. They will therefore diverge after reflection, as from some point M in the line GD produced.

PROP. LII. Diverging rays reflected from a convex surface are made more diverging.

Plate 6. Fig. 28. Let GB, GF, be the incident rays. These having larger angles of incidence than the parallel ones AB and EF in the preceding case, their angles of reflection will also be larger than theirs; they will therefore diverge after reflection, suppose in the lines BP and FO, as from some point N farther from C than the point M; and the degree wherein they will diverge, will exceed that wherein they diverged before reflection.

PROP. LIII. Converging rays reflected from a convex surface, if they tend toward the focus of parallel rays, will become parallel;—if to a point nearer the surface than that focus, will converge less than before reflection;—if to a point between that focus and the centre, will diverge as from a point on the contrary side of the centre farther from it than the point toward which they converged;—if to a point beyond the centre, they will diverge as from a point on the contrary side of the centre nearer to it than the point toward which they first converged;—and if toward the centre, they will proceed, on reflection, as from the centre.

Plate 6. Fig. 28. Let the converging rays be KB, LF, tending toward M, the focus of parallel rays; then, as the parallel rays AB, EF, were reflected into the lines BK and FL, those rays will now on the contrary be reflected into them.

Let them converge in the lines PB, OF, tending toward N, a point nearer the surface than the focus of parallel rays; they will then be reflected into the converging lines BG and FG, in which the rays GB, GF, proceeded, which were shown to be reflected into them in the proposition immediately foregoing; but the degree wherein they will converge, will be less than that wherein they converged before reflection.

Let them converge in the lines RB and SF proceeding toward X, a point between the focus of parallel rays and the centre; their angles of incidence will then be less than those of the rays KB and LF, which became parallel after reflection; their angles of reflection will therefore be less, on which account they must necessarily diverge, suppose in the lines BH and FI, from some point, as Y; which point will fall on the contrary side of the centre with respect to X, and will be farther from it than X.

If the incident rays tend toward Y, the reflected ones will diverge as from X, those which were the incident ones in one case, being the reflected ones in the other.

And lastly, if the incident rays converge toward the centre, they fall in with their respective perpendiculars; on which account they proceed after reflection, as from thence.

Exp. Illustrate the three preceding propositions by receiving upon a convex mirror, a solar ray passing through an open orifice, or a concave or convex lens.

PROP. LIV. When rays fall upon a plane surface, if they diverge, the focus of the reflected rays will be at the same distance behind the surface, that the radiant point is before it;—if they converge, it will be at the same distance before the surface, that the imaginary focus of the incident rays is behind it.

Plate 6. Fig. 26. Case 1. Of diverging rays. Let AB, AC, be two diverging rays incident on the plane surface DE, the one perpendicularly, the other obliquely; the perpendicular one AB will be reflected to A proceeding as from some point in the line AB produced; the oblique one AC will be reflected into some line CF, such that the point G, where the line FC produced intersects the line AB produced also, shall be at an equal distance from the surface DE with the radiant A. For the perpendicular CH being drawn, ACH and HCF will be the angles of incidence and reflection, which being equal, their complements ACB and FCE are so too; but the angle BCG is equal (El. I. 15.) to FCE; therefore in the triangles ABC and GBC the angles at C are equal, the side BC is common, and the angles at B are also equal to

each other as being right ones; therefore (El. I. 26.) the lines AB and BG, opposite to the equal angles at C, are also equal, and consequently the point G, the focus of the incident rays AB, AC, is at the same distance behind the surface, that the point A is before it.

Case 2. Of converging rays. Supposing FC and AB to be two converging incident rays, CA and BA will be the reflected ones, (the angles of incidence in the former case being now the angles of reflection, and the reverse) having the point A for their focus; but this, from what was demonstrated above, is at an equal distance from the reflecting surface with the point G, which in this case is the imaginary focus of the incident rays FC and AB. What is here demonstrated of the ray AC, holds equally of any other, as AI, AK, &c.

Schol. The case of parallel rays incident on a plane surface, is included in this proposition; for in that case we are to suppose the radiant to be at an infinite distance from the surface, and then by the proposition, the focus of the reflected rays will be so too; that is, the rays will be parallel after reflection, as they were before.

PROP. LV. When parallel rays are incident upon a spherical surface, the focus (or imaginary radiant) of the reflected rays will be the middle point between the centre of convexity and the surface.

Case 1. Of parallel rays falling upon a convex surface. Let AB, DH, represent two parallel rays incident on the convex surface BH, the one perpendicularly, the other obliquely; and let C be the centre of convexity; suppose HE to be the reflected ray of the oblique incident one DH proceeding as from F, a point in the line AB produced. Through the point H draw the line CI, which will be perpendicular to the surface at that point, and the angles DHI and IHE, being the angles of incidence and reflection, will be equal. But HCF is equal (El. I. 29.) to DHI, and CHF (El. I. 15.) to IHE; wherefore the triangle CFH is isosceles; and consequently, the sides CF and FH are equal; but supposing BH to vanish, FH is equal to FB, and therefore upon this supposition FC and FB are equal; that is, the imaginary radiant of the reflected rays is the middle point between the centre of convexity and the surface. Plate 7. Fig. 1.

Case 2. Of parallel rays falling upon a concave surface. Let AB, DH, be two parallel rays incident, the one perpendicularly, the other obliquely, on the concave surface BH, whose centre of concavity is C. Let BF and HF be the reflected rays meeting each other in F; this will be the middle point between B and C For drawing through C the perpendicular CH, the angles DHC and FHC, being the angles of incidence and reflection, will be equal, to the former of which the angle HCF is equal, as alternate; and therefore the triangle CFH is isosceles. Wherefore CF and FH are equal; but if we suppose BH to vanish, FB and FH are also equal, and therefore CF is equal to FB; that is, the focal distance of the reflected rays is the middle point between the centre and the surface.* Plate 7. Fig. 2.

Schol. The converse of these two cases may be demonstrated in a similar manner, by making the incident rays the reflected ones.

PROP. LVI. When rays fall upon any spherical surface, if they diverge, the distance of the focus of the reflected rays from the surface is to the distance of the radiant point from the same (or, if they converge, to that of the imaginary focus of the incident rays) as the distance of the focus of the reflected rays from the centre is to the distance of the radiant point (or imaginary focus of the incident rays) from the same.

* It is here observable, that the farther the line DH is taken from AB, the nearer the point F falls to the surface. For the farther the point H recedes from B, the larger the triangle CFH will become; and consequently, since it is always an isosceles one, and the base CH, being the radius, is every where of the same length, the equal legs CF and FH will lengthen; but CF cannot grow longer unless the point F approach toward the surface. And the farther H is removed from B, the faster F approaches to it. This is the reason, that whenever parallel rays are considered as reflected from a spherical surface, the distance of the oblique ray from the perpendicular ray is taken so small with respect to the focal distance of that surface, that without any physical error, it may be supposed to vanish. Hence it follows, that if a number of parallel rays, as AB, CD, EG, &c. fall upon a convex surface, and if BA, DK, the reflected rays of the incident ones, AB, CD, proceed as from the point F, those of the incident ones CD, EG, namely, DK, GL, will proceed as from N, those of the incident ones, EG, HI, as from O, &c. because the farther the incident ones CD, EG, &c. are from AB, the nearer to the surface are the points F, f, f, in the line BF, from which they proceed after reflection; so that properly the imaginary radiants of the reflected rays BA, DK, GL, &c. are not in the line AB produced, but in a curve line passing through the points F, N, O, &c. Plate 7. Fig. 3.

The same is applicable to the case of parallel rays reflected from a concave surface, as expressed by the dotted lines on the other half of the figure, where PQ, RS, TV, are the incident rays; QF, Sf, Vf, the reflected ones intersecting each other in the points X, Y, and F; so that the foci of those rays are not in the line FB, but in a curve passing through those points.

Plate 7. Fig. 4. Case 1. Of diverging rays falling upon a convex surface. Let RB, RD, represent two diverging rays flowing from the point R as from a radiant, and falling the one perpendicularly, the other obliquely, on the convex surface BD, whose centre is C. Let DE be the reflected ray of the incident one RD; produce ED to F, and through R draw the line RH parallel to FE, till it meets CD produced in H. Then will the angle RHD be equal (El. I. 29.) to EDH the angle of reflection, and therefore equal also to RDH, which is the angle of incidence; wherefore the triangle DRH is isosceles, and consequently DR is equal to RH. Now the lines FD and RH being parallel, (El. VI. 2.) FD is to RH, or its equal RD, as CF to CR; but BD vanishing, FD and RD differ not from FB and RB; wherefore FB is to RB also, as CF to CR; that is, the distance of the imaginary radiant from the surface is to the distance of the real radiant point from the same, as the distance of the imaginary radiant from the centre is to the distance of the real radiant from thence.

Plate 7. Fig. 4. Case 2. Of converging rays falling upon a concave surface. Let KD and CB be the converging incident rays, having their imaginary focus in the point R, which was the radiant in the foregoing case. Then as RD was in that case reflected into DE, KD will in this be reflected into DF; for since the angles of incidence in both cases are (El. I. 15.) equal, the angles of reflection will be so too; so that F will be the focus of the reflected rays; but it was there demonstrated, that FB is to RB as CF to CR, that is, the distance of the focus from the surface is to the distance (in this case) of the imaginary focus of the incident rays, as the distance of the focus from the centre is to the distance of the imaginary focus of the incident rays from the same.

Plate 7. Fig. 5. Case 3. Of converging rays falling upon a convex surface, and tending to a point between the imaginary radiant of parallel rays and the centre. Let BD represent a convex surface, whose centre is C, and focus of parallel rays is P; and let AB, KD, be two converging rays incident upon it, and having their imaginary focus at R, a point between P and C. Now because KD tends to a point between the focus of parallel rays and the centre, the reflected ray DE will diverge from some point on the other side of the centre, suppose F; as was shown Prop. LIII. Through D draw the perpendicular CD, and produce it to H; then will KDH and HDE be the angles of incidence and reflection, which being equal, their vertical ones RDC and CDF will be so too, and therefore the vertex of the triangle RDF is besected by the line DC; whence (El. VI. 3.) FD and DR, or, BD vanishing, FB and BR are to each other as FC to CR; that is, the distance of the imaginary radiant of the reflected rays is to that of the imaginary focus of the incident ones, as the distance of the former from the centre is to the distance of the latter from the same.

Plate 7. Fig. 5. Case 4. Of diverging rays falling upon a concave surface, and proceeding from a point between the focus of parallel rays and the centre. Let RB, RD, be the diverging rays incident upon the concave surface BD, having their radiant in the point R, the imaginary focus of the incident rays in the foregoing case. Then as KD was in that case reflected into DE, RD will now be reflected into DF. But it was there demonstrated that FB and RB are to each other as CF to CR; that is, the distance of the focus is to that of the radiant; as the distance of the former from the centre is to the distance of the latter from the same.*

Schol. 1. If the reflected ray be made the incident one, those cases which are respectively the converse of the foregoing may be demonstrated in the same manner.

Schol. 2. Let it be required to find the distance of the imaginary radiant of diverging rays incident upon a convex surface, whose radius of convexity is 5 parts, and the distance of the real radiant from the surface is 20.

Call the focal distance sought x; then will the distance of the focus from the centre be $5 - x$, and that of the radiant from the same 25; therefore (by Prop. LVI.) we have the following proportion, $x : 20 :: 5 - x : 25$, and $x = \frac{100}{45}$.

If in any case it should happen, that the value of x should be a negative quantity, the focal point must then be taken on the contrary side of the surface.

Plate 7. Fig. 6. * In the case of diverging rays falling upon a convex surface, the farther the point D is taken from B, the nearer the point F, the imaginary radiant of the reflected rays, approaches to B, while the radiant R remains the same. For it is evident from the curvature of a circle, that the point D may be taken so far from B, that the reflected ray DE shall proceed as from F, G, H, or even from B, or from any point between B and R, and the farther it is taken from B, the faster the point, from which it proceeds, approaches toward R. The like is applicable to any of the other cases of diverging or converging rays incident upon a spherical surface. This is the reason that, when rays are considered as reflected from a spherical surface, the distance of the oblique rays from the perpendicular one is taken so small, that it may be supposed to vanish. From hence it follows, that if a number of diverging rays are incident upon the convex surface BD at the several points B, D, D, &c. they shall not proceed after reflection as from any point in the line RB produced, but as from a curve line passing through the several points F, f, f, &c. The same is applicable in all the other cases.

SECTION II.

Of Images produced by Reflection.

PROP. LVII. When any point of an object is seen by reflected light, it appears in the direction of that line which the ray describes after its last reflection.

Since reflection gives the same direction to the rays as if they had originally come from the place from which the reflected rays diverge, an object seen by reflection must appear in that place. The visible image must therefore consist of imaginary radiants diverging from thence.

PROP. LVIII. In all mirrors, plane or spherical, the place of the imaginary radiant, when it is determined, is the intersection of the perpendicular from the radiant to the mirror, and any reflected ray.

Let D be a radiant in any object DE, and DF a ray from this radiant reflected in the line FC; draw DI the perpendicular from the radiant to the mirror, and produce CF, DI, till they meet in L; this point will be the imaginary radiant. Because the ray DI falls perpendicularly upon the mirror, it will be reflected back in the same line ID, and therefore will appear to come from some point in DI produced. And since (from Prop. LVII.) all rays which diverge from the same real radiant before reflection, must diverge from the same imaginary radiant after reflection, any other ray from D, as FC, must appear to diverge from the same imaginary radiant with the ray DI, that is, from some point in DI; but the ray FC (by Prop. LVII.) appears after reflection to proceed in the line FC; it must therefore appear to come from some point in FC, and also from some point in DI, that is, from the point L, in which DI intersects FC. The imaginary radiant of the rays DI, DF, after reflection is therefore L, the intersection of the perpendicular and the reflected ray. Plate 7. Fig 7.

Def. XXI. The *passage of reflection* is the incident ray added to the reflected ray; as DF + FC.

PROP. LIX. In plane mirrors, the distance of the last image from the mirror is equal to the distance of the object from it, and the distance of any point in the last image from the eye is equal to the passage of reflection.

The distance of the imaginary radiants L, M, behind the mirror, are (by Prop. LIV.) respectively equal to the distances of the corresponding real radiants D, E, before the mirror; therefore the distance of the last image L, M, made up of imaginary radiants between L and M, corresponding to real radiants in the object DE, is equal to the distance of that object. The distance of L, the highest point of the image, from C, any given place of the eye, is CFL, equal to DFC the passage of reflection, because (by Prop. LIV.) LF is equal to DF. The same may be shown of M, or any other point in the image. Plate 7, Fig. 7.

PROP. LX. In plane mirrors, the image is equal and similar to the object.

If D be the highest point of the object. the highest point of the image is (by Prop. LVII.) in the perpendicular DIL; and if E be the lowest point of the object, the lowest point of the image is in the perpendicular EZM. But DIL and EZM are parallel (El. XI. 6.) because they are both perpendicular to the plane surface AB. Consequently, the distance between these lines, that is, the heights of the object and image, DE, LM, are equal. In like manner it may be shown, that any diameter of the object is equal to its corresponding diameter in the image; whence the object and image are in all respects equal, and consequently similar surfaces.

CHAPTER. IV.

Of Vision.

SECTION I.

OF THE LAWS OF VISION.

PROP. LXI. When the rays which come from the several points in any object enter the eye, they will paint an inverted image upon the *retina*.

Plate 7. Fig. 10. Let ABA be a section of an eye. AB, BA, is the *tunica sclerotica*, a white coat which encompasses the globe of the eye, except the fore part between A and A. The fore part AA is covered by a transparent coat, a little more protuberant than any other part of the eye called the *tunica cornea*. In the cavity of the eye is placed a convex lens C *e*, consisting of a hard transparent substance, called the *crystalline humour*. This humour is kept in its place and fixed to the coats by certain ligaments all around it at *e e*, called *ligamenta ciliaria*. Under the *tunica cornea*, and at some little distance from it, is the *iris*, *o*, *o*, which has in it a small orifice, called the pupil of the eye. This iris is tinged with a variety of colours, from which the eye is said to be blue, hazel, black, &c. It consists of muscular fibres, which can contract or dilate the pupil. The remaining part between the *cornea* and the crystalline humour is filled with a thin transparent fluid, like water, called the *aqueous humour*. S *c* N is a white coat, which consists of the fibres of the optic nerve woven together like a net; this coat is called the *retina*. Between the *sclerotica* and the *retina* is another coat, which is called the *choroides*. The cavity of the eye, between the crystalline humour and the *retina*, is filled with a transparent substance, neither so fluid as the aqueous humour nor so hard as the crystalline, called the *vitreous humour*. The *optic nerve* MBB is inserted at N.

Mr. Harris has given a table of the dimensions of the human eye, of which the following are the principal particulars.

	In.
Diameter from the cornea to the choroides	.95
Radius of the cornea	.335
Distance of the cornea from the first surface of the crystalline	.106
Radius of the first surface of the crystalline	.331
Radius of the back surface of the crystalline	.25
Thickness of the crystalline	.373

Fig. 11. Through the pupil the rays which diverge from the several points of any object ABC pass into the cavity. The rays emitted from any point of an object beyond the nearest limit of distinct vision, on entering the cornea are rendered converging. The aqueous humour, having about the same density and degree of convexity as the anterior surface of the cornea, probably does not much change their convergency; but after passing through the pupil they are rendered more and more converging at both surfaces of the crystalline humour, and are thus finally thrown upon a single point of the retina. Consequently at DEF, or somewhere upon the retina, as upon a piece of white paper in a dark room, an inverted image of the object (by Prop. XXV.) will be painted.

Schol. The refractive powers of the aqueous and vitreous humours have been found by experiment to be about the same as common water; and that of the crystalline is a little greater; that is, the sine of incidence is to that of refraction, out of air into the aqueous humour, as 4 to 3, out of the aqueous into the crystalline as 13 to 12, and out of the crystalline into the vitreous as 12 to 13.

Exp. Take off the *sclerotica* from the back part of the eye of an ox, or other animal, and place the eye in the hole of the window shutter of a dark room, with its fore part toward the external objects; a person in the room will, through the transparent coat, see the inverted image painted upon the *retina*.

Def. XXII. The optic axis, is the axis of the crystalline humour continued to the object at which we look. The axis PO of the crystalline humour GPH, continued to the point B, is the optic axis directed toward that point.

Def. XXIII. That point of the *retina*, upon which the optic axis continued back would fall, is called the middle of the *retina*. If OP be continued back to E, the point E is the middle of the retina.

PROP. LXII. The images upon the *retina* are the cause of vision.

It is found from experience, that when the image upon the *retina* is bright, the object is clearly seen; and when the image is faint, the object appears faint; also, that when the image is distinct, the object is seen distinctly; and when the image is confused, the object appears confused. Hence it may be concluded, that these images are the cause of vision.

Cor. It is manifest that a different conformation of the eye, or some parts of the eye, is necessary for distinct vision at different distances. Some think the change is in the length of the eye; others, that it is a change in the figure or position of the crystalline humour, and others that it is a change in the cornea. Any of these changes would produce the effect.

PROP. LXIII. The point in any object toward which the optic axis is directed, is seen more distinctly than the rest.

It is known from experience, that when the eye is turned directly toward any part of an object, that is, when the optic axis is directed toward that point, though the whole object, if it be not very large, will be seen, that part on which we look directly will appear most distinct; and the other portions of the object, being drawn on parts of the *retina*, somewhat nearer to the crystalline humour than the middle point of the retina, will appear a little confused.

PROP. LXIV. Objects appear erect, although their images on the *retina* are inverted.

This is known by experience, and is not inconsistent with the explanation above given of vision. For it is not the image on the *retina*, but the object itself, which we see, and we judge of its relative position, by moving the point of distinct vision along the object, and determine that part to be the highest which requires the eye to be the most lifted up in order to see it distinctly.

PROP. LXV. An object may appear single, although it is seen by both eyes at once.

If both eyes are turned directly to the object C, that is, if the optic axes AC, BC, meet in the object, it will appear single. But if, whilst one eye is turned toward C, the corner of the other is pressed with the finger so as to alter the position of its optic axis, the object will then appear double. For when one eye is turned toward an object, and the other turned a different way, the same object will be seen by each eye in a different direction; that is, one eye will see it in one place, and the other in another, from whence it must appear double. But, if both eyes are directed the same way, that is, to the place of the object, though two objects may be said to be seen, yet as both of them are alike, and seen in the same place, they will appear but as one. If whilst both eyes are directed toward C, another object D be placed at some considerable distance directly beyond it, the object D will appear double; for since the eyes see the object D without being turned directly toward it, the place of D is indeterminate; to the right eye it appears on the right hand of C, and to the left eye on the left of C; that is, being seen in two places, it must appear double. If the sight be directed to the farther object D, the nearer object C will appear double. For, the object C is seen by the right eye in the direction of a line which passes on the left side of D, and by the left eye in the direction of a line which passes on the right of D. In both cases, one of the objects appears double, when the eyes are not directly fixed upon it, that is, when the optic axes do not meet in it: and the other object appears single, when the eyes are both directly fixed upon it, that is, when the optic axes meet in the object. Plate 7. Fig. 12

Exp. 1. View a nearer and a more distant object at the same time, according to the proposition.

2. Let a pasteboard, having a hole in it, be fixed between the eyes of a spectator and two candles, so placed, that when the right eye is shut, the left eye may see only the one candle, and when the left eye is shut, the right eye can see only the other; although both candles are visible, if both eyes be fixed steadfastly upon the hole, they will appear as one candle placed at the hole.

PROP. LXVI. There is one part of the *retina* where no perception of the object is conveyed to the mind by the image formed upon it.

This is found by experiment. Place two small circles of white paper upon a dark coloured wall at the height of the eye, and at the distance of near two feet from each other. If the spectator, at a proper distance, shuts his right eye and looks with the left directly at the paper on his right hand, he will not see the left hand paper, although the objects round it are visible. Hence it is to be inferred, that the rays from the left hand paper fall upon a part of the *retina* which is insensible.

Schol. It is supposed that this part of the *retina* is that where the optic nerve is inserted; and because the coat called the choroides touches the *retina* in all other parts, but is discontinued here, it has been conjectured that the seat of vision is not the optic nerve, but the choroides; but this point remains undetermined.

Cor. Hence an object cannot become invisible to both eyes at once; because the image cannot fall upon the optic nerve of each eye at the same time. An object seen with both eyes, is said to appear about $\frac{1}{10}$ or $\frac{1}{12}$ brighter than with one eye alone. Harris's Optics, p. 116.

PROP. LXVII. If the crystalline humour has either too much or too little convexity, the sight will be defective.

In persons who are shortsighted, the humours of the eye are too convex, and bring the rays to a focus before they reach the retina, unless the object be brought near to it; in which case (by Prop. XXVII.) the image is cast farther back. In others, the humours of the eye have so little convexity, that the focal point lies behind the retina; whence, unless the object is removed to a great distance from the eye, the vision will be indistinct.

Plate 7. Fig. 11. Def. XXIV. The *optic angle* in viewing any object is the angle at the eye subtended by the diameter of the object. The angle AOC, subtended by AC, the diame- of the object, is the optic angle.

PROP. LXVIII. The apparent diameter of any object is proportional to the diameter of the image of that object on the *retina*.

Plate 7. Fig. 11. To an eye placed at O, the apparent magnitude of the object ABC is that visible extension which lies between A and C. If two lines AO, CO, are drawn from these points crossing one another at the eye, they will be the axes of the pencils which come from A and C, and will contain between them the diameter AC; and the points A, C, will be represented on the *retina* (by Prop. LXI.) at F and D; consequently, DF will be the diameter of the image; and this diameter is contained between the two lines AO, CO, produced to the retina. Now it is manifest, that the visible extension contained between AO and CO, that is, the apparent diameter of the object, is as the angle AOC; and that the diameter of the image contained between DO and FO is as the angle DOF. But the angles DOF, AOC, (El. I. 15.) are equal. Therefore the apparent diameter of the object, and the diameter of the image, are each of them proportional to the same angle, and consequently proportional to each other.

Schol. 1. When we speak of an optic angle, it is not meant that we see the point in which the optic axes meet; but, since by experience we learn to judge of such distances as are not very great, by the sensations accompanying the different inclinations of the eye, which are analogous to the optic angle, we express these different inclinations of the eye, by that angle. In like manner, although the eye does not see a pencil of rays, whilst the breadth of the pupil bears a sensible proportion to the distance of the focus from which the rays diverge to the eye, we have sensations from which experience enables us to judge of the place of that focus. So, the magnitude of an image upon the retina being always proportional to that of the visual angle of the object, though that angle is not actually measured by the eye, the difference of sensations accompanying different magnitudes of the image enable us to distinguish different visual angles. Thus it appears, that the use of lines and angles in optics, has its foundation in nature.

Schol. 2. The angle subtended by the least visible object, called by the writers on optics the *minimum visibile*, cannot be accurately ascertained, as it depends upon the colour of the object, and the ground upon which it is seen; it depends also upon the eye. Mr. Harris thinks the least angle for any object to be about 40″; and at a medium not less than two minutes.

To the generality of eyes the nearest distance of distinct vision is about 7 or 8 inches. Taking 8 inches for that distance, and 2 minutes for the least visible angle, a globular object of less than $\frac{1}{300}$ part of an inch cannot be seen. Harris's Optics, p. 120—124.

PROP. LXIX. When the diameter of an object is given, its apparent diameter is inversely as its distance from the eye.

The apparent diameter of an object (Prop. LXVIII.) is as the diameter of its image upon the retina; and (Prop. XXXI.) the diameter of the image, when the object is given, is inversely as the distance of the object. Therefore the apparent diameter of the object is also inversely as the distance of the object. The same may be proved of any apparent length whatsoever.

Cor. 1. Hence the apparent diameter of an object may be magnified in any proportion; for the less

the distance of the eye from the object, the greater will be its apparent diameter. But without the help of glasses, an object brought nearer the eye than about five inches, though it appears larger, will at the same time be seen confusedly.

COR. 2. Hence parallel lines, as the sides of a long room, or two rows of trees, as ABC, DEF, seen obliquely, appear to converge more and more, as they are farther extended from the eye; for the apparent magnitude of their perpendicular intervals, as AD, BE, CF, &c. are perpetually diminished. Plate 7. Fig. 15.

COR. 3. An horizontal plane AI seems to ascend. For the visual rays cut a perpendicular DA to the horizon, in points that are higher and higher, or nearer to the horizontal line OG, according as they proceed from points in AI that are more remote from A.

COR. 4. It is for a like reason that a ceiling DH appears to descend, and that faster than the floor ascends, as the distance of the eye of the spectator from the ceiling is greater than the distance of the eye from the floor.

PROP. LXX. The apparent diameter of an object, whose distance is given, is directly as its real diameter.

The apparent diameter of an object (by Prop. LXVIII.) is as the diameter of its image; and the diameter of the image (by Prop. XXXIII.) when the distance of the object is given, is as the diameter of the object. Therefore the apparent diameter of an object, whose distance is given, is as its real diameter.

PROP. LXXI. The apparent diameters of different objects at different distances from the eye will be equal, if their real diameters are as their distances.

For (by Prop. XXXIV.) the diameters of their respective images upon the *retina* will be equal; and their apparent diameters (by Prop. LXVIII.) are as the diameters of their images.

PROP. LXXII. The apparent length of an object, seen obliquely, is as the apparent length of a subtense of the optic angle perpendicular to the optic axis.

If DF be an object seen obliquely, and DG an object seen directly, that is, if DF be oblique, and DG perpendicular to the optic axis OR, then supposing them to subtend the same angle DOF, their images upon the *retina* (Def. XXIV.) will be equal, whence (by Prop. LXVIII.) their apparent diameters will be equal. Consequently, the greater the subtense GD is, the greater will be the apparent length of the object DF; and the reverse. Plate 7. Fig. 13.

COR. Hence an object appears shortened by being seen obliquely.

PROP. LXXIII. When equal objects in the same right line are seen obliquely, their apparent lengths are inversely as the squares of their distances from the eye.

Let the eye be at O; and in the line BC at different distances from the eye take equal spaces DF, df. The apparent length of DF (by Prop. LXXII.) is proportional to the apparent length of GD, the subtense of the optic angle DOF, perpendicular to the optic axis OR. In like manner, the apparent length of df is proportional to that of gd, the subtense of dOf. Now GD, gd, are subtenses also of the angles GFD, gfd; and as the side Of is to the side OF, so is the sine of the angle OFf, that is, of its supplement OFB, to the sine of the angle OfF, or OfB. Hence, since small angles are to one another nearly as their sines, if these are small angles, Of will be to OF, as the angle OFB to the angle OfB; that is, Of will be to OF, as the subtense GD to the subtense gd, or GD is to gd inversely as OF to Of; that is, the subtenses of the optic angles, and consequently, from what has been shown, the apparent diameters DF, df, are inversely as their distances from the eye. This proportion arises from the different degrees of obliquity at which the eye sees the equal spaces DF, df. But if their obliquities with respect to the eye were the same, the apparent length of DF (by Prop. LXIX.) would be to that of df inversely as their distances. Since then the apparent lengths of DF, df, are inversely as their distances on account of their different obliquities, and also inversely as their distances on account of their different distances; on both accounts taken together, they are in the ratio compounded of the inverse ratio of their distances, and the same, that is, inversely as the squares of their distances. Plate 7. Fig. 13.

PROP. LXXIV. The apparent diameter of an object is not changed by contracting or dilating the pupil.

For when the distance is given, the diameter of the image (by Prop. XXXVI. Schol. 2.) remains the

same, whatever be the area of the pupil, and consequently (by Prop. XXXIII.) the apparent diameter of the object.

PROP. LXXV. An object appears larger when it is seen confusedly, than when it is seen distinctly.

For the confused image (by Prop. XXIX.) is larger than the distinct image, and consequently (by Prop. LXVIII.) the apparent magnitude of the object is greater when it is seen confusedly than when it is seen distinctly.

Cor. Hence objects appear magnified when seen through a mist; the drops of which refract the rays so differently, that they cannot be collected into one focus.

PROP. LXXVI. A spectator in motion sees an object at rest, moving the contrary way.

Plate 7. Fig. 14. If while an object at P is at rest, the eye be carried parallel to PQ in the direction from Q toward P, its image on the *retina* will move from p to q; the same effect will be produced, if the object move from P toward Q; and if the velocity of the object and the eye, in each case, be the same, the apparent velocity will be the same also.

Cor. 1. An object moving along a line PK will appear at rest to a spectator moving along the line QG, parallel to PK; if the motion of the object be quicker or slower than that of the spectator, it will have an apparent motion direct or retrograde; if the two motions are in contrary directions, the apparent motion of the object will vary with the real motion of the spectator.

Cor. 2. If the earth be supposed to move round its axis from west to east, while the heavenly bodies are at rest, they appear to us to move the contrary way, there being nothing in this case to indicate to us our own motions. And therefore no argument drawn from the apparent diurnal motions of the stars and planets can be of any support to either the *Ptolemaic* or *Pythagorean* systems.

Schol. A person riding, or walking slowly, though he perceives the change of situation of adjacent bodies, yet being sensible of his own motion, and having time to reflect in the intervals of these apparent changes, those bodies appear to keep their places. But if he runs or rides very swiftly, he cannot help fancying, that the bodies, which he is looking at, are moving toward him. The deception is still stronger when he sits at his ease in a swift sailing vessel.

PROP. LXXVII. The same degree of velocity appears greatest, when the motion is in a line perpendicular to the optic axis; and when the motion is in other directions, the apparent velocity will be as the cosine of the angle of inclination to the said perpendicular.

Plate 7. Fig. 14. If two bodies set out at the same time from P, the one moving along the line PQ, perpendicular to the optic axis Q q, and the other along the line PS, oblique to it, and if their velocities be such, as to pass over the lines PQ, PS, in the same time, it is manifest, that their apparent velocities will be the same; for the images of each will pass over the same space $p\,q$, on the *retina*, in the same time. The real velocities being, in this case, as PQ to PS, it is manifest, that when the velocities are equal, the apparent velocity of the body which moves in PQ is to that of the body moving in PS, as PS to PQ; that is, as radius to the cosine of the angle QPS; but PS is always greater than PQ; whence the proposition is manifest.

PROP. LXXVIII. If objects at different distances from the eye move in parallel lines, nearly at right angles to the optic axis, and if their velocities are proportional to these distances, their apparent velocities will be equal; and if their real velocities are equal, their apparent velocities will be reciprocally as their distances from the eye.

Plate 7. Fig. 14. Let an object move from Q to P, in the same time that another moves from G to H, their real velocities are as QP to GH, that is, (El. VI. 2.) as QO to GO, the distances from the eye; and their apparent velocities will be equal; for the space $q\,p$ upon the *retina* will be passed over in the same time by the image of each. If the velocities of the objects G, Q, be equal, the object G will arrive at K, and its image describe the space $q\,k$ upon the *retina*, in the same time that the image of the object Q describes the space $q\,p$; whence the apparent velocities of these two objects are as $q\,k$ and $q\,p$, or as GK (or QP) and GH; that is, (El. VI. 2.) the apparent velocity of the object G is to that of Q, as QO to OG.

Schol. It is here supposed, that the spectator makes no allowance for the different distances.

PROP. LXXIX. The apparent velocities of bodies moving in parallel lines at different distances from the eye, are directly as the real velocities, and reciprocally as the distances.

Let two bodies move from Q, G, in parallel lines QP, GH; let the velocity of the object Q be called V, and that of G, v; and let their apparent velocities be called A, a. If the two objects be conceived to move in the same line GK, whatever be their velocities, $V : v :: A : a$; and supposing the velocity of the object Q, to be the same in QP as before in GK, A in QP : A in GK :: GO : QO, by Prop. LXXVIII. and A in GK : a :: V : v; whence, compounding these ratios, $A : a :: GO \times V : QO \times v$, that is, $:: \frac{V}{QO} : \frac{v}{GO}$. Plate 7. Fig. 14.

Schol. 1. We judge of the *distance* of any object by the visible length of the plane which lies between the eye and the object. When this method fails us, we compare the known magnitude of the object, with its present apparent magnitude; or we compare the degrees of distinctness with which we see the several parts of an object; or we observe whether the change of the apparent place of an object when viewed from different stations, or its *parallax*, be great or small, this change being always in proportion to the distance of the object. On this principle, we may judge of the distance of a near object by observing the change which is made in its apparent situation, upon viewing it successively, with each eye singly. Or since it is the difference of the apparent place of an object, as viewed by each eye separately, which makes an object to be seen double unless we turn both eyes directly toward it, and since in doing this, where the distance is very small, we turn the eyes very much toward each other, and less at a greater distance; the different sensations accompanying the different degrees in which the eyes are turned toward each other, afford, by habit, a rule for judging of the distance of objects.

Schol. 2. In objects placed at such distances as we are used to, and can readily allow for, we know by experience how much an increase of distance will diminish their apparent *magnitude*, and therefore instantly conceive their real magnitude, and neglecting the apparent, suppose them of the size they would appear if they were less remote; but this can only be done, where we are well acquainted with the real magnitude of the object; in all other cases, we judge of magnitudes by the angle which the object subtends at the known, or supposed distance; that is, we infer the real magnitude from the apparent magnitude in comparison with the distance of the object.

SECTION II.

Of Vision as affected by Refraction.

PROP. LXXX. When any small object is seen through a refracting medium, it appears in the direction of that line which the rays describe after their last refraction.

The ray DE from any small object D, in passing through a glass prism, the end of which is ACB, will be refracted toward a perpendicular, and will describe the line EF; and when it goes out of the prism, it will be refracted from a perpendicular into the line FG, which is its direction after its last refraction; and the object D will be seen in this direction at L instead of D. For the image in the *retina* will be in the place in which it would have been, if the naked eye had been looking at an object really placed at L, the last direction of the rays. Plate 7. Fig. 16.

Cor. Hence an object seen through a glass cut into different surfaces inclined to one another, will appear at one view in many different places. The object A seen from the point F through the glass CEDB, will appear at H, A, G; the last direction of the rays AC, AD, AB, after refraction. Fig. 17.

Exp. View any object through a multiplying glass.

Schol. If a hair be placed across a small hole made in a thin board, and an eye situated, in the dark, look through the hole at a number of candles, the hair will appear multiplied; for a shadow of the hair is cast upon the eye by each of the candles.

PROP. LXXXI. An object seen through a denser medium terminated by plane and parallel surfaces, appears nearer, brighter, and larger, than with the naked eye.

Let AB represent the object, CDEF the medium, and GH the pupil of an eye. Let RK, RL, be two rays proceeding from the point R, and entering the denser medium at K and L; these rays will here by refraction be made to diverge less (by Prop. XVII.) and to proceed afterward, suppose in the lines K a, L b; at a and b, where they pass out of the denser medium, they will be as much refracted the con- Plate 7. Fig. 18.

trary way, proceeding in the lines $o\ c$, $b\ d$, parallel to their first directions (by Prop. XI.); produce these lines back till they meet in e, this will be the apparent place of the point R, and it is evident from the figure, that it must be nearer the eye than that point; and because the same is true of all other pencils flowing from the object AB, the whole will be seen in the situation $f\,g$, nearer to the eye than the line AB. As the rays RK, RL, would not have entered the eye, but have passed by it in the directions K r, L t, had they not been refracted in passing through the medium, the object appears brighter. The rays A h, B i, will be refracted at h and i into the less converging lines $h\ k$, $i\ l$, and at the other surface into k M, l M, parallel to A h aud B i produced (by Prop. XI.), so that the extremities of the object will appear in the lines M k, M l, produced, namely, in f and g, and under as large an angle f M g, as the angle A q B, under which an eye at q would have seen it, had there been no medium interposed to refract the rays; and therefore it appears larger to the eye at GH, being seen through the interposed medium, than otherwise it would have done. But it is here to be observed, that the nearer the point e appears to the eye on account of the refraction of the rays RK, RL, the shorter is the image $f\,g$, because it is terminated by the lines M f and M g, upon which account the object is made to appear less; and therefore the apparent magnitude of an object is not much argumented by being seen through a medium of this form.

Farther, it is apparent from the figure, that the effect of a medium of this form depends wholly upon its thickness; for the distance between the lines R r and $e\ c$, and consequently the distance between the points e and R depends on the length of the line K a; again, the distance between the lines AM and f M, depends on the length of the line $h\ k$, but both K a and $h\ k$ depend on the distance between the surfaces CE and DF, and therefore the effect of this medium depends upon its thickness.

PROP. LXXXII. In viewing objects through a convex or concave lens, the object itself is not seen, but its last image, consisting of all the imaginary radiants from which the rays appear to diverge after the last refraction.

Plate 7. Fig. 8. If AC be an object nearer the convex lens GIL than its principal focus, the rays which diverge from any point B in this object, will in passing through the lens (by Prop. XVII.) be made to diverge less, and the imaginary radiant (by Prop. VI.) will be more distant than the real one. Hence the rays BG, BL, after refraction will appear to diverge from the radiant E, farther from the lens than the real radiant B. The same happening to the rays from every other point of the object, there will be in DEF as many imaginary radiants as there are real radiants in the object ABC, which, taken together, will compose the last image. And since all the rays fall upon the eye as if they had diverged from this last image, the eye will be affected by the object ABC in the same manner in which it would be affected without the lens, by an object in all respects like DEF, that is, the eye perceives the last image.

PROP. LXXXIII. An object seen through a convex or concave lens will appear erect, if the object and its image are on the same side of the lens, but inverted if they are on contrary sides.

Plate 7. Fig. 8, 9. It appears from the last Prop. that all the rays which diverge from B before refraction, will appear to diverge from E after refraction; and the like may be said of any other points, A and D, or C and F. Now AI is the axis of the beam which comes from A, and therefore, with the rest of the rays of the beam after refraction, will appear to diverge from the point D in the same right line with A. The same may be shown of FCI. Now these right lines only cross one another at the lens. Consequently, the highest point both of the object and image is in DI, the upper side of the angle DIF, and the lowest points of both in FI, the lower side of the same angle; that is, the image, which is visible, is erect, or in the same position with the object.

If the object ABC is more remote than the principal focus of the convex lens E, there will be (by Prop. XXIV.) a distinct image formed on the other side of the lens. If the rays thus collected are not received upon a plane surface, they will proceed straight forward; those which had converged diverging from the focus; whence an inverted image will be presented to an eye placed beyond the focus.—In the same manner this, and the preceding proposition, may be proved concerning an object seen through a concave lens.

Exp. 1. Observe the image of a candle whose rays have passed through a convex lens, and are received at the focus on a white surface, whilst the object is on the same, or on the contrary side of the lens.

2. An inverted image will be produced without a lens, by solar rays passing through a very small hole into a darkened room; and if the edge of a knife be applied to one side of the hole, and moved slowly over it, the parts of the image situated opposite to the covered side will be first concealed; from whence it is manifest. that the rays cross one another in passing through the hole.

PROP. LXXXIV. The apparent magnitude of an object seen through a lens placed close to the eye, or to the object, is equal to its apparent magnitude when seen without the lens.

If the eye be placed close to the lens at I, the diameter of the object of refracted vision DF, is to the diameter of the object of plane vision AC (by Prop. XXVIII.) as EL to IB, that is, as their respective distances from the vertex of the lens. Therefore (as appears from Prop. LXIX.) their apparent diameters, when seen from I, are equal. Plate 7. Fig. 8.

If the lens be placed close to the object, the real radiants touching the lens, the imaginary radiants, that is, the last image, will also touch the lens; whence their diameters, or apparent magnitudes will be equal.

PROP. LXXXV. If an object seen through a convex lens is nearer to the lens than its principal focus, it will appear brighter than to the naked eye, distinct, and in an erect position.

In this case, the rays which come from any point (by Prop. XVII.) will be brought nearer by refraction, and consequently a greater number will enter the eye, than if there had been no refraction; whence it is manifest, that the object will appear brighter. Because the rays of each pencil diverge after refraction, but less than before, they come to the eye, as they would if the object were at a moderate distance and no lens were used, and therefore will be seen distinctly. And because the refracted rays (by Prop. XVII.) diverge less than the incident rays, that is, (by Prop. VI.) the imaginary radiants are more remote than the real ones, the last image, as DEF, which is formed by these imaginary radiants, is farther from the lens than the object, and on the same side of the lens; and consequently, since the extreme axes DAI, FCI, only cross one another at the lens, the image will be in the same position with the object and appear erect.

PROP. LXXXVI. If an object seen through a convex lens be farther from the the lens than its principal focus, the object will appear brighter than to the naked eye, confused, and in an erect position.

If the eye be placed between the lens and the distinct image, whilst the eye is nearer the lens than the place of the image, the rays being made convergent by the lens, will be closer together, and therefore a greater number of them will enter the eye, and the object will appear brighter, than if it had been seen without a lens; because the rays come to the eye in a converging state, which from one and the same point they do not in plane vision, they will give a confused image. And because no image is formed till the rays come to the *retina*, the object will appear erect.

PROP. LXXXVII. If an object seen through a convex lens be in the principal focus, it will appear brighter than to the naked eye, and erect.

The rays of each particular beam, becoming in this case (by Prop. XVII.) parallel after refraction, are brought nearer together than if there had been no lens; consequently, more rays will enter the eye from every point, and the object will appear brighter; and no image being formed before the rays come to the *retina*, the object will appear erect.

PROP. D. A minute object, when seen through a lens of *very small* principal focal length, appears magnified and distinct, if the object be placed in the principal focus.

The angle under which the object appears, will be to that which it subtends, when seen by the naked eye; as the distance at which it is viewed by the naked eye distinctly, is to the principal focal length of the lens.

If the principal focal length of the lens be $\frac{1}{30}$ of an inch, and the distance at which the eye can see distinctly be eight inches, it follows that the lens will magnify 240 times in diameter.

PROP. LXXXVIII. When an object, seen through a convex lens, is nearer than the principal focus, it is magnified, unless the lens touches the eye or the object; and as the eye departs from the lens, its apparent magnitude will decrease.

If the object ABC continue in the same place, or do not change its distance BI from the lens, the last image DF will (by Prop. XXVII.) remain at the same distance EI; therefore the real diameter Plate 7. Fig 8.

of DF (by Prop. XXXIII.) wil be invariable, wherever the eye is placed. If the eye be at the vertex of the lens I, the apparent diameters of the last image DEF, seen through the lens, and that of the object ABC, seen with the naked eye, are manifestly the same. Both these apparent diameters decrease as the eye recedes from the lens toward O; but the apparent diameter of the last image decreases in the inverse ratio of OE to IE, and that of the object in the inverse ratio of OB to IB. But OE has a less ratio to IE than OB to IB; for IE and IB are unequal quantities, of which IE, the distance of the image, is always (by Prop. XVII. and VI.) the greater, which are equally increased, but not proportionally; therefore the apparent diameter of the image decreases slower than that of the object, as the eye recedes from the lens. Consequently, when the eye is at any distance from the lens, the last image, or the object of refracted vision, will appear greater than the object seen by the naked eye. As the eye departs from the lens, the apparent magnitude of the object, from what has been said, must continually decrease, till at an infinite distance it vanishes.

PROP. LXXXIX. If an object seen through a convex lens is more remote than the principal focus, and the eye on the other side of the lens is nearer than the place of the image, the object appears magnified, and its apparent magnitude will be inversely as the distance of the eye from the image.

Plate 7. Fig. 9. Suppose the eye at the side of the lens GL; it might successively see both the object and the image without looking through the lens; and in this situation (by Prop. XXVIII.) the real diameter of the object is to that of the image, as their respective distances from the lens or the eye; consequently (by Prop. XXXIV.) their apparent diameters will be equal. Next, suppose the eye close to the image at F, E, or D, the apparent diameter of the image would manifestly be infinite. Also in this situation of the eye, the apparent diameter of the object would be infinite; for, if the eye be at F, the rays from the point C are the only rays collected into the eye, which appear diffused over the whole surface, and would do so if the lens were ever so large; and the same would be true of the points B or A, if the eye were at E or D; that is, the apparent diameter of the object seen through the lens is infinite. Since then the object and the image appear equal when the eye is close to the lens, and both appear infinite when the eye is close to the image, they must have increased equally as the eye was moving from the lens to the image, and their apparent diameters must always have been equal. Hence, the object in every station of the eye, when it does not touch the lens, is magnified. And because the apparent diameter of the object seen through the lens is every where equal to that of the image, and that of the image (by Prop. XXXI.) inversely as the distance of the eye from the picture, the apparent diameter of the object seen through the lens, is inversely as the distance of the eye from the image.

PROP. XC. If an object seen through a convex lens be placed in the principal focus, its apparent magnitude will not be altered by withdrawing the eye from the lens.

Since in this case the rays from the object come parallel to the eye, both the imaginary radiants and the image (by Prop. XVII.) are infinitely distant. Therefore the apparent magnitude of the object cannot be diminished by receding from the imaginary radiants, nor increased by approaching to the image, but will always remain the same.

PROP. XCI. If a convex lens be moved whilst the eye and object remain fixed, the apparent magnitude of the object will increase, till the lens is at the middle point between them, after which it will decrease till the lens reaches the object; provided the eye is never farther from the lens than the place of the image.

When the lens is at either extreme (by Prop. LXXXIV.) the object is not magnified; but between the extremes (by Prop. LXXXVIII. and LXXXIX.) it is magnified; therefore when it is equally distant from the two extremes, it is most magnified, and must increase in its apparent magnitude as the lens moves from the eye toward the middle station, and decrease, as it moves from that middle station toward the object. *

PROP. XCII. The apparent magnitude of an object, seen through a concave lens, decreases as the eye, or the object, departs from the lens.

* The reasoning here appears not perfectly conclusive.

If the eye touches the vertex of the lens I, the apparent diameters of the object and the last image are equal. As the eye recedes from the lens, its distance both from the object ABC and last image DEF increases, and consequently, the apparent magnitude of both decreases. But the distance IE from the last image increases faster than the distance IB from the object, as was shown in Prop. LXXXVIII. Therefore (by Prop. LXIX,) the apparent diameter of the last image, or the object of refracted vision, is diminished as the eye recedes from the lens, more than that of the object seen by the naked eye. —Again, as the object departs from the lens, the image departs with it; whence its visible diameter decreases. * Plate 7. Fig. 9.

PROP. XCIII. When the eye and object are fixed, if a concave lens be moved from the eye, the apparent magnitude of the object will decrease till it reaches the middle point between them, and increases as it moves on toward the object.

When the lens is at each extréme, the apparent magnitude of the object seen through the lens (by Prop. LXXXIV.) is the same as when seen with the naked eye. In all other stations of the lens, the object appears diminished; therefore it must appear most of all diminished when the lens is in the middle station, and it must decrease whilst it is approaching to that station, and increase whilst it is departing from thence toward the object.

EXP. View a candle through a convex or concave lens, in the manner described Prop. XLII. varying the position of the object, or lens, according to the preceding Propositions, from Prop. LXXXIV.

PROP. XCIV. Convex lenses assist the sight of those persons whose eyes are not sufficiently convex, and concave lenses, that of those whose eyes are too convex.

For convex lenses enable the former to bring the rays from objects to a focus nearer to the crystalline than can be done by their eyes; and concave lenses enable the latter to bring the rays to a focus at a greater distance; and thus to produce a distinct image upon the *retina*.

SCHOL. Were there no other use of the science of dioptrics, says Mr. Molyneux, than that of spectacles, the advantage that mankind receives thereby is inferior to no other benefit, not absolutely requisite to the support of life. For as the sight is the most noble and extensive of all our senses;—as we make the most frequent use of our eyes in all the actions and concerns of life, surely that instrument which relieves the eyes when decayed, and supplies their defects, must be estimated as the greatest of all advantages. Forlorn must have been the situation of many young, and almost all old people, before this admirable invention. The same author concludes, that spectacles were first used about the end of the 13th century; and he ascribes the invention to Friar Bacon.

SECTION III.

Of Vision as affected by Reflection.

PROP. XCV. If a plane mirror and the object seen in it are both perpendicular to the horizon, the object appears erect.

The object DE, and the mirror AB, being both perpendicular to the horizon, the lines DI, EZ, in which (by Prop. LVII.) the highest and the lowest points of the object appear, being both perpendicular to the surface AB, are parallel to each other, and do not meet. Therefore the line DI, which is highest at the object, is also highest at the image, and EZ will be the lowest at both; therefore the image is not inverted with respect to the object; and each point of the image LM (by Prop. LIV.) is equally distant from the surface of the mirror with its corresponding point in the object DE; therefore LM, DE, are parallel (El. I. 30.), and since the object is erect, the image will be so too. Plate 7. Fig. 7.

PROP. XCVI. When the object is parallel to a plane mirror, the length or breadth of that part of the mirror, upon which the image appears, is to the length or breadth of the object, as any reflected ray is to the passage of reflection.

* The reasoning here appears not perfectly conclusive.

Plate 7. Fig. 7. If the object DE is parallel to the mirror AB, and the image LM is seen by the eye at C, then FN, the length of that part of the mirror which is taken up by the image, subtends the angle LCM, under which the image appears. For since all the visible length of the image is manifestly included within the angle LCM, there cannot be more of the mirror taken up by that visible length, than is included within the same angle. Now the length of the image LM is equal (by Prop. LX.) to the length of the object DE. And (El. VI. 2.) FN is to LM, as FC to CL, or (by Prop. LIX.) CFD; that is, the length of that part of the mirror which is taken up by the image is to the length of the image, or (by Prop. LX.) the length of the object, as any reflected ray is to the passage of reflection of that ray. In the same manner it may be shown, that the breadth of that part of the mirror taken up by the image, is to the breadth of the object in the same ratio.

Cor. Hence, in plane mirrors the object, and the part of the surface on which it appears, are similar.

PROP. XCVII. If, at a certain distance from the mirror, the whole object cannot be seen by reflection, the whole will become visible either by bringing the eye nearer to the mirror, or removing the object farther from it.

Plate 7. Fig 7. For (by Prop. XCVI.) the less the ratio of the reflected ray is to the passage of reflection, so much the less will be the ratio of the length of that part of the mirror on which the whole object will appear, to the length of the object. If therefore the reflected ray CF decreases by bringing the eye nearer to the mirror, since it is diminished faster than the greater quantity DFC, the passage of reflection, when equal parts are taken from each, the ratio of the reflected ray CF to the passage of reflection DFC (El. V. 8.) diminishing, the ratio of the length of that part of the mirror, on which the whole object will be seen, to the object, is diminished; that is, (the object being given) the length of the part of the mirror on which the whole object is seen, will be diminished; consequently, the whole object, which at a certain distance of the eye from the mirror was not visible, on a nearer approach of the eye may become visible.

If the object DE be removed farther from the glass, the ratio of the reflected ray FC to the passage of reflection DFC will also be diminished, because DFC will be increased whilst FC remains the same; and consequently the length of the part of the mirror, on which the whole image is seen, is diminished, and a less surface of glass is required in order to see the whole image.

PROP. XCVIII. If a spectator sees himself entirely in a plane mirror placed parallel to him, the mirror must be half as long as himself.

When a spectator is looking at himself, the incident ray is his distance from the mirror, and the reflected ray is equal to it, and is the distance of the mirror from him. The passage of reflection is therefore equal to twice his distance from the mirror; and consequently the reflected ray is to the passage of reflection as 1 to 2; whence (by Prop. XCVI.) the length of the glass, in which he can see himself entirely, must be to his own length as 1 to 2, or the length of the glass must be half his own length.

If the mirror be at all shorter than this, the spectator will not be able to see himself, whether he is nearer to the glass or farther from it. If he approaches toward the glass, the object, being himself, approaches as fast as the eye, so that, though (by Prop. XCVII.) he might see more of himself by the approach of the eye, he will see just as much less of himself on account of the approach of the object. In the same manner it may be shown, that if he recedes from the mirror, he will not be able to see himself entirely.

PROP. XCIX. Objects perpendicular to the horizon, seen in a plane mirror parallel to the horizon, appear inverted.

By Prop. LIV. each imaginary radiant is at the same distance behind the mirror, that the real radiant is before it; hence, if the mirror be below the object and the eye, the object will have its lowest part nearest the surface of the mirror, and its highest part farthest from it, and therefore will in this situation appear inverted; if the mirror be above the object and the eye, the object will have its highest part nearest the surface, and its lowest part farthest from it, and therefore will, in this situation also, appear inverted.

PROP. C. If a plane mirror be inclined to the horizon at an angle of forty-five degrees, an object parallel to the horizon will appear erect in the mirror, and an object perpendicular to the horizon will appear parallel to it.

Let the object CD parallel to the horizon, be seen in AB, a mirror so placed as to incline to the horizon in an angle of forty-five degrees. At whatever distance any radiant C is from the mirror, at the same distance (by Prop. LIV.) is the corresponding radiant *c* in the image, or CE will be equal to *c* E. In like manner, DB will be equal to *d* B. Thus every radiant in the image is at the same distance behind the mirror, as the object is before it; whence the image *c d* makes half a right angle *c* BA with the mirror on one side, whilst the object makes with it half a right angle CBA on the other; whence *c*BC is a right angle; that is, the image is perpendicular to the object or horizon, and appears erect in the mirror. Plate 7. Fig. 19.

By making *c d* the object, and CD the image, it may be shown in like manner that when the object is erect, it will appear in the mirror parallel to the horizon.

PROP. CI. If an object be placed between two plane mirrors inclined to one another at any angle, several images may be seen.

Let the object F be placed between the two plane mirrors CB, CA, making with each other the angle BCA. From the object F, draw FD perpendicular to the mirror CA, meeting CA in K, and make KD equal to FK. The image of F will (by Prop. LIV.) appear at D. In like manner, if FG be drawn perpendicular to CB, the object will be seen in G, as far behind the mirror as F is before it. Thus two images of the same object, but of different sides or surfaces of it, will be seen. Plate 7. Fig. 2.

Again, since some of the reflected rays, which diverge from the image D in all directions fall upon the opposite mirror CB, the image D may be considered as an object placed before the mirror CB; and consequently, when the rays which diverge from D are reflected from CB, if DHE be drawn perpendicular to CB, and if EH is taken equal to HD, these reflected rays will (by Prop. LVIII.), represent the image of D at E, as far behind the mirror as D is before it. In like manner, some of the rays from this second image E will fall upon the mirror CA, and the image of E, or a third image of the object, will appear. And thus, as long as the image represented in one mirror is before the other, so long a new image of the last image will be produced. And all these images, beginning from CA, and being successive representations of D, will be images of the side of the object F toward CA. Besides these, there will be another set of images beginning from CB, which will be formed in the same manner, and represent the side of the object toward CB, the first of which will be G, and the second L.

PROP. CII. The images which appear in two plane mirrors inclined to one another, are in the circumference of a circle, the radius of which is the distance of the object from the vertex of the angle contained between the mirrors.

Since FK is equal to KD, KC common, and the angles at K right angles, CF, the distance of the object from the vertex of the angle made by the inclination of the mirrors to one another, is (El. I. 4.) equal to CD. In like manner it may be proved that CE is equal to CD. Therefore CF and CE are equal to one another. Thus all the straight lines drawn from C to G, L, or any other image in the mirrors, may be proved to be equal to CF. Consequently, if C be made the centre of a circle, and CF its radius, the circumference will pass through the points D, E, G, L, and every other image which appears in the mirrors, that is, all the images are in the circumference of a circle, whose radius is the distance of the object from the vertex of the angle made by the inclination of the mirrors to one another. Plate 7 Fig. 20.

PROP. CIII. If two plane mirrors are parallel to one another, and an object is placed between them, innumerable images of that object may be seen in each, standing in a right line.

If the two mirrors CA, CB, were separated at C, so as to be less inclined, or more nearly parallel to one another, the angle of inclination being diminished, it is manifest from Prop. CII. that the number of images will be increased. At the same time the circumference of the circle in which the images are placed will be enlarged, because the vertex C is farther removed from F, or FC is increased. Consequently, if the mirrors CA, CB, are so far separated, that the vertex is infinitely distant, the images become innumerable, and they are placed in a straight line. Plate 7. Fig. 20.

PROP. CIV. In spherical mirrors, concave or convex, when the place of the image is determinate, the object and the image are in the same situation, if they are both on the same side of the centre, and in contrary situations if they are on opposite sides.

Plate 7. Fig. 23. Let AFB be an object placed nearer the concave mirror SGV than its principal focus, and let C be the centre of concavity. The rays from A, F, B, being reflected, will (by Prop. LII.) diverge, and the distances of the corresponding imaginary radiants I, E, M, may be determined by Prop. LVI. The real and imaginary radiants are, in this case, on the same side of C the centre of concavity. Now, the imaginary radiant which corresponds to the real one A, is (by Prop. LVIII.) in CAI the perpendicular drawn from A to the surface: and the same with respect to B, and all the other radiants. And these perpendiculars, and all the rest, being drawn from the centre, do not cross each other but at the centre; consequently they are in the same position with respect to each other at the object and the image, and that which is the highest at the object will be the highest at the image, and the reverse. Since therefore (by Prop. LVIII.) every point of the object appears in its perpendicular at the image, the highest point in the object will appear the highest in the image, and the reverse; that is, the object and image will be in the same situation. In like manner it may be shown, that if the object AFB is placed before a convex mirror, and its image IM is on the same side of the centre, they will be both in the same situation. If the object AFB be farther from the concave mirror SGV than its principal focus, and if M, E, I, be the places of the several foci, to which the rays from A, F, B, (by Prop. L.) will converge, a distinct image of the object will appear upon a paper placed at M, E, I; and if the paper be taken away, and the eye be more remote from the mirror than MEI, the rays will diverge from these foci and become the last image. But, because the extreme perpendiculars ICH, MCK, in which (by Prop. LVIII.) the points A and B will appear, cross each other at the centre C, between the object and image, A, the highest point of the object, will appear at I, the lowest point of the image, and the reverse; that is, the image with respect to the object, will be inverted, or they will be in contrary situations. The same may be shown in like manner with respect to the convex mirror.

Plate 7. Fig. 22.

Fig. 21.

PROP. CV. In spherical mirrors, concave or convex, the diameter of the object is to the diameter of the image, as the distance of the object from the centre, to the distance of the image from the centre, and also as the distance of the object from the surface to the distance of the image from the surface.

Plate 7. Fig. 21. 23. If the eye is any where in the line FG, or that line produced, FG is the optic axis; whence the visible length of the object AB, and also of the image IM, is proportional (as appears from Prop. LXVIII.) to a subtense of the optic angle perpendicular to FG. The visible extensions or lengths of the object and of the image being then perpendicular to the same line FG, are parallel to one another. Hence in all the cases, the angles ACB, ICM, are equal, and also the angles CAB, CIM. Therefore (El. VI. 4.) AB, the visible length of the object, is to MI, that of the image, as AC, the distance of the object from the centre, is to IC, the distance of the image.

Again, since the object AFB consists of real radiants, and the image MEI of imaginary radiants when the rays diverge, and of foci when they converge, after reflection; and since when they diverge, FG, the distance of the object from the surface, is (by Prop. LVI.) to EG, the distance of the image from the surface, as FC, the distance of the object from the centre, is to EC, the distance of the image from the centre: but (by El. VI. 2.) AB is to MI, as FC to EC; therefore AB is likewise to MI, as FG to EG; that is, the diameter of the object is to that of the image, as the distance of the object from the surface to that of the image from the surface.

PROP. CVI. If the eye is close to a convex or concave mirror, the apparent diameter of the object is equal to the apparent diameter of the image.

If the eye is at G, the real diameters of the object AFB, and image MEI are (by Prop. CV.) as their respective distances from the eye. Therefore (by Prop. LXXI.) their apparent diameters will subtend the equal angles AGB, IGM, and will be equal.

PROP. CVII. If the eye is placed in the centre of a concave mirror, it can see nothing in the mirror but its own image.

For the eye, in this situation, is in the place of its own image, and therefore rays will be reflected to it from every point of the surface.

PROP. CVIII. If an object is nearer to a concave mirror than its principal focus, the image appears behind the mirror, farther from the mirror and larger than the object, erect, and distinct.

The object AFB being nearer to the concave mirror than the principal focus, the rays which diverge from each point in the object before the mirror will diverge after reflection (by Prop. L.) less than before from the imaginary radiants M, E, I; whence the image formed by them will (Prop. VI.) be farther from the mirror than the object; and consequently (by Prop. CV.) it will be larger than the object. Because the object is nearer the mirror than its principal focus, it is likewise nearer than the centre; whence (by Prop. CIV.) the image will be erect. Lastly, the image will be seen distinctly, because the rays from it diverge as from objects at a moderate distance. Plate 7. Fig. 23.

PROP. CIX. If an object touch a concave mirror, the image will touch it likewise, and they will be equal.

For the real radiants being close to the mirror, the imaginary radiants are so too; whence (by Prop. CV.) their diameters will be equal.

PROP. CX. If an object is placed in the focus of a concave mirror, the image is at an infinite distance behind the miror, larger than the object, erect, and distinct.

When the object AFB is in the principal focus of the concave mirror SGV, the image is at an infinite distance. It will (by Prop. CV.) be larger than the object, on account of its remoteness. It will be seen erect (by Prop. CIV.) because it is on the same side of the centre with the object. And since the rays of each beam are parallel, it will be seen as distinctly, as the naked eye sees very remote objects.

PROP. CXI. If the object be farther from a concave mirror than its focus, and the eye be nearer than the place of the image, the object will appear confused, behind the mirror, erect, and magnified.

The rays which diverge from A, F, B, in an object more remote from the concave mirror SGV, than its focus, will (by Prop. L.) be collected, and on a surface of white paper form an inverted image. If the eye is any where between B and G, the rays from every radiant are converging when they come from the mirror to the eye, whence it will appear confused, because the eye is not accustomed to see rays in this state. The rays AH, AG, diverging from A, will, after reflection, converge toward I; but if the eye be nearer to the mirror than I, the reflected ray GI will not cross its perpendicular HI in any place before the eye, since they are in a state of convergency; consequently, the apparent place of this, or any other point of the image, will be indeterminate. It will be seen erect, because MEI, the inverted image of the object, will be drawn inverted on the *retina*. Lastly, because the rays of each beam converge after reflection, as HI, GI, they will appear to come, not from points, but from circular spots larger than the points of the object, the image will appear confused, and be larger than the object. Plate 7. Fig. 21.

PROP. CXII. If an object is farther from a concave mirror than its principal focus, and the eye is farther from the mirror than the place of the image, the image appears before the mirror, inverted, and distinct.

Rays coming from M, E, I, an object at a greater distance than the principal focus from the mirror SGV, will (by Prop. L.) converge to AFB, and would paint an inverted image upon a surface of white paper placed there. From thence they will diverge, (the paper being taken away); whence to the eye placed any where beyond AFB, the image will appear inverted. And it will be seen distinctly, because the rays come to the eye diverging, as from an object at a moderate distance. Plate 7. Fig. 21.

SCHOL. 1. The inverted images of objects may be represented in a dark room by a concave mirror which receives rays passing from external objects through a hole in a window-shutter, and collects them into a focus on a surface of white paper.

SCHOL. 2. A concave mirror, collecting the parallel rays of the sun into a focus, will act as a burning glass.

PROP. CXIII. When an object is placed before a convex mirror, its image appears behind the mirror, nearer the mirror and less than the object, distinct, and erect.

If AFB be an object placed before the convex mirror SGV, the rays which before reflection diverge from A, F, B, will (by Prop. LVII.) after reflection diverge from as many radiants I, E, M, behind the mirror, forming the image. And because the reflected rays (by Prop. LII.) diverge more than the incident ones, the image IM (by Prop. VI.) will be nearer the mirror than the real radiants or object AFB; whence (by Prop. CV.) the image will be less than the object. And because the reflected rays come to the eye in a state of divergency, the image will be seen as distinctly as any visible object, seen by Plate 7. Fig. 22.

such diverging rays, at the same distance. Lastly, if the object AFB were at an infinite distance from the mirror, the rays proceeding from any point in it would fall parallel upon the mirror, and therefore would upon reflection form the image in the principal focus, that is, in the middle point between G and C, or on the same side of C with the object; whence (by Prop. CIV.) it would be erect. At any finite distance of the object, the image, being still nearer the surface, must therefore be erect.

PROP. CXIV. When either the eye or the object departs from a convex mirror, the apparent diameter of the image decreases.

Plate 7. Fig. 22. If the object AFB continue in its place, the image IM will (by Prop. LVI.) be always at the same distance from the mirror; and (by Prop. CV.) the real diameter will be invariable; consequently (by Prop. LXIX.) the apparent diameter of the image will be inversely as the distance of the eye.

If the object AFB depart from the mirror, the ratio of FG, the distance of the object, to GE, that of the image, (El. V. 8.) will increase; whence (by Prop. CV.) the ratio of the diameter of the object to that of the image will likewise increase; that is, the image will become less with respect to the object; but the eye remaining in the same place, the apparent diameter of the image will (by Prop. LXX.) be as its real diameter; consequently, the apparent diameter will decrease.

Exp. The Propositions in this section may be confirmed, by placing an object before a Plane, Concave, or Convex Mirror, according to the terms of their respective propositions.

CHAPTER. V.

Of Colours.

SECTION. I.

OF THE DIFFERENT REFRANGIBILITY OF LIGHT.

Def. XXV. Rays of light are *differently refrangible*, when at the same or equal angles of incidence, some are more turned out of the way than others.

Def. XXVI. Rays are *differently reflexible*, when some are more easily reflected than others.

Def. XXVII. Light is called *homogeneous*, when all the rays are equally refrangible; and *heterogeneous*, when some rays are more refrangible than others.

Def. XXVIII. The *Colours* of homogeneous rays are called *primary* or *simple* colours; those of heterogeneous, *secondary* or *mixed*.

PROP. CXV. The rays of the sun are not all equally refrangible; and those rays which have a different degree of refrangibility, have likewise a different colour.

Plate 7. Fig. 24. If a beam of light SF from the sun pass into a dark room through F, a round hole in a window-shutter EG, and be received upon a white surface, a white round image will be seen. If a glass prism ABC be so placed as to receive the beam of light, the rays of this beam, from their refraction in passing through the prism, will be turned upward, and the refracted image PT will be oblong, having its breadth equal to the diameter of the circular picture O. If all the rays were equally refracted upward, it is manifest that such a refraction would not change the form of the picture. Since therefore the refracted image is oblong, it must be formed by rays differently refrangible, which fall with equal angles of obliquity upon BC, the first side of the prism, but are some of them, in refraction, turned more out of the way than others; those rays which go to P, the upper part of the image, being most refrangible, and those which go to T, the lower part, being least refrangible.

This oblong image is of different colours in different parts, the whole image being made up of rays of seven different colours, in the following order, beginning with those which are most refrangible;

violet, indigo, blue, green, yellow, orange, red. This refracted picture consists of several round pictures so near each other, that each higher circle mixes in part with that below it, whence the colours near the the upper and lower edge of each circle are blended. The sides of these circles being very near to each other appear like right lines.

EXP. 1. Observe the prismatic image formed by the refraction of the rays passing through a single prism.

2. To separate the several colours as much as possible, make the hole F in the window-shutter very small, and collect the rays which pass through it, into a focus L, by a convex lens MN. Let the rays which have passed through the lens be now received upon a prism placed near the lens; the rays will be refracted upward into an oblong image. Plate 7. Fig. 27.

3. To prove that the prismatic image is produced by the different refrangibility of the rays, and by no other cause, let a second prism DH be placed beyond the first *a b c*, at right angles to it. The rays passing through this second prism are refracted sideways; those which were most refracted upward by the first prism, are most refracted sideways by the second; but, the rays not being spread in breadth, the image remains of the same form. Plate 7. Fig. 25.

PROP. CXVI. Those rays of light, which are most refrangible, are also most reflexible.

If the beam of light passing through F fall upon a prism ABC, whose sides AC, AB, are equal, and the angle at A a right angle; when the obliquity of these rays, as they are to pass out of the prism at its base BC, is less than 40 degrees (by Cor. 3. Prop. XIII.), the greater part of the beam will pass through, but some rays will be reflected at the surface BC. Those rays which pass through the base (by Prop. CXV.) form an oblong coloured image at HG, between the most refrangible ray MH, and the least refrangible ray MG. If the rays which are reflected from M are made to pass through another prism XYV, they will also form a faint oblong coloured image *p t*. Now, if the prism ACB be turned slowly round upon its axis in the direction ACB, the obliquity of the rays FM to the base BC, will continually increase till all the rays will be reflected at M. Consequently, the image *p t* will become much brighter than before. And this total reflection will not be produced at once, but the most refrangible rays MH will be first entirely reflected; for the violet colour in HG will first disappear, and the same colour at *p* will first become brighter. In like manner, as the prism ABC is turned round, each different sort of ray will be reflected sooner, as it has a greater degree of refrangibility. Hence it appears, that the rays of the sun have different degrees of reflexibility, and that those which are most refrangible are also most reflexible. Plate 7. Fig. 26.

PROP. CXVII. Homogeneous light is refracted regularly without any dilatation of the rays.

EXP. When the rays of any colour in the oblong image, as green, are separated from the rest, in the manner described in Prop. CXV. if some of these rays are transmitted through a small hole in a thin board, and refracted by a prism placed on the other side, the image formed by these rays after refraction will not be oblong, but circular.

PROP. CXVIII. The confused appearance of objects seen through refracting bodies is owing to the different refrangibility of light.

EXP. Small objects placed in a sun-beam and viewed through a prism, will be seen confusedly; but if they are placed in a beam of homogeneous light, separated by a prism, they will appear as distinct through a prism, as when viewed by the naked eye.

SCHOL. 1. Although the 13th Proposition (in which it was shown that when a ray of the sun is passing out of one medium into another, the ratio of the sine of incidence to the refracted sine will not be changed by changing the obliquity of the incident ray) proceeds upon the supposition, that all rays are equally refrangible, and therefore is not exactly true; the demonstration is strictly applicable to any one sort of rays, as the red ones, which are equally refrangible.

SCHOL. 2. Since, all other circumstances being equal, the same cause, namely, the passing of the rays out of one given medium into another, will turn the violet rays more out of the way than the red rays; the attracting force which acts upon both being the same, it is probable that any single ray of the least refrangible sort contains a greater quantity of matter than any single ray of the most refrangible sort.

PROP. CXIX. The colours of homogeneous light can neither be changed by refraction nor reflection.

Exp. 1. Let a beam of homogeneous light pass through a round hole in a pasteboard, and then be refracted by a prism on the other side; the colour of the rays will remain the same.

2. Red lead, viewed in homogenous red light, will be red, but if placed in green, or any other homogeneous light, it will take the colour of the rays which fall upon it.

PROP. CXX. The whiteness of the sun's light arises from a due mixture of all the primary colours.

Plate 7. Fig. 28. Exp. If the oblong picture PT fall upon the convex lens MN, the rays, which were separated at PT, will, by passing through the lens, be collected into a focus at G, and form a round image of the sun upon a piece of paper DE. This image, formed of all the primary sorts of rays, is white. That the whiteness of the image is owing to the due mixture of all the sorts of rays, appears from hence, that, if any of the colours be intercepted at the lens, the image loses its whiteness. The paper being removed from DE to *d e*, the rays, having crossed at G, will form the prismatic image *t p*, inverted, but distinct; from whence it appears, that the colours are not changed by being mixed at the focus.

PROP. CXXI. The colours of all bodies are either the simple colours of homogeneous light, or such compound colours as arise from a mixture of homogeneous light.

Each sort of light having a peculiar colour of its own, which no refraction or reflection can alter, since bodies appear coloured only by reflected light, their colours can be no other than the colour of some single homogeneous light, or of a mixture of different sorts of light.

PROP. CXXII. Water, air, glass, or any other transparent substance, when drawn into thin plates, becomes coloured.

Exp. 1. If a soap-bubble be blown up, and set under a glass that the motion of the air may not affect it, as the water glides down the sides and the top grows thinner, several colours will successively appear at the top, and spread themselves from thence in rings down the side of the bubble, till they vanish in the same order in which they appeared. At last a black spot appears at the top, and spreads till the bubble bursts.

2. If a piece of plane polished glass is placed upon the object glass of a long telescope, and the interval between them is filled up with water, as the glasses are pressed together the same colours arise at the point of contact, and spread themselves in circular rings round that point in the same order as in the soap-bubble.

3. A convex and concave lens, of nearly the same curvature, being pressed closely together, exhibit rings of colours about the point where they touch. Between the colours there are dark rings, and when the glasses are very much compressed, the central spot is dark. Sir I. Newton found the thickness of the air between the glasses, where the colours apeared, to be as 1, 3, 5, 7, &c. and the thickness where the dark rings appeared, to be as 0, 2, 4, 6, 8, &c. The coloured rings must have appeared from the reflection of the light; and the dark rings from the transmission of it. Consequently, the rays were transmitted when the thickness of the air was 0, 2, 4, 6, 8, &c. and reflected at the thicknesses 1, 3, 5, 7, &c. Sir I. Newton, therefore, supposed, that every ray of light, in its passage through any refracting surface, is put into a certain state, which, in the progress of the ray, returns at equal intervals, and disposes the ray, at every return, to be easily transmitted through the next refracting surface, and between the returns, to be easily reflected by it. These he calls fits of easy transmission and reflection. See Schol. Prop. XLVI.

4. Two pieces of plate glass wiped clean, and rubbed together, will soon adhere with a considerable force, and exhibit various ranges of colours. One of the most remarkable circumstances attending this experiment, is the facility with which the colours may be removed, or even made to disappear, by heats too low to separate the glasses. A touch of the finger immediately causes the irregular rings of colours to contract toward the centre, in the part touched.

Cor. 1. From these experiments it appears plain, that the colours of bodies depend, in some degree, upon the thickness and density of the particles that compose them.

Cor. 2. Hence, if the density, or size of the particles in the surface of a body be changed, the colour is likewise changed.

Schol. 1. When the thickness of the particles of a body is such, that one sort of colour is reflected, other colours will be transmitted, and therefore the body will appear of the first colour. And, in general, a less thickness is found to be nescessary to reflect the most refrangible rays, as violet and indigo, than those which are least refrangible, as red and orange.

Schol. 2. Sir I. Newton, from a great variety of experiments on light and colours, concludes that

every substance in nature is transparent, provided it be made sufficiently thin. Gold, when reduced into thin leaves, transmits a bluish-green light. If we suppose any body, therefore, as gold, for instance, to be divided into a vast number of plates, so thin as to be almost perfectly transparent; it is evident, that all, or the greater part of the rays will pass through the upper plates, and when they lose their force, will be reflected from the under ones. They will then have the same number of plates to pass through, which they had penetrated before; and thus, according to the number of those plates, through which they are obliged to pass, the object appears of one colour or another, just as the rings of colour appeared in Exp. 3. according to the distance of the glasses, or the thickness of the plates of air between them.

The philosopher who has of late years most distinguished himself on the subject of light and colours is Mr. Delaval, who, by a great variety of well conducted experiments, has shown that colours are exhibited, not by reflected, but by transmitted light. This he proved by covering coloured glass, and other transparent coloured media, on the further surface, with some substance perfectly opaque, when he found that they reflected no colour, but appeared perfectly black.

He concludes, therefore, that, as the fibres of mineral and animal substances are found, when cleared of heterogeneous matters, to be perfectly white, the rays of light are reflected from these white particles, through the coloured media with which they are covered; that these media serve to intercept and impede certain rays in their passage through them, while a free passge being left to others, they exhibit, according to these circumstances, different colours.

Mr. Delaval instituted other experiments with coloured fluids put into phials of flint glass, in the form of a parallelopiped. The bottom, and three sides of each phial, were covered with a black varnish, the neck and the front being left uncovered. On exposing them to the incident light, he found that from the parts of the phials which were covered, no light was reflected, but it was perfectly black, while the light transmitted through the uncoated parts of the phials, was of different colours. The same fluids spread thinly on a white ground, exhibited their proper colours; the light, indeed, being in this case reflected from a white ground, and transmitted through a coloured medium.

From these, and many other experiments, Mr. Delaval concludes, (1.) That the colouring particles do not reflect any light. (2.) That a medium, such as Sir I. Newton describes, is diffused over the anterior and further surfaces of the plates, whereby objects are reflected equally and regularly, as in a mirror.

Our author next considers the colouring particles themselves, unmixed with other media. For this purpose, he reduced several transparent coloured lipuors to a solid consistence by evaporation with a gentle heat, which does not injure the colouring matter; and in this state also the colouring particles reflect no light, but are perfectly black.

To determine the principle on which opaque bodies appear coloured, it must be recollected, first, that all the coloured liquors appeared such only by transmitted light; and secondly, that these liquors spread thinly upon white ground, exhibited their respective colours; he therefore concludes, that all coloured bodies, which are not transparent, consist of a substratum of some white subtance, which is thinly covered with the colouring particles.

On extracting, by means of spirits of wine, the colouring matter from the leaves, wood, and other parts of vegetables, he found that the basis was a substance perfectly white. He also extracted the colouring matter from different animal substances, as flesh, feathers, &c. when the same conclusion was obtained. Flesh consists of fibrous vessels, containing blood, and is perfectly white when divested of blood by ablution, and the red colour proceeds from the light which is reflected from the white fibrous substance, through the red transparent covering formed by the blood. The result was the same from an examination of the mineral kingdom.

Some portions of light are reflected from every surface of a body, or from every different medium into which it enters. Thus transparent bodies reduced to powder, and water in the shape of froth, appear white, which is no other than a copious reflection of light from all the surfaces of the minute parts, and from the air interposed between them.

For a full investigation of this curious and interesting subject, the reader must be referred to the Memoirs of the Manchester Society, vol. ii.

SECTION II.

Of the Rainbow.

PROP. CXXIII. When the rays of the sun fall upon a drop of rain and enter into it, some of them, after one reflection and two refractions, may come to the eye of a spectator, whose back is toward the sun, and his face toward the drop.

Plate 8. Fig. 1. If the sun shine upon XY, a drop of rain, in any lines SF, SD, SA, &c. the greater part of the rays enter the drop, and passing on to the second surface, will be transmitted through the drop. But at PG in the second surface some few rays will be reflected, and proceed in some such lines as NR, NQ; and coming out of the drop in the lines RV, QT, they may fall upon the eye of a spectator, placed in those lines with his face toward the drop. These rays are refracted when they enter the drop, reflected from the second surface, and again refracted when they come out of the drop.

Def. XXIX. When rays of light reflected from a drop of rain come to the eye, those rays, which excite a perception of light, are called *effectual.*

PROP. CXXIV. When rays of light come out of a drop of rain, they will not be effectual, unless they are parallel and contiguous.

Most of the rays, which enter the drop between X and A, passing out of the hinder surface between P and G, only a few rays are reflected, and come out of the drop through the nearer surfaces between A and Y. None of these, only the rays which are parallel to one another will be effectual, because if they diverge, they will be so far asunder when they come to the eye, that only a very few of them can enter the pupil, and no perception of colours will be excited. Also unless several parallel rays be very near each other, the rays will be too few to create any perception.

PROP. CXXV. When rays of light come out of a drop of rain after one reflection, those will be effectual which are reflected from the same point, and enter the drop near one another.

Plate 8. Fig. 2. Any rays AB, CD, when they have passed out of the air into a drop of water, will be refracted toward the perpendiculars BL, DL, (by Prop. XI.) And as the ray AB falls farther from the axis than the ray CD, AB will be more refracted than CD; so that these rays, though parallel to one another at their incidence, may describe the lines BE, DE, after refraction, and be both of them reflected from one and the same point E. Now all rays which are thus reflected from one and the same point, when they have described the lines EF, EG, and after reflection emerge at F and G, will be so refracted, when they pass out of the drop into the air, as to describe the lines FH, GI, parallel to one another. If these rays were to return from E in the lines EB, ED, and were to emerge at B and D, they would be refracted into the lines of their incidence BA, DC, (by Prop. XII.) But if these rays, instead of being returned in the lines EB, ED, are reflected from the same point E, in the lines EG, EF, the lines of reflection EG and EF, will be inclined both to one another and to the surface of the drop, just as much as the lines EB and ED are. First, EB and EG make just the same angle with the surface of the drop; for the angle BEX, which EB makes with the surface of the drop, is the complement of incidence, and the angle GEY, which EG makes with the surface is the complement of reflection; and these two are equal to one another, by Prop. XLV. In the same manner we might prove that ED and EF make equal angles with the surface of the drop. Secondly, the angle BED is equal to the angle FEG, or the reflected rays EG, EF, and the incident rays BE, DE, are equally inclined to each other. For the angle of incidence BEL is equal to the angle of reflection GEL, and the angle of incidence DEL is equal to the angle of reflection FEL, by Prop. XII. Consequently, the difference between the angles of incidence is equal to the difference between the angles of reflection, or BEL — DEL is equal to GEL — FEL, or BED, to GEF. Since therefore either the lines EG, EF, or the lines EB, ED, are equally inclined both to one another and to the surface of the drop, the rays will be refracted in the same manner, whether they were to return in the lines EB, ED, or are reflected in the lines EG, EF. But if they were to return in the lines EB, ED, the refraction, when they emerge at B and D, would make them parallel. Therefore if they are reflected from one and the same point E in the lines EG, EF, the refraction, when they emerge at G and F, will likewise make them parallel.

Farther, in order to render the rays which emerge at F and G effectual, they must not only emerge in a direction parallel to each other, but must enter the drop nearly at the same place.

Plate 8. Fig. 1. Let XY be a drop of rain, AG the axis or diameter of the drop, and SA a ray of light, that comes from the sun and enters the drop at the point A. This ray SA, because it is perpendicular to both the

surfaces, will pass straight through the drop in the line AGH without being refracted, (by Prop. XIV.) But any collateral rays that fall about SB, as they pass through the drop, will be made to converge to their axis, and passing out at N will meet the axis at H, (by Prop. XI.) Rays which fall farther from the axis than SB, such as those which fall about SC, will likewise be made to converge; but then their focus will be nearer to the drop than H, (by Prop. XI.) Suppose therefore I to be the focus to which the rays that fall about SC will converge; any ray SC, when it has described the line CO within the drop, and is tending to the focus I, will pass out of the drop at the point O. The rays, that fall upon the drop about SD more remote still from the axis, will converge to a focus still nearer than I, suppose at K, (by Prop. XXI. note.) These rays therefore go out of the drop at P. The rays, that fall still more remote from the axis, as SE, will converge to a focus nearer than K, as suppose at L; and the ray SE, when it has described the line EO within the drop, and is tending to L, will pass out at the point O. The rays that fall still more remote from the axis, will converge to a focus still nearer. Thus the ray SF will, after refraction, converge to a focus at M, which is nearer than L, and having described the line FN within the drop, it will pass out at the point N. Now here we may observe, that as any rays SB or SC fall farther above the axis SA, the points N, or O, where they pass out behind the drop will be farther above G, or, that as the incident ray rises from the axis SA, the arc GNO increases, till we come to some ray SD, which passes out of the drop at P, and this is the highest point where any ray, that falls upon the side AX, can pass out; for any rays SE, or SF, that fall higher than SD, will not pass out in any point above P, but at the points O, or N, which are below it. Consequently though the arc GNOP increases, whilst the distance of the incident ray from the axis SA increases, till we come to the ray SD; yet afterward the higher the ray falls above the axis SA, this arc PONG will decrease.

As there are many rays which pass out of the drop between G and P, so (by Prop. XLIII.) some few rays will be reflected from thence; and consequently, the several points between G and P, which are the points where some of the rays pass out of the drop, are likewise the points of reflection for the rest which do not pass out. Therefore in respect of those rays which are reflected, we may call GP the arc of reflection, and may say that this arc of reflection increases, as the distance of the incident ray from the axis SA increases, till we come to the ray SD; the arc of reflection is GN for the ray SB, it is GO for the ray SC, and GP for the ray SD. But after this, as the distance of the incident ray from the axis SA increases, the arc of reflection decreases; for OG, less than PG, is the arc of reflection for the ray SE, and NG is the arc of reflection for the ray SF.

From hence it is obvious, that some one ray, which falls above SD, may be reflected from the same point with some other ray, which falls below SD. Thus, for instance, the ray SB will be reflected from the point N, and the ray SF will be reflected from the same point; and consequently, when the reflected rays NR, NQ, are refracted as they pass out of the drop at R and Q, they will be parallel, by what has been shown in the former part of this Prop. But since the intermediate rays which enter the drop between SF and SB, are not reflected from the same point N, these two rays alone will be parallel to one another when they come out of the drop, and the intermediate rays will not be parallel to them. And consequently these rays, RV, QT, though they are parallel, after they emerge at R and Q, will not be contiguous, and for that reason will not be effectual, (by Prop. CXXIV.) The ray SD is reflected from P, which has been shown to be the limit of the arc of reflection; such rays, as fall just above SD and just below SD, will be reflected from nearly the same point P, as appears from what has been already shown. These rays therefore will be parallel, because they are reflected from the same point P; and they will likewise be contiguous, because all of them enter the drop at one and the same place, very near to D. Consequently such rays, as enter the drop at D and are reflected from P, the limit of the arc of reflection, will be effectual (by Prop. CXXIV.), since when they emerge at the part of the drop between A and Y, they will be both parallel and contiguous.

PROP. CXXVI. When rays which are effectual emerge from a drop of rain after one reflection and two refractions, those which are most refrangible will, at their emersion, make a less angle with the incident rays than those do which are least refrangible; by which means, the rays of different colours will be separated from one another.

Let FH, GI, be effectual violet rays emerging from the same drop at F, G; and FN, GP, effectual red rays emerging from the same drop at the same points. The violet rays (by Prop. CXXIV.) are parallel among themselves, because they are effectual; for the same reason the red rays are parallel among themselves; but on account of the difference of refrangibility of the violet and red rays, the violet ray GI is not parallel to the red ray GP, but they diverge from the point G; and so of the rest. Both the violet ray GI and the red ray GP are refracted from the perpendicular LO, but (by Prop. Plate 8. Fig. 2.

CXV.) GI more than GP; whence the angle IGO is greater than the angle PGO. If the incident ray AB be continued in the direction ABK, and if IG and PG be continued backward till they meet AB in K and W, the angle IKA is that which the violet or most refrangible ray makes at its emersion with the incident ray, and PWA that which the red or least refrangible ray makes with the same. And the angle IKA (El. I. 16.) is less than the exterior angle PWA. The same may be proved concerning the rays FH, FN, or any other rays which emerge respectively parallel to GI, and GP. But (by Prop. CXXIV.) all the effectual violet rays are parallel to GI, and all the effectual red rays are parallel to GP. Therefore the effectual violet rays at their emersion make a less angle with the incident rays than the effectual red rays. And universally the more refrangible rays, at their emersion, make a less angle with the incident rays, than those which are less refrangible. And since the effectual rays GI, GP, of different colours make different angles with the incident ray AK at their emersion, they will be separated from one another; so that if the eye were placed in the beam FGHI, it would receive only rays of one colour from the drop XY, and in FGNP only rays of another colour.

Schol. The angle which the effectual red rays make with the incident rays is found to be 42° 2′, that of the violet rays 40° 17′.

Exp. Let a glass glass globe filled with water be exposed to the rays of the sun; let the eye of the spectator be so situated, that the least refracted ray from the drop, coming to the eye, shall make an angle of about 42° with the line passing through the eye and the sun, the red rays only will be seen; if the place of the eye be changed so as to enlarge this angle, the red will disappear; but if the angle be lessened, the colours of the more refrangible rays will appear.

PROP. CXXVII. If a line is supposed to be drawn from the centre of the sun through the eye of the spectator, the angle which any effectual ray after two refractions and one reflection makes with the incident ray, will be equal to the angle which it makes with that line.

Plate 8. Fig. 2. Let I be the place of the eye of the spectator; QT a line drawn from the centre of the sun through the eye; and AB a ray coming from the centre of the sun. These two lines AB, QT, on account of the great distance of the sun, may be looked upon as parallel. Therefore (El. I. 29.) the alternate angles AKI, KIT, or GIT, are equal.

PROP. CXXVIII. When the sun shines upon the drops of rain as they are falling, the rays which come from those drops to the eye of the spectator, after one reflection and two refractions, produce the innermost or primary rainbow.

Plate 8. Fig. 3. Let TFY be the innermost or primary rainbow, the outer part of which TFY is red, the inner part VDX violet, and the intermediate parts, reckoning from the red to the violet, orange, yellow, green, blue, indigo. Suppose the spectator's eye at A; and let AI be an imaginary line from the centre of the sun to the eye of the spectator. If a beam of light S coming from the sun falls upon any drop F, and the effectual rays which emerge at F make an angle FAI of 42° 2′ with the line AI, these rays (by Prop. CXXVII.) make the same angle with the incident rays, and consequently are red. Hence the drop F will appear red; for all the other rays which emerge from F, and would be effectual if they fell upon the eye, being refracted more than the red rays, will pass above the eye. If another beam of light S falls upon the drop D, and the effectual rays emerging at H make an angle of 40° 17′ with the incident rays, the drop D will be of a violet colour; for all the other rays which emerge from H, and would be effectual if they came to the eye, being refracted less than the violet rays, will pass below the eye. The intermediate drops between F and D will for the same reasons be of the intermediate colours. And that which has been proved concerning the drops in the line FD, may be shown of any other set of drops in which the angles made by the emerging and incident rays are equal. Thus, wherever a drop of rain is placed, if the angle which the effectual rays make with AI is equal to the angle FAI, or is 42° 2′, any such drop will appear red. If FAI was turned round upon the line AI, so that one end of this line should always be at the eye, and the other at I opposite to the sun, in this revolution the drop F would describe a circle, of which I would be the centre, and TFY an arc. And since in this revolution the angle FAI continues the same, if the sun were to shine upon this drop as it revolves, the effectual rays (by Prop. CXXVII.) would make the same angle with the incident rays in whatever part of the arc TFY the drop may happen to be; and consequently in whatever part of the arc the drop F is, it will appear red. Now as innumerable drops are falling at once in right lines from the cloud, whilst one drop is at F, there will be others at T, Y, and every other part of the arc, which will appear red in the same manner that F would have done in the supposed circular revolution. Therefore, when the sun shines upon the rain, there will be a red arc TFY produced opposite to the sun. In like manner a violet arc VDX will be produced, and other intermediate arcs of the several intermediate colours, which will together make up the primary rainbow.

SCHOL. Cascades and fountains, whose waters, in their fall, are divided into drops, will exhibit rainbows to a spectator, properly situated, during the time of the sun's shining. This appearance is also seen by moonlight, though seldom sufficiently vivid to render the colours distinguishable. Dr. Gregory, in his excellent Economy of Nature, says, that he once saw a lunar bow; it was in autumn, the night was uncommonly light but showery, and the colours much more vivid than he could have conceived. There were not so many colours distinguishable as in the solar bow. Coloured bows have been seen on grass formed by the refraction of the sun's rays in the morning dew.

Artificial rainbows may be produced by candle light on the drops of water ejected by a small fountain, or *jet d' eau*, or from the stream emitted from an æolipile. But the most natural and pleasing is by means of the air fountain, the jet of which is perforated with a great number of very fine holes, from which the water spouts so as to form a kind of fluted column. The rainbow is formed by the sun's rays; for the spectator has only to place the spouting streams directly in the sun's beams, with his own back to the sun, and being in a direct line with the sun and centre of the jet, by stooping his head to a certain degree, he will discover the beautiful appearance of the natural prismatic colours, and a small rainbow, on the same principle as those which are seen in the time of rain and sunshine.

PROP. CXXIX. The primary rainbow is never a greater arc than a semicircle.

Since the line AI is drawn from the sun through the eye of the spectator, and through I the centre of the rainbow, this centre is always opposite to the sun. And since the angle FAI is an angle of 42° 2′, F, the highest part of the bow, is 42° 2′ from I, its centre. If therefore the sun is more than 42° 2′ above the horizon, I, which is opposite to it, must be more than 42° 2′ below the horizon, and no primary rainbow will be seen. As much as the altitude of the sun is less than 42° 2′, so much will the highest point F of the rainbow be above the horizon; and when the sun is in the horizon, I, the centre of the bow, will also be in the horizon on the opposite side, and half the circle will be visible; but when the sun is set, no bow can be seen. Plate 8. Fig. 3.

PROP. CXXX. When the rays of the sun fall upon a drop of rain, some of them after two reflections and two refractions may come to the eye of a spectator, who has his back toward the sun and his face toward the drop.

If parallel rays from the sun, ZV, YW, fall upon the lower part of the drop of rain BGW, they will be refracted toward the perpendicular VL, WL, in entering the drop, and proceed in the direction VH, WI. At HI some part of these rays will (by Prop. XLIII.) be reflected into the directions HF, IG. And some of these rays will be again reflected at F, G, into the directions FD, GB; which rays, when they emerge out of the drop at B and D, will be refracted from the perpendiculars, and may come to the eye of a spectator whose back is toward the sun and his face toward the drop. Plate 8. Fig. 5.

PROP. CXXXI. Those rays which are parallel to one another after they have been once refracted and once reflected in a drop of rain, will be effectual when they emerge after two refractions and two reflections.

The contiguous rays ZV, YW, being refracted toward the perpendiculars VL, WL, when they enter the drop, will (by Prop. XVIII.) become convergent; and because these rays fall upon the drop very obliquely, their focus will not be far from the surface VW. If this focus be at K, the rays, after they have passed the focus, will diverge from thence in the directions KH, KI; and if KI be the focal distance of the concave reflecting surface HI, the reflected rays HF, IG, (by Prop. L.) will be parallel. These rays are reflected again from the concave surface FG, and will meet in a focus at E, so that GE will be the focal distance of this reflecting surface; and because HI, FG, are parts of the same sphere, the focal distances GE, KI, are equal. When the rays have passed the focus E, they will diverge in the lines EB, ED. Now, if the rays VK, WK, when they have met at K, were to be turned back in the directions KV, KW, on emerging at V and W, they would (by Prop. XX.) be refracted into the lines of incidence, and become parallel. But since GE is equal to IK, the rays ED, EB, which diverge from E, fall in the same manner upon the drop at D and B, as the rays KV, KW, would fall upon it at V and W, and ED, EB, have the same inclination to the refracting surface DB, as KV, KW, would have to VW; whence the rays ED, EB, emerging at D and B, will be refracted in the same manner, and will have the same situation with respect to one another, as KV, KW, would have, that is, will be parallel to one another; having been contiguous before their entrance into the drop, they will therefore (by Prop. CXXIV.) be effectual. Plate 8. Fig. 5.

PROP. CXXXII. When effectual rays emerge from a drop of rain after two reflections and two refractions, those which are most refrangible will at their emersion

make a greater angle with the incident rays than the least refrangible will make with them; by which means the rays of different colours will be separated.

Plate 8. Fig. 5 Let BM, BA, a violet and a red ray, emerge from B; the angle which the violet ray BM makes with the incident ray YW is Y *r* M; and that which the red ray BA makes with the same is YSA. And sine BSY, the external angle of the triangle B *r* S, is (El. I. 16.) greater than the internal angle B *r* S or B *r* Y; Y *r* M, the complement of B *r* S, is greater than YSA, the complement of BSY. Consequently, since the emerging rays make different angles with the same incident ray, the refraction which they suffer at emersion will separate them from one another.

Schol. The angle which the violet rays make with the incident ones is found to be 54° 7′, and that of the red rays 50° 57′.

PROP. CXXXIII. If a line be supposed to be drawn from the centre of the sun through the eye of the spectator, the angle which, after two refractions and two reflections, any effectual ray makes with the incident ray, will be equal to the angle which it makes with that line.

If YW be an incident ray, and BA an effectual ray, and AO a line drawn from the centre of the sun

Plate 8. through A, the eye of the spectator, YW and AO may be considered as parallel; whence the alter-

Fig. 5. nate angles YSA, SAO, (El. I. 29.) will be equal.

PROP. CXXXIV. When the sun shines upon the drops of rain as they are falling, the rays which come from those drops to the eye of the spectator, after two reflections and two refractions, produce the outermost or secondary rainbow.

Plate 8. Fig. 4. When the sun shines upon a drop of rain E in the outer edge of the secondary rainbow CBD, the effectual violet ray EA (by Prop. CXXXII. Schol.) makes an angle EAI of 54° 7′ with AI, a line drawn from the sun through the eye of the spectator, and therefore (by Prop. CXXXIII.) makes the same angle with the incident ray SB. Therefore if the spectator's eye be at A, all the rays except the violet will (by Prop. X.) make a less angle with AI, than EA, and fall above the spectator's eye. In like manner it may be shown, that from the drop F, only red rays will come to the spectator's eye, the rest falling below it; and that the rays emerging from the intermediate drops between E and F, and coming to A, will emerge at intermediate angles, and present to the eye the intermediate colours. If EAI be conceived to turn round upon the line AI, in such a revolution of the drop E, the angle EAI would remain the same, and consequently the emerging rays would make the same angle with the incident rays. But in such a revolution the drop E would describe a circle, of which I would be the centre, and CBD an arc. Consequently, since the emerging rays make the same angle with the incident ones when the drop is at any other part of the arc, as at E, the colour of the drop will be violet to an eye placed at A, in whatever part of the arc the drop is placed. Now, since there are innumerable drops of rain falling at once, while one drop is at E, there will be others in all parts of the arc, which will all appear violet-coloured, for the same reason that E would have appeared of this colour in any other part of the arc. In like manner, as the drop F appears red at F, and at any part of the arc FD, so will any other falling drop when it comes to any part of that arc. The intermediate arcs are formed in the same manner with the violet arc CBD, and the red arc FD; and thus the whole secondary rainbow is produced.

PROP. CXXXV. The colours of the secondary rainbow are fainter than those of the primary, and are ranged in the contrary order.

At every reflection many rays pass out of the drop without being reflected; consequently the secondary rainbow, which is produced after two reflections, is formed by fewer rays than the first, which is produced after one reflection.

Plate 8. Fig. 3. Again, in the primary bow, the violet rays, when they emerge effectually, make a less angle with the incident rays (by Prop. CXXVI.) and therefore (by Prop. CXXVII.) with the line AI, than the red rays. But the rays are here only once reflected, and the angle which the effectual rays make with AI is the distance of the coloured drop from I, the centre of the bow. Therefore the violet arc in the primary bow will be nearer to the centre of the bow than the red arc; that is, the innermost colour will be violet, and the outermost red. But in the secondary rainbow, the rays are twice reflected; and (by Prop. CXXXII.) the violet rays, which emerge so as to be effectual after two reflections,

make a greater angle with the incident rays, that is, with the line AI, than the red ones; which angle is the distance of the violet arc from I, the centre of the bow. Therefore the violet arc in the secondary bow will be farther from the centre of the bow than the red arc; that is, the outermost colour is violet and the innermost red.

PROP. CXXXVI. The secondary rainbow is never a greater arc than a semicircle.

This is proved in the same manner as Prop. CXXIX. with this difference, that, since the rays of the highest colour in the secondary bow make an angle of 54° 7′ with AI; this bow will begin to appear when the altitude of the sun is less than 54° 7′; and when the sun is in the horizon on one side, this bow will have its centre in the horizon on the other side at the distance of 54° 7′ from its highest point.

CHAPTER VI.

Of Optical Instruments.

SECTION I.

Of Telescopes.

DEF. XXX. An *Astronomical Telescope* consists of two convex lenses, whose distance from each other is equal to the sum of their principal foci; that lens which is toward the object, is called the *object-glass;* that which is next the eye, is called the *eye-glass.*

If NL is one convex lens, whose focal distance is MF, and BD another, whose focal distance is CF; and if these are so placed that the distance between them is equal to MF added to CF, that is, MC, they form an astronomical telescope. Plate 8. Fig. 7.

PROP. CXXXVII. Very remote objects, seen through an astronomical telescope, appear distinct and inverted.

Let PM, PL, PN, be rays coming (by Prop. VIII.) parallel from the middle point in a very distant object; let AN, AM, AL, come from the lowest point, and QN, QM, QL, come from the highest point. These parallel rays will (by Def. XVIII.) be collected into the focus, and there form an image of the object, which (by Prop. LXXXII.) forms the object of refracted vision. But, by the construction of the telescope, GFE is the focus of the eye-glass. Consequently the rays which diverge from any point G in this image will, (by Prop. XX.) after they have passed through the eye-glass, become parallel. Therefore if the eye is at any point on the other side of the eye-glass, the object of refracted vision may be seen as distinctly as any very remote object can be seen by the naked eye; and because the image is the object of vision, (by Prop. XXV.) it will be seen inverted. Plate 8. Fig. 7.

PROP. CXXXVIII. The apparent diameter of an object, seen through an astronomical telescope, is to the apparent diameter of the same object seen by the naked eye at the station of the object-glass, as the distance of the image from the object-glass is to its distance from the eye-glass.

If the image, formed by the object-glass NL, were received upon a paper at EFG, the apparent diameter of the object seen by the naked eye at M, the station of the object-glass, would be (by Prop. LXXXIV.) equal to the apparent diameter of the image seen from the same station. Now the real diameter of the image is given, because its distance MF from the lens is given. Consequently, the apparent diameter of the image (by Prop. LXIX.) will be inversely as the distance of the eye from it. If the eye be placed at C, the station of the eye-glass, and consequently its distance from the image be FC, the image will appear to the eye in that station bigger than at the station M (by Prop. LXXXIX.) in the inverse ratio of the distances FC, MF; that is, the apparent magnitude of the image at C will be to Plate. 8. Fig. 7.

that at M, as MF to FC. But the apparent magnitude of the image seen from M is equal to that of the object seen by the naked eye. Therefore the image seen from C appears bigger than the object, in the ratio of MF to FC. This would still be the case (by Prop. LXXXIV.) if the eye-glass were placed between the eye and the image, touching the eye. And since the image is in the focus of the eye-glass, the apparent magnitude (by Prop. XC.) is the same, whether the eye is close to the lens, or at any distance from it. Therefore wherever the eye is, the apparent diameter of the object, seen with the telescope, is to the apparent diameter of the same object seen by the naked eye at the station of the object-glass, as MF to FC, or as the distance of the distinct image from the object-glass, to its distance from the eye-glass; that is, as the focal distance of the object-glass is to the focal distance of the eye-glass; consequently if the former be divided by the latter, the quotient will express the magnifying power; thus, if MF : FC :: 10 : 1, the telescope will magnify 10 times in diameter.

PROP. CXXXIX. A telescope will not magnify an object, unless the focal distance of the object-glass be greater than the focal distance of the eye-glass.

Plate 8. Fig. 7. The rays which come from distant objects being nearly parallel, the image GFE (by Def. XVIII.) will be in the focus of the object-glass, which by the construction of the telescope, is also the focus of the eye-glass. But the apparent diameter of an object seen through a telescope, is to its apparent diameter when seen by the naked eye, (by Prop. CXXXVIII.) as the distance of the image from the object-glass, to its distance from the eye-glass; that is, by what has been just proved, as the focal distance of the object-glass, to the focal distance of the eye-glass. Consequently, if MF, the focal distance of the object-glass, is greater than FC, the focal distance of the eye-glass, the object will be magnified; if MF be equal to FC, the object will appear as to the naked eye; if MF be less than FC, the object will appear diminished.

Cor. 1. Hence the object-glass of a telescope should be less convex than the eye-glass.

Cor. 2. An object will be equally magnified by two telescopes of very different lengths, if the ratio of the focal distances of the object-glass and eye-glass be the same in each.

Cor. 3. If a telescope be inverted, objects seen through it will be diminished; for the object-glass, which has the greater focal distance, then becomes the eye-glass.

PROP. CXL. The visible area, or space which may be seen at one view through a telescope, is as the area of the eye-glass.

Plate 8. Fig. 7. If GFE is any image, its distance from the object-glass being equal to the focal distance of the lens, the area of the image (by Prop. XXXVI.) is given; but the quantity of this image which can be seen at one view must be greater or less, according to the magnitude of the hole through which it is seen; that is, must be as the area of the eye-glass.

PROP. CXLI. The brightness of an object seen through a telescope depends upon the area of the object-glass, but not the visible area.

The brightness of the image, that is, of the object of refracted vision, is (by Prop. XXXVIII.) as the area of the lens which forms it, that is, of the object-glass. But (by Prop. XXXVI. Schol. 2.) the magnitude of the image is the same, whether the area of the object-glass is great or small; and consequently, if we look at it through an eye-glass of a given area, the quantity to be seen at once will not be altered by any change in the area of the object-glass.

PROP. CXLII. The distance of the eye from the eye-glass, should be equal to the principal focal distance of the eye-glass.

Plate 8. Fig. 7. Since the image GFE is in the focus of the lens DCB, wherever the eye is placed on the other side of the glass, the image will appear equally magnified. But when the eye is just as far from the eye-glass as its focal distance, the visible area will be the greatest; for, in that case, (by Def. XVIII.) none but rays parallel, before the refraction, to MC the axis of the telescope, and therefore to the sides of the cylindrical tube in which the lenses are placed, can reach the eye, and consequently, no rays can come from the inner surface of this tube to the eye to make it visible; whereas in any other station of the eye, oblique rays from that surface would make the sides of the tube visible; whence the area of the vision, which remains the same, being (by Prop. CXL.) always as the area of the eye-glass, will be in part occupied by the sides of the tube, and the object will be seen only through the remaining part.

Def. XXXI. A telescope, consisting of four convex lenses, is a *double Astronomical Telescope.*

Let the two lenses NML, and B, placed at the distance MB, equal to the sum of their focal distances, form one telescope, and two lenses, C, D, placed at the distance CD, equal to the sum of their focal distances, form another. If these two telescopes are fixed at the distance CB from each other, so as to be both used together, they form a double telescope; the lens LMN next to the object, is called the object-glass, and the lens B next the object-glass is called the first eye-glass, C the second, and D next to the eye the third. Plate 8. Fig. 8.

PROP. CXLIII. An object seen through a double telescope appears distinct and erect.

The parallel rays which fall upon the object-glass NML (by Prop. CXXXVII.) form a distinct inverted image at GFE, the focus of the object-glass. This image being also in the focus of the first eye-glass B, the rays of each beam from the several points of this image will become parallel by passing through B; whence, falling parallel on the second eye-glass C, they will form a distinct inverted image at KIH, the focus of this second eye-glass; and because KIH is also the focus of the third eye-glass D, the rays from this image, after passing through this third eye-glass, will come to the eye parallel to each other. Consequently, the object will be seen distinctly; and because the second image is inverted with respect to the first, which is inverted with respect to the object, the second image, or object of refracted vision, is in the same situation as the object itself. Plate 8. Fig. 8.

PROP. CXLIV. A double telescope magnifies an object in the ratio of the focal distance of the object-glass, to the focal distance of the first eye-glass.

The first telescope MB magnifies the object in the ratio of MF to FB; and the second telescope CD is commonly made up of two lenses of equal convexities, which will not alter the apparent magnitude of the objects. Therefore, when both are used together, the object is only magnified by the first in the ratio of MF, the focal distance of the object-glass, to FB, the focal distance of the first eye-glass; consequently, the magnifying power is found by dividing MF by FB. Plate 8. Fig. 8.

Schol. 1. The different refrangibility of the rays of light makes refracting telescopes imperfect; for those rays which are most refracted by passing through the lens, will be brought to a focus and form an image nearer to the object-glass than those which are less refracted; and consequently the several sorts of rays are not properly collected in one focus to produce a perfectly white image, but each has its own focus, producing a confused and coloured image.

Of two refracting telescopes which magnify equally, the shorter will give a more imperfect image than the longer. For the image appearing equal in both, but being farther from the object-glass in the longer than the shorter, must be in reality larger or more magnified; whence the defect arising from the different refrangibility of the rays will be more visible in the longer than in the shorter telescope. Hence, reflecting telescopes are more perfect than refracting ones; for when all the rays are reflected, their angles of incidence and reflection being equal, they will all meet in a focus at the same distance.

Schol. 2. To remedy the defect of refracting telescopes, arising from the different refrangibility of rays of light, a compound object-glass was invented by Mr. Dollond, consisting partly of white flint glass, and partly of crown glass, which have different refracting powers. These refract contrary ways; and the excess of refraction in the crown glass is made such, as to destroy the colour caused by the flint glass. A telescope thus formed is called achromatic. Let ABED represent a double concave lens of white flint glass, and AGDF a double convex of crown glass; then the part of the lenses which are on the same side of the common axis, viz. ACB and AFG may be conceived to act like two prisms which refract contrary ways; and if the excess of refraction in the crown glass be such as precisely to destroy the divergency of colour caused by the flint glass, the incident ray SH will be refracted to X without any production of colour; the same is true of the ray *sh*, and of all the other incident rays; and consequently the whole focal image, formed by this compound object-glass, will be achromatic, or free from colour which might arise from refraction. It will therefore bear a larger aperture, and greater magnifying power, and of course enlarge objects much more than a common refracting telescope of the same length. The great impediment to the construction of large achromatic telescopes, is the want of a flint glass of an uniform density. Dr. Blair has, within these few years, discovered that certain fluids, particularly those which contain the muriatic acid, may be formed into Plate. 12. Fig. 10.

lenses. With these he has produced achromatic telescopes, which seem as perfect as the thing will admit of. See Transactions of the Edinburgh Royal Society. Also, Encyc. Brit. Art. Telescope, Vol. XVIII. Part i.

Schol. 3. The construction of the eye has excited a suspicion, that that might be an achromatic instrument; but as the successive refractions are all in the same direction, that notion cannot be maintained, unless one of the humours is found to refract the red rays more than the violet, or all the humours refract all the rays equally.

PROP. CXLV. To explain the construction and use of several kinds of telescopes.

I. Of Galileo's Telescope.

Galileo's telescope consists of a convex object-glass and a concave eye-glass, so placed that the distance between them is the difference of their focal distances.

Plate. 8. Fig. 9. In this telescope ZYX, a convex lens, is placed at the distance from BA, a concave lens, of YC, the difference between YF, the focal distance of ZX, and CF, the focal distance of BA.

From a distant object let rays fall upon the convex lens YZ, from which they will proceed toward the focus of this lens at FG. But the concave lens AB, the focus of which is at FG, renders the converging rays parallel when they reach the eye; whence an image will be formed upon the *retina*. And the pencils of rays being made more diverging by passing through the concave lens, the visible image is seen under a larger angle than the object, and appears magnified. Also, because the pencils which form the image only cross one another once, the image appears erect.

II. Of Sir Isaac Newton's Telescope.

Plate. 8. Fig. 10. In the tube ABCD, toward the end BC, let the concave mirror GH be placed perpendicular to DC, the lower side of the tube. If an object, which is at such a distance that rays coming from the same point may be considered as parallel to one another, be placed before the open end of the tube AD, these parallel rays will be reflected from the concave mirror GH, and becoming convergent, would (by Prop. CXII.) form an inverted picture of the object upon a paper held at the focus of the mirror. But if the converging rays, before they reach the focus, fall upon a plain mirror K, placed at an angle of 45 degrees with DC, the side of the tube, or with the axis of the telescope, they will be reflected from thence, and meet before it at L, forming an image perpendicular to the object, or parallel to the axis of the telescope. If this image be placed in the focus of the convex lens L, fixed in the side of the telescope, the eye will see it distinctly through the lens.

The image seen from the station of the eye-glass L, either with or without the glass, will (as in the refracting telescope, see Prop. CXXXVIII.) appear as much larger than when seen from the concave mirror, that is, as much larger than to the naked eye, as the distance of the image from the eye-glass is less than its distance from the mirror, or as its distance from the mirror is greater than its distance from the lens.

III. Of Gregory's Telescope.

Plate 8. Fig. 11. In the tube TTYY, let a concave mirror EA be placed. Any parallel rays OO, PP, form an object A, falling upon this mirror, will, after reflection, (by Prop. CXI.) form an inverted image at C, its focus. Let C be more remote from a second smaller concave mirror PO (placed parallel and opposite to the first mirror EA in such manner that their axes shall be in the same straight line) than its focus. The rays which diverge from the several points of the image at C, and fall upon the mirror PO, will (by Prop. L.) converge after reflection; and consequently, if they pass through a hole NM in the first mirror EA, they will form a second image, which will be inverted in respect of the first, and in the same position with the object. If, whilst these rays are converging, they pass through a plano-convex lens *S p o* (placed in a smaller tube joined to the larger), they will be brought to a focus sooner than they would otherwise have been, forming the second image F. This erect image is seen by the eye at O, through a meniscal eye-glass LL, whose convexity is greater than its concavity. For the magnifying power of this telescope, see Musschenbroek. Introd. ad Phil. Nat. or Priestley's Optics, page 376.

Schol. 1. In the telescopes made by Dr. Herschel, the object is reflected by a mirror as in the Gregorian telescope, and the rays are intercepted by a lens at a proper distance, so that the observer has his back to the object, and looks through the lens at the mirror. The magnifying power will be the same as in the Newtonian telescope; but there being no second reflector, the brightness of the object viewed in the Herschel telescope, is greater than that in the Newtonian telescope.

The tube of Dr. Herschel's grand telescope is 39 feet 4 inches in length, 4 feet 10 inches in diame-

ter, every part of which is made of iron. The concave surface of the great mirror is 48 inches of polished surface in diameter, its thickness is $3\frac{1}{2}$ inches, and its weight is upward of 2000lb. This noble instrument was, in all its parts, constructed under the sole direction of Dr. Herschel; it was begun in the year 1785, and completed August 28, 1789, on which day was discovered the sixth satellite of Saturn. It magnifies 6000 times.

SCHOL. 2. Dr. Priestley observes, that the easiest method of finding the magnifying power of any telescope, by experiment, is to measure the diameter of the aperture of the object glass, and that of the little image of it which is formed at the place of the eye. For the proportion between these gives the ratio of the magnifying power, provided no part of the original pencil be intercepted by the bad contruction of the telescope. For, in all cases, the magnifying power of telescopes or microscopes, is measured by the proportion of the original pencil, to that of the pencil which enters the eye. Another method, is to observe at what distance you can read any book with the naked eye; and then removing the book to the farthest distance at which you can distinctly read it by the help of the telescope. The book chosen for this purpose should be such, that the connexion of the subject should not assist the observer; as tables of logarithms, &c. Much depends on the steadiness with which the instrument is fixed.

SECTION II.

Of Microscopes.

DEF. XXXII. A *single Microscope* is one convex lens placed between a small object and the eye. Plate 8. Fig. 6.

DEF. XXXIII. A *double Microscope* consists of two convex lenses, of which the object-glass is more convex than the eye-glass; and the distance between them is equal to the distance of the image from the object-glass, added to the focal distance of the eye-glass.

Let AB, a convex lens, be the object-glass, and EF, another convex lens, be the eye-glass. Let the small object KL be farther from the object-glass than its focus; an image MDN of the object will (by Prop. XXIV.) be formed behind the glass; let the distance of this image from the object-glass be ID, and let its distance from the eye-glass be equal to the focal distance of the eye-glass; the distance of the two glasses from each other will be ID + DX, or IX, that is, the distance of the image from the object-glass, added to the focal distance of the eye-glass. Plate 8. Fig. 12.

Some compound microscopes are made with three glasses, so that the rays after passing through AB the object-glass, and EF the eye-glass, are again made converging by a second eye-glass, and therefore brought sooner to a focus, than by the first, and the field of vision will be much greater than if only one lens were used.

COR. Hence it appears, that the difference between the microscope and telescope is, that in the telescope the rays of each pencil fall upon the object-glass nearly parallel, and are united in its focus; but in the microscope they fall upon it very much diverging from one another, and therefore form the image in a place beyond the focus, and consequently larger than the object.

PROP. CXLVI. An object seen through a double microscope appears distinct and inverted.

The pencils of rays issuing from the objects KL, being transmitted through the object-lens AB, their foci will be in MN; where there will be an inverted image of the object, which is viewed through another lens, or eye-glass EF, the focus of which is at MN; hence a distinct and direct image is formed upon the *retina*, and it is seen inverted. Plate 8. Fig. 12.

PROP. CXLVII. The apparent diameter of an object seen through a double microscope is to that of the same object seen by the naked eye at the limit of distinct vision, in the compound ratio of the distance of the image from the object-glass, to its

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distance from the eye-glass, and of the limit of distinct vision to the distance of the object-glass from the the object.

The first part of this Proposition is demonstrated as Prop. CXXXVIII. And it is manifest (from Prop. LXIX.), that if CI, the distance of the object-glass from the object, be less than the limit of distinct vision, the apparent diameter of the object will be as much greater than that of the object at the distance at which the naked eye can see it distinctly, as IC is less than that distance. Therefore the object is magnified, because the distance of the image from the object-glass is greater than its distance from the eye-glass, and also because the distance from the object is less than the limit of distinct vision. The magnifying power of the microscope is then in the ratio compounded of these two ratios. Suppose ID $=6$IC ; and the eye-glass EF to be one inch focus, and the limit of distinct vision to be seven inches, then the diameter of the object KL will be magnified $6 \times 7 = 42$; consequently 1764 times in surface.

PROP. CXLVIII. When the same eye-glass is used, the magnifying power of the microscope will be increased by increasing the convexity of the object-glass.

For in order to keep the image in the focus of the eye-glass, when the convexity of the object-glass is increased, the object-glass must be brought nearer to the object ; the consequence of which will be, that the ratio of the limit of distinct vision to the distance of the object-glass from the object will (El. V. 8.) be increased ; whence (by Prop. CXLVII.) the ratio of the apparent diameter of the object of refracted vision to that of the object seen by the naked eye will also be increased.

Schol. The aperture of the object-glass in a microscope must be small, else the outermost rays, diverging too much, will hinder the distinctness of vision ; but, on account of the smallness of the aperture, the object will appear faint, and it will appear necessary, in order to remedy this, to illuminate the object as much as possible.

PROP. CXLIX. To describe the construction and use of the Solar Microscope.

In a dark room, let a round hole be made in a window-shutter about three inches in diameter, through which the sun may cast a cylinder of rays into the room. In this hole let a tube be fixed, containing a convex lens of about two inches in diameter, and three inches focal distance ; the object, placed between two concave glasses, at the distance of about two inches and a half from the first convex lens ; and a second convex lens, whose focal distance is a quarter of an inch, placed at this distance from the object. Let a plane mirror, connected with the tube, and moveable by means of a wheel, receive the sun's rays on the outside of the shutter, and convey them into the tube. The rays, passing through the first lens, will strongly illuminate the object, from which they will pass through the second lens, and form an inverted image of the object, magnified in the ratio of the distance of the object from the lens to that of the image from the lens, that is, in this case supposing the distance of the distinct picture to be twelve feet, or 144 inches, in the ratio of $\frac{1}{4} : 144$. Consequently, the diameter of the object will be magnified 576 times.

SECTION. III.

Of the Magic Lantern.

PROP. CL. To describe the construction of the Magic Lantern.

Plate 8. Fig. 13. In the side of a lantern let a tube be inserted, consisting of two parts, one moveable upon the other. In the moveable part let a convex lens GG be fixed ; in the immoveable part let an object EE, painted with transparent colours upon a piece of thin glass, be placed ; and in the fixed part of the tube, a convex lens DD. This lens will cast a strong light from the candle upon the object EE. And when the rays which diverge from the several points of the object are, by the lens GG, made to converge, they will (by Prop. XXV.) form an inverted image of the object at KL, upon any white surface ; provided that the object is farther from the lens than its focus, and that the whole apparatus is placed in a dark room. The image KL will be larger than the object EE, in proportion as the distance of the image from the lens is greater than the object. A concave reflector AB may be placed within the lantern, behind the candle, to increase the illumination of the picture EE. If the object be placed in an inverted position, its image will appear erect

SECTION. IV.

Of the Camera Obscura.

PROP. CLI. To describe the construction and use of the Camera Obscura.

Let CD be a convex lens, and HK a plane mirror inclined at an angle of 45 degrees. An inverted image of the object AB would be formed at EG, where the foci of the rays from the objects are found after refraction; but the rays being intercepted by the plane mirror HK, are reflected (by Prop. C.) to MM, the focal distance before it, making an angle with the mirror of 45 degrees; whence the image will be in a position perpendicular to the object, at the top of the box, where, if the rays be received on a sheet of oiled paper, or a plate of glass unpolished on one side, it will be distinctly visible. Plate 8. Fig. 14.

BOOK VII.

OF ASTRONOMY.

PART I.

OF THE MOTIONS OF THE HEAVENLY BODIES.

CHAPTER I.

Of the Solar System in General.

Def. I. THE *Solar System* consists of the Sun, which is a luminous body; seven primary planets, Mercury, Venus, the Earth, Mars, Jupiter, Saturn, and the Herschel; eighteen secondary planets, the Earth's Moon, Jupiter's four Satellites, Saturn's seven, and six belonging to the Herschel; and an uncertain number of comets; all which are opaque.

Schol. Upon entering the subject of Astronomy, it will be proper briefly to describe the different systems which have been invented, in order to solve the natural appearances of the heavenly motions.

Ptolemy supposed the earth to be perfectly at rest, and the sun, moon, planets, comets, and fixed stars to revolve about it every day; but that besides this diurnal motion, the sun, moon, planets, and comets, had a motion in respect to the fixed stars, and were situated in respect to the earth, in the following order; the Moon, Mercury, Venus, the Sun, Mars, Jupiter, Saturn. The revolutions of these bodies he supposed to be made in circles about the earth placed a little out of the centre. This system will not solve the phases of Venus and Mercury, and therefore cannot be true.

The system received by the Egyptians was this; the earth was supposed immoveable in the centre, about which revolved, in order, tbe Moon, Sun, Mars, Jupiter, and Saturn; and about the Sun revolved Mercury and Venus. This disposition will account for the phases of Mercury and Venus, but not for the apparent motions of Mars, Jupiter, and Saturn.

Another system was that of Tycho Brahe, a Danish nobleman, who was anxious to reconcile the appearances of nature with some passages of the Scriptures, taken in their literal interpretation. In his system, the earth is placed immoveable in the centre of the orbits of the sun and moon, without any rotation about its axis; but he made the sun the centre of the orbits of the other planets, which, therefore, revolved with the sun about the earth. Objections to this system are, the want of that simplicity by which all the apparent motions may be solved; and the necessity of supposing that all the heavenly bodies revolve about the earth every day; also to suppose that a body should revolve in a circle about its centre without any central body is physically impossible.

Some of Tycho's followers, seeing the absurdity of a diurnal revolution of the heavenly bodies about the earth, gave a rotatory motion to the earth, and this was called the Semi-Tychonic system.

The system, which is now universally received, is called the Copernican. It was formerly taught by Pythagoras, 500 years before Christ; and afterward rejected, till revived by Copernicus in the sixteenth century. Here the sun is placed in the centre of the system, about which the planets revolve from west to east, in the following order; Mercury, Venus, the Earth, Mars, Jupiter, Saturn, and the Herschel planet; beyond which, at immense distances, are placed the fixed stars. The moon revolves round the earth; and the earth turns about an axis. The other secondary planets move round their respective primaries from west to east at different distances, and in different periodical times.

According to this doctrine, the Sun S is the centre of the system; Mercury *a*, Venus *b*, the Earth *t*, Mars *c*, Jupiter *f*, and Saturn *h*, revolve in elliptical orbits round the sun; the moon *d*, revolves about the earth, and the satellites of Jupiter, Saturn, and the Herschel, revolve about their primaries; and the planes of their orbits are inclined to one another.* Plate 9. Fig. 1.

This doctrine, being admitted as true, will account for the apparent motions, and other phenomena, of the heavenly bodies, as will be seen in the following Chapters.

CHAPTER II.

Of the Earth.

SECTION I.

Of the Globular Form of the Earth, and its diurnal Motion about its Axis, and of the Appearances which arise from these.

PROPOSITION I.

The earth is of a globular form.

For, 1. The shadow of the earth projected on the moon in an eclipse is always circular; which appearance could only be produced by a spherical body. 2. The convexity of the surface of the sea is visible; the mast of an approaching ship being seen before its hull. 3. The north pole becomes more elevated by travelling northward, in proportion to the space passed over. 4. Navigators have sailed round the earth, and by steering their course continually westward, arrived, at length, at the place from whence they departed.

DEF. II. The *Axis* of the earth is an imaginary line passing through the centre, about which its diurnal revolution is performed.

DEF. III. The *Poles* of the earth are the extremities of this axis.

DEF. IV. The *Equator* is the circumference of an imaginary great circle passing through the centre of the earth, perpendicular to the axis, and at equal distances from the poles.

DEF. V. If the axis of the earth be produced both ways, as far as the concave surface of the heavens, in which all the heavenly bodies appear to be placed, it is then called the *Axis of the Heavens*; its extremities are called the *Poles of the Heavens;* and the circumference produced by extending the plane of the equator to the same concave surface, is called the *Equator in the Heavens.*

DEF. VI. Circles drawn through the poles of the earth or heavens, perpendicular to the plane of the equator, are called *Secondaries* of the equator.

DEF. VII. The *sensible Horizon* is an imaginary circle, which, touching the surface of the earth, separates the visible part of the heavens from the invisible. The *rational Horizon* is a circle parallel to the former, the plane of which passes through the centre of the earth.

* Dr. Herschel, on the 13th of March, 1781, while pursuing a plan which he had formed of observing, with telescopes of his own construction, every part of the heavens, discovered, in the neighbourhood of H. Geminorum, a planet far beyond the orbit of Saturn, which had never before been visible to mortal eyes. He has since discovered six secondaries belonging to this new planet, which planet is called either the HERSCHEL, from the name of its indefatigable and truly great discoverer; or the Georgium Sidus, or Georgian Planet, in honour of the late king, who distinguished himself as the patron of Dr. Herschel. The planet is denoted by this character ♅; an H, as the initial of the name, the horizontal bar being crossed by a perpendicular line, forming a kind of cross, the emblem of Christianity, denoting, perhaps, that its discovery was made in the Christian era, as all the other planets were known long before that period.

Plate 9. Fig. 4.

Schol. Since (by Book VI. Prop. LXIX.) the apparent diameter of an object is inversely as its distance, if the distance be increased in such manner that it may be looked upon as infinite, the apparent magnitude becomes a point. Hence AF, the semidiameter of the earth, viewed at the different distances *o*, O, R, diminishes, till at the distance of O, a fixed star, it becomes a point, and the star appears in the same place in the heavens, whether viewed from the visible horizon SET, or rational horizon HBR.

Def. VIII. The *Poles* of the *Horizon* are two points, the one of which, over the head of the spectator, is called the *Zenith;* the other, which is under his feet, is called the *Nadir.*

Def. IX. Circles drawn through the zenith and nadir of any place, cutting the horizon at right angles, are called *Vertical Circles.*

Def. X. A vertical circle passing through the poles of the heavens, is a *Meridian,* and is said to be the meridian of any place through which it passes.

Def. XI. The meridian of any place passing through the poles, and falling perpendicularly upon the horizon, cuts it in two opposite cardinal points, called *North* and *South.*

Def. XII. A *Meridian Line* is the common intersection of the plane of the meridian and the plane of the horizon.

Cor. Hence any line which lies due north and south in a horizontal plane, may be considered as part of the meridian line.

Schol. 1. To draw a meridian line; perpendicular to a horizontal plane, erect a wire, or stile, seven or eight inches long, and as it is not easy to determine precisely the extremity of the shadow, it will be best to make the stile flat at top, and to drill a small hole through it, noting the lucid point projected by it; mark, at several different times before noon, these lucid points, and through them draw concentric circles about the middle point of the wire's station; observe in the afternoon when the lucid points again touch these circles; and find the middle point of each arc between the points already taken; a line drawn through these middle points, and the common centre, will be the meridian line; for, since at equal distances from noon, the sun is at the same height, or in verticals equally distant from the meridian, the circle drawn through the zenith at equal distances from these verticals is the meridian. This should be done about the summer solstice, between the hours of 9 and 11 in the morning and 1 and 3 in the afternoon.

Schol. 2. To observe the transit of any heavenly body over the plane of the meridian; place in this plane a telescope, having two cross hairs before its object-glass, one vertical, the other horizontal, and observe when the vertical hair passes through the centre of the heavenly body; or hanging two plumb-lines exactly over the meridian line, place your eye close to one of the threads in such manner, as that it shall cover the other thread, and observe when the body is behind the threads.

Def. XIII. The *Altitude* or *Depression* of any heavenly body above or below the horizon, is the arc of a vertical circle intercepted between the body and the horizon, or the angle at the centre measured by that arc.

Schol. The altitude of any heavenly body is found by the help of a quadrant thus; bring the quadrant into such a situation that the star may be seen through the sights; then the angle, contained between the string of the plummet and the side of the quadrant, on which the sights are not placed, is the altitude of the star.

Def. XIV. The *Prime Vertical* is that which crosses the meridian at right angles in the zenith and nadir, cutting the horizon in the cardinal points *East* and *West.*

Def. XV. The *Azimuth* of a heavenly body, is the arc of the horizon intercepted between the meridian and vertical circle passing through that body; it is eastern or western, as the body is east or west of the meridian.

SCHOL. The azimuth of any star may be thus found. Let AC be a given meridian line. Above any point A in this line, let a cord with a plummet be hung; let another cord with a plummet be hung at E, so that the star and the two cords shall lie in one and the same right line. Let the perpendiculars AD, BE, represent the cords, and draw AB. From the point B to any point C, in the meridian line AC, taken at pleasure, draw the right line BC; then with a scale of equal parts measure the three lines AB, AC, BC. In the triangle, therefore, ABC, there will given all the sides, from whence will be found the angle BAC, equal to the azimuth required. Plate 9. Fig. 5.

For if the meridian line be supposed to be continued to F, and the line BA to G, the angle FAG will be the azimuth of the star; but the angle FAG will be equal to the angle at the vertex BAC; therefore the angle BAC will be equal to the azimuth.

DEF. XVI. The *Amplitude* of a heavenly body at its rising is the arc of the horizon intercepted between the point where the body rises, and the east; its amplitude at setting is the arc of the horizon intercepted between the point where the body sets and the west; it is northern, or southern, as the body rises, or sets, to the north or south of east or west.

DEF. XVII. If a heavenly body rises, or sets, when the sun rises, it is said to rise or set *cosmically*; if it rises, or sets, when the sun sets, it is said to rise or set *achronically*; it is said to set or rise *heliacally*, when it approaches so near the sun as to become invisible, or recedes so far from him as to become visible.

DEF. XVIII. The *Latitude* of a place upon the surface of the earth is its distance from the earth's equator; it is measured by the arc of the geographical meridian of the place intercepted between the place and the equator: latitude is either northern or southern.

DEF. XIX. *Parallels of Latitude* are circles on the surface of the earth, drawn parallel to the equator.

PROP. II. A degree in the equator is to a degree in any parallel of latitude as radius to the cosine of latitude.

Let EPQ be a geographical meridian, EQ the equator, and FB a parallel of latitude. The circumference EQ is to the circumference FB, and any part of EQ, to any similar part of FB, as CQ, or CB the radius of EQ, to AB the radius of FB; and AB is the cosine of the arc BQ, which is the latitude of the parallel FB. Therefore a degree in EQ is to a degree in FB, as radius to the cosine of latitude. Plate 9. Fig. 3.

DEF. XX. The *Longitude* of a place is the distance between the meridian of that place, and the meridian of some other place, taken at pleasure, and called the first meridian; it is measured by the arc in the equator intercepted between these two meridians. Longitude is either eastern or western, and is measured 180 degrees each way.

PROP. III. The altitude of one pole, and the depression of the other, at any place on the earth's surface, is equal to the latitude of that place.

Let R be a place on the earth's surface; Z, N, its zenith and nadir; P, S, the poles of the heavens, and F, *s*, the poles of the earth; EE, the celestial equator, *ee*, the terrestrial, and HO the horizon. The latitude of the place is *e* R, or the equal arc EZ; and PO is the elevation of one pole, and HS the depression of the other. Because ZO is the distance of the zenith from the horizon, it is an arc of 90 degrees; and because EP is the distance of the pole from the equator, it is also an arc of 90 degrees; ZO and EP are therefore equal. Take from each of these the common arc ZP, and the remainders EZ and PO are equal. But HS and PO are equal, because they subtend the equal angles HTS, PTO; therefore the elevation of one pole PO, and the depression of the other HS, are equal to the latitude of the place EZ. Plate 9. Fig. 2.

COR. Hence the circumference of the earth may be measured, by measuring the length on the surface of the earth passed over in a line which lies north and south, while the pole gains one degree of elevation, and multiplying this length by 360. A degree of latitude contains $69\frac{1}{4}$ English miles; whence 24930 miles is the measure of the circumference of the earth, and the radius 3956; but, as will be shown hereafter, the earth is a spheroid, whose polar diameter is to the equatorial, as 311 to 312.

PROP. IV. The elevation of the equator at any place is equal to the complement of its latitude.

Plate 9. Fig. 2. Because ZO is equal to EP (each being an arc of 90 degrees) EZ is equal to PO, that is, (by Prop. III.) to the latitude of the place. But EH, the elevation of the equator, is the complement of EZ; it is therefore equal to the complement of the latitude of the place.

PROP. V. The earth revolving daily round its axis from west to east, the heavenly bodies will appear to a spectator on the earth to revolve in the same time from east to west.

Let RCBF be the earth, T its centre, HTO the rational horizon to a spectator at R, whose zenith is Z; let a star appear in the horizon at H. The earth revolving from west to east, that is, in the order of the letters, R, C, B, F, in a fourth part of one revolution, the spectator will be carried from R to C; consequently, his horizon will become ZN, and the star which appeared in his horizon at H, when he was at R, will now appear nearly in the zenith. When another fourth part of the revolution is completed, the spectator will be at B, and N being now his zenith, and HO his horizon, the star will be set with respect to him, and will not rise till he is again in the station R, that is, till the earth has completed one revolution. Thus whilst the earth has turned once round upon its axis from west to east, all the heavenly bodies in the concave sphere of the heavens will appear to have turned round from east to west.

PROP. VI. The alternate succession of day and night is the effect of the revolution of the earth round its axis.

For, all the heavenly bodies appearing (Prop. V.) to move from east to west, while the earth revolves from west to east, the sun will appear, in each revolution, to rise above the horizon in the east, and after describing a portion of a circle, to set in the west, and will continue below the horizon, till, by the revolution of the earth, it again appears in the east; and thus day and night will be alternately produced.

Schol. The time of noon is found, by observing the instant when the centre of the sun is cut by the perpendicular hair in a meridian telescope, as described (Def. XII. Schol. 2.), or by a sun-dial.

SECTION II.

Of the Annual Motion of the Earth round the Sun.

PROP. VII. The earth revolving round the sun in 365 days, 6 hours, 9 minutes, 12 seconds, the sun appears to revolve round the earth in the same time.

Plate 9. Fig. 8. Let S represent the sun, BAC the orbit of the earth, and FGHE the starry concave. Whilst the earth is moving from A through B to C, it is manifest that, to a spectator on the earth, the sun must appear to move from E through F to G, in the great circle of the heavens formed by the plane of the earth's orbit. In like manner, while the earth is passing from C to A, the sun will appear to pass from G to E.

Schol. 1. It is manifest that the circle in which the sun appears to move is the same, in which the earth would appear to move to a spectator in the sun. Hence the apparent place of the sun being found, the true place of the earth in its orbit is known.

Schol. 2. The orbit in which the earth revolves round the sun is not a circle but an ellipse, having the sun in one of its foci. For the computations of the sun's place, upon this supposition, allowing for the disturbing forces of the planets, are found to agree with observations. See Prop. XXXIII.

Def. XXI. The circle which the sun appears to describe annually in the concave sphere of the heavens, is called the *Ecliptic.*

Def. XXII. A portion of the heavens, about 16 degrees in breadth, through the middle of which passes the ecliptic, is called the *Zodiac.*

Schol. Within this zone lie the orbits of all the planets.

DEF. XXIII. The stars in the Zodiac are divided into 12 *Signs*; Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius, Pisces. Figures, representing these signs, are drawn upon the celestial globe, in that portion of its spherical surface, which corresponds to the portion of the concave sphere of the heavens, in which the stars belonging to each sign are respectively placed. Plate 9, Fig. 8.

PROP. VIII. The axis of the earth in every part of the earth's revolution about the sun, makes, with the plane of its orbit, that is, of the ecliptic, an angle of $66\frac{1}{2}$ degrees.

Let BA represent the plane of the ecliptic or earth's orbit, seen edgewise; S the sun; and P p produced the axis of the equator. If the earth be at S, its axis is not perpendicular to the plane of the ecliptic, but makes an angle with it, PSA, of about 66° 30′. In any other part of its orbit, as M, or X, the axis of the earth is still inclined to the plane of the ecliptic in the same angle. Plate 9. Fig. 7.

COR. 1. The axis, in any one part of the orbit, is in a position parallel to that in which it was at any other part of the orbit. Supposing the line FG to represent the situation of the axis of the earth when at DFG, and to be parallel to the line HI; then when the earth is at dfg, or any other part of its orbit, its axis fg will still be parallel to the same line HI; therefore fg is parallel to FG. Plate 9. Fig 6.

COR. 2. The planes of the equator and ecliptic make with each other an angle of $23\frac{1}{2}$ degrees nearly.

SCHOL. The obliquity of the ecliptic is not permanent, but is continually diminishing, by the ecliptic approaching nearer to a parallelism with the equator, at the rate of about $\frac{1}{2}$ a second in a year, or from 50″ to 55″ in 100 years. The inclination, in 1820, was 23° 27′ 57″, nearly. The diminution of the obliquity of the ecliptic to the equator is owing to the action of the planets upon the earth, especially the planets Venus and Jupiter. The whole variation, it is said, can never exceed 2° 42′, when it will again increase.

The obliquity of the ecliptic may be thus found. Observe with a good instrument, very accurately divided, the meridian altitude of the sun's centre, on the days of the summer and winter solstice, then the difference of the two will be the distance between the tropics; the half of which will be the obliquity sought.

By the same method, the declination of the sun for every day in the year may be found, and a table constructed. See Prop. XVIII.

DEF. XXIV. The ecliptic being divided into twelve equal parts, each of these parts is called a *Sign;* and the names of the signs in the ecliptic are the same with those in the zodiac, but do not exactly correspond with them.

DEF. XXV. The two points in which the ecliptic cuts the equator are called the *Equinoctial Points;* the vernal equinox is at the first degree of Aries in the ecliptic; the autumnal, at the first of Libra.

SCHOL. The moment of time in which the sun enters the equator may be found by observation, the latitude of the place of the observer being known. For in the equinoctial day, or near it, with an instrument exactly divided into degrees, minutes, and parts of minutes, take the meridian altitude of the sun; if it be equal to the altitude of the equator, or to the complement of the latitude, the sun is then in the equator; but if it is not equal, mark the difference, which will be the declination of the sun. The next day, again observe the meridian altitude of the sun, and gather from thence his declination. If these two declinations be of different kinds, as the one south and the other north, the equinox happens some time between the two observations; if they be both of the same sort, the sun has either not entered the equinoctial, or has passed it. And from these two observations of the sun's declination, the moment of the equinox is thus investigated.

Let CAB be a portion of the ecliptic, EAQ an arc of the equator, and let their intersection be in A. Let CE be the declination of the sun at the time of the first observation, OD his declination in the second observation; the arc CO will be the motion of the sun in the ecliptic for one day. In the spherical triangle AEC, right-angled at E, we have the angle A which the equator and the ecliptic make, as also CE, the declination of the sun, known by observation, by which may be found the arc CA. And in the same manner in the triangle AOD the side AO is found; and thence the arc CO, which is the sum or difference of the arcs CA, AO. Therefore as CO is to CA, so is 24 hours to the time between the first observation, and the moment of the ingress of the sun to the equinox. Plate 9. Fig. 9.

Def. XXVI. The points of the ecliptic which are at the greatest distance from the equator, are called the *Solstices;* and the circles which pass through these points parallel to the equator, are called the *Tropics;* the summer solstice is at the first of Cancer, the winter solstice at the first of Capricorn; the northern tropic is called the tropic of Cancer, the southern, of Capricorn.

Cor. The sun is once in the year at each of the tropics, and twice at the equator.

Def. XXVII. Circles which pass through the poles at right angles to the equator, or any other great circles, are called *Secondaries* to that circle; the secondary which passes through the equinoctial points, is called the *Equinoctial Colure.*

Def. XXVIII. That pole which is nearest the tropic of Cancer, is called the *North Pole;* that which is nearest the tropic of Capricorn, is called the *South Pole.*

Def. XXIX. An imaginary line passing through the centre of the ecliptic, and perpendicular to the plane of it, is the *Axis of the Ecliptic;* its extremities are the *Poles of the Ecliptic*, and all circles, passing through these poles, and perpendicular to the ecliptic, are its secondaries.

Cor. The axis of the ecliptic makes an angle of $23\frac{1}{2}$ degrees nearly with that of the equator. Compare Prop. VIII. Cor. 2. and Schol.

Def. XXX. The *Polar Circles* are described by the revolution of the poles of the ecliptic about the poles of the equator; that which is next to the north pole, is called the *Arctic* circle; the opposite, the *Antarctic* circle.

Def. XXXI. The *Declination* of any heavenly body is its distance from the equator; this is either northern or southern. The degrees of declination of any body are reckoned upon a secondary of the equator passing through that body.

Def. XXXII. The *Right Ascension* of any heavenly body is its distance from the first of Aries reckoned upon the equator; this is measured, by observing the arc which is intercepted between Aries and a secondary to the equator passing through the sun or star.

Def. XXXIII. The *Latitude* of any heavenly body is its distance from the ecliptic, and the degrees of latitude are reckoned on a secondary of the ecliptic, passing through the body.

Def. XXXIV. The *Longitude* of any heavenly body is its distance from the first of Aries; and is measured on the ecliptic by the arc intercepted between the first of Aries and the secondary of the ecliptic, which passes through the body; the longitude increases, as the body recedes from Aries, through the whole revolution, till it reaches 360°, or comes again to Aries.

Def. XXXV. Two bodies are said to be in *Conjunction* with each other, when they have the same longitude, or are in the same secondary of the ecliptic on the same side of the heavens, though their latitude be different; they are said to be in *Opposition,* when their longitudes differ half a circle, or they are on opposite sides of the heavens.

PROP. IX. The axis of the heavens is perpendicular to the planes of all the circles which the heavenly bodies describe in their apparent diurnal motions.

For the heavenly bodies, from the revolutions of the earth round its axis, appear to move from east to west in circles perpendicular to the axis.

Cor. 1. The planes of all these circles are parallel to the equator.

Cor. 2. The axis passes through the centres of the circles.

Def. XXXVI. The celestial sphere is called *right, oblique,* or *parallel,* as the celestial equator is at right angles, oblique, or parallel to the horizon.

PROP. X. In all places on the equator, the poles lie in the horizon, and all the circles of daily motion make right angles with the horizon.

For these places (by Def. XVIII.) having no latitude, the poles (by Prop. III.) are neither elevated above nor depressed below the horizon; and since the equator is 90 degrees from the poles, it is at right angles to the horizon, and also all circles parallel to it.

PROP. XI. Those who live at the equator are in a right sphere; and, consequently, their days and nights are always equal.

The great circle of the celestial equator and its parallels (by last Prop.) make right angles with the horizons of all places in the earth's equator; therefore (by Def. XXXVI.) the inhabitants of those places live in a right sphere. Hence, because the celestial axis PTS is in the plane of their horizon, and this axis is at right angles to the plane of the equator, and (by Prop. X.) passes through its centre and through that of all circles parallel to the equator, the plane of the horizon also passes through the centres of these circles; and consequently divides the equator and its parallels into two equal parts. One half of these circles will therefore always be above the horizon, and the other half below it. But each of the heavenly bodies in its daily motion describes some one of those circles, and the diurnal motion of the earth is uniform; therefore any heavenly body will, in this situation, be just as long above the horizon as below it. And because this will be the case with respect to the sun, as well as any other body, in whatever part of the heavens he is seen, the days and nights at the equator will always be of equal length. Plate 9. Fig. 10.

PROP. XII. At the poles of the earth, one celestial pole is in the zenith, and the other in the nadir; the equator coincides with the horizon, and all the circles of daily motion are parallel to the horizon.

For the latitude of the poles is 90 degrees from the equator, and the circles of daily motion are parallel to the equator.

PROP. XIII. Those who live at either pole are in a parallel sphere; they see the heavenly bodies carried round them in circles parallel to the horizon, and their day and their night continues each half a year.

An inhabitant at P has the equator EQ in the horizon, and all its parallel circles also parallel to the horizon. Therefore each of the heavenly bodies, in its apparent daily motion, being in some one of these circles, must describe a path parallel to the horizon; so that those which are above the horizon will never set by this motion, and those which are below it will never rise. The sun, therefore, in this situation, will not rise or set by the diurnal motion of the earth. But from the annual motion of the earth, the sun daily changes its apparent place in the heavens till it has described the circle of the ecliptic CL; one half of which is above the horizon, and the other half below it, because these circles have a common centre T, the centre of the earth. Therefore, for one half of the year the sun will be in some part of CT, that half of the ecliptic which is above the horizon, and will daily revolve in circles above the horizon; and for the other half, it will be in some part of TL, and will perform its daily revolutions in circles below the horizon. Plate 9. Fig. 10.

PROP. XIV. In any place between the poles and the equator, one celestial pole will be elevated, and the other depressed, at an angle less than a right angle; and the celestial equator will make an angle less than a right angle with the horizon.

For, since the place is not in the equator, it has some latitude; and since it is not at either of the poles, its latitude is less than 90 degrees; whence (by Prop. III.) the poles are elevated, or depressed, in an angle less than a right angle; and consequently the equator, which is perpendicular to the axis, makes an angle of less than 90 degrees with the horizon.

PROP. XV. Those who live on any part of the surface of the earth between the equator and either pole, are in an oblique sphere, and have all the circles of daily motion oblique to their horizon.

Plate 9. Fig. 10. Let HO be the horizon of a place which lies between the earth's equator and either of its poles; the celestial equator EQ, and all its parallel circles, will be oblique to the horizon; and therefore each of the heavenly bodies, being in some one of these circles, will appear to move in a path oblique to the horizon.

PROP. XVI. When the sun, in his annual apparent course, is in the points in which the ecliptic cuts the equator, the day and night will be of the same length at all places on the surface of the earth; but when the sun is in any other part of the ecliptic, the days will be longer as the sun's declination toward the elevated pole increases, and shorter as its declination toward the depressed pole increases.

Plate 9. Fig. 10. The plane of the horizon HO, of any place, passing through T, the centre of the sphere and also through the centre of the equator, divides the equator EQ into two equal parts, one half above, and the other half below the horizon. When therefore the sun has no declination, or is in the equator, it will appear in its daily revolution to describe the equator EQ, and, therefore, during one half of the revolution, it will be above the horizon, and, during the other half, below it.

But suppose the sun to have its declination toward P, the elevated pole, equal to E m; its diurnal apparent revolution will be in the circle $m\,m$, the centre of which is in a part of the axis above the horizon; whence the plane of the horizon does not pass through the centre, and consequently the circle $m\,m$ is divided into two unequal parts, the greater above the horizon, and the less below it. Therefore the sun, describing the circle $m\,m$, with an uniform velocity, in its apparent diurnal revolution will be longer in describing the part above the horizon, than the part below it. And this difference manifestly increases, as the circle of the sun's apparent diurnal motion recedes from the equator, that is, as the sun's declination toward P increases. In like manner, it may be shown that, the days will be shorter, as the sun's declination toward the depressed pole increases.

Plate 9. Fig. 7. Or thus. Let AB represent the plane of the ecliptic seen edgewise; S the sun in the focus of the orbit; MO, KL, XY, the earth in different parts of its orbit. If FI, the axis of the ecliptic BA, were also the axis of the earth, that is, if the planes of the equator and ecliptic were coincident, it is manifest that the sun, the apparent annual motion of which is in the plane of the ecliptic, would at all times of the year appear to move in the circle of the equator, and to be equally distant from the poles, and consequently could produce, by its apparent motion, no varieties in the length of the days and nights. But the earth's axis being inclined to the plane of its orbit, as P p, when the earth is at MO, the pole P will be toward the sun, and the pole p turned from it, and the reverse when the earth is arrived at XY. When the earth is in the middle station between B and A, in either part of its orbit, both the poles will be in the circle illuminated as at KL.

In the position MO, since the sun must always illuminate one half of the globe, the light will pass beyond the pole P as far as F, and will extend toward the pole p no farther than I. Consequently, in the diurnal revolution of the earth round its axis, while the earth remains in this position, all the parts of the globe between F and G will be illuminated, and all the parts between I and H will be dark. Farther, in this position greater portions of those parallels, which lie between the equator and the circle FG, will at any instant be in the illuminated, than in the dark, hemisphere; and, on the contrary, greater portions of those which lie between the circle HI and the equator, will at any instant be in the dark, than in the enlightened hemisphere. Consequently, any given place on the side of the equator toward P, will, in one diurnal revolution, be longer in the light than in the dark, and the reverse on the side toward p. The difference between the length of daylight and night will decrease on either side of the equator, as we approach toward it; and at the equator, the illuminated and dark portions of the circle being always equal, the days and nights will be of equal length. The contrary to all this will take place in the situation XY. Continual variations will take place, while the earth passes from MO to KL, and from KL to XY. But in the situation KL, the illumination extending exactly to both poles, all the parallel circles are half illuminated, and half dark; consequently, any place upon the globe will, in a diurnal revolution, have equal portions of light and darkness; that is, day and night will be every where of equal length. This must happen twice in every annual revolution.

Cor. 1. All bodies, which are on the same side of the equator with the spectator, continue longer above the horizon than below it, and *vice versâ*.

Cor. 2. As the orbits of the moon and planets are inclined to the equator, a variation of the times of their continuance above and below the horizon will take place.

Schol. 1. When the sun is very near either of the tropics, the days do not appear of different lengths, for the circles of apparent diurnal motion are so near to each other, that they cannot be sensibly distinguished.

Schol. 2. The different degrees of heat at different seasons of the year are owing partly to the different lengths of the days, and partly to the different degrees of obliquity with which the rays fall upon the atmosphere at different altitudes of the sun.

PROP. XVII. When the sun, or any other heavenly body, is in the equator, it rises in the east, and sets in the west.

For it then rises and sets in the points in which the equator cuts the horizon; that is, because the equator is at right angles to the meridian, which passes through the north and south points, in the points of east and west.

Cor. In north latitude, those bodies, which have north declination, rise between the east and north; those, which have south declination, rise between the east and south.

PROP. XVIII. When the declination of the sun is toward the elevated pole, its meridian altitude is equal to its declination added to the elevation of the celestial equator; when its declination is toward the depressed pole, its meridian altitude is equal to its declination, substracted from the elevation of the equator.

Let HO be the horizon, T the earth, P and S the celestial poles, Z the zenith, N the nadir, EQ the equator. If the sun be at C, having its declination toward the elevated pole P, when it arrives at the meridian PS, its meridian altitude CH is equal to the sum of CE, its declination, and EH, the elevation of the equator. If the sun be at I, having its declination toward the depressed pole S; when it arrives at the meridian, its altitude IH is equal to the difference of EH, the elevation of the equator, and EI, the sun's declination, as appears from the figure. Plate. 9. Fig. 10.

PROP. XIX. When the declination of a heavenly body toward the elevated pole is equal to the latitude of any place, the body will pass through the zenith of that place; and when its declination toward the depressed pole is equal to the latitude, it will pass through the nadir.

Any star or planet which passes through Z, the zenith, in its apparent diurnal revolution, must describe the circle Z z; whence its distance from the equator or declination will be EZ. But EZ is the distance of the zenith from the equator, which, because the elevation of the equator is equal to the complement of latitude, (Prop. IV.) is equal to the latitude. In like manner the reverse may be proved.

PROP. XX. A heavenly body, seen from any place, will never set from the diurnal motion of the earth, if the complement of its declination toward the elevated pole be equal to, or less than, the latitude of the place; and it will never rise, if the complement of its declination toward the depressed pole be equal to, or less than, the latitude.

Let PD, which is the complement of declination of a body at D, and also the distance of the body at D from the pole, be equal to PO, the elevation of the pole, or (by Prop. III.) the latitude; it is manifest that the body at its lowest depression will be no farther from the pole than the horizon is; that is, will never be below it. In like manner the reverse may be shown. A parallel of declination, as DO, at a distance from the elevated pole equal to the latitude of the place, is called *the circle of perpetual apparition;* and a parallel, as H h, at the same distance from the other pole, *the circle of perpetual occultation.* Plate 9. Fig. 10.

Schol. The latitude of a place may be found, by observing the greatest and least altitude of a fixed star that never sets.

Let A be a star near the north pole, which in its daily motion describes the circle AB without setting. A quadrant being placed in the plane of the meridian, or along the meridian line, observe its altitude when it is at A, and afterwards when at B; the difference of these altitudes is AB. And since the star, in its revolution about the pole, is always at equal distances from it, if AB be bisected in P, this point will be the pole, and consequently PO will be the elevation of the pole. But since the lengths of the arcs AO, BO, have been found by observation, their difference AB, and the half of this difference, AP, or BP, is known; and PO is equal to BP + BO, or to AO — PA. Whence the elevation of the pole, that is, the latitude, is equal to the sum of the least altitude added to half the difference of the greatest and least altitude, or it is equal to the remainder arising from subtracting half the difference of the greatest and least altitudes from the greatest altitude. Plate. 9. Fig. 10.

Or, the latitude may be found from the sun's meridian altitude and declination. If the sun's meridian altitude, found by a quadrant, be CH, this altitude is equal to the sun's declination CE, added to the

elevation of the equator EH. Therefore, if CE, the declination toward the elevated pole, be taken from the meridian altitude, the remainder EH will be the elevation of the equator. But since the elevation of the equator is the complement of latitude, the latitude is the complement of the elevation of the equator. This elevation, therefore, being found, the latitude of the place is known.

Or, the latitude of a place is equal to the sun's meridian zenith distance, added to his declination, when he passes the meridian between the zenith and the equator.

Exp. 1. To find the latitude from an observation of the sun's altitude, Aug. 7, 1776, at the Observatory at Cambridge [in England].

Apparent meridian altitude of the sun's lower limb	53°	46′	8″
Sun's apparent semi-diameter, from the Ephemeris	0	15	50
Apparent altitude of the sun's centre	54	1	58
Deduct for refraction	0	0	41
Altitude of the sun's centre	54	1	17
Zenith distance of the sun's centre is found by subtracting the last altitude from 90°	35	58	43
Add the sun's declination	16	13	57
Latitude of the place	52	12	40

Exp. 2. Dec. 1, 1793. The observed meridian altitude of Sirius was 59° 50′; required the latitude.

Observed latitude	59°	50′	S.
Therefore, Zenith distance	30	10	N.
Declination of Sirius	16	27	S.
Consequently, the latitude required	13	13	N.

Def. XXXVII. The two tropics and two polar circles upon the surface of the earth, divide it into five parts, called *Zones*; the torrid zone lies between the two tropics; the temperate zones between the tropics and polar circles; and the frigid zones between the polar circles and the poles.

PROP. XXI. At any place in the torrid zone the sun is vertical twice every year.

The sun in passing from the equator to the tropic of Cancer, 23½ degrees from the equator, has every northern declination from 0 to 23½; and every place between the equator and the tropic of Cancer has some northern latitude between 0 and 23½; therefore, in some part of its course from the equator to the tropics of Cancer, the sun must have a declination equal to the latitude of every place between the equator and the tropic; whence it must be once in the zenith of every such place in its course toward the tropic of Cancer. For the same reason it must be once in the zenith of every such place in its course from the tropic to the equator. The like may be shown on the southern side of the equator.

PROP. XXII. The sun is vertical once every year at the places which lie in the tropics.

For the sun's declination is then 23½ degrees, equal to the latitude of the tropics.

PROP. XXIII. At the polar circles, the longest day and the longest night is 24 hours.

When the sun is in the tropic of Cancer, the complement of its declination toward the elevated pole is 66½ degrees, equal to the latitude of the arctic polar circle; on this day, therefore, (by Prop. XX.) the sun will not set. When the sun is in the tropic of Capricorn, the complement of its declination toward the depressed pole will be 66½ degrees, equal to the latitude of the arctic pole; whence the sun will not rise during that day. The same may be shown with respect to the antarctic circle.

PROP. XXIV. The longest day, and the longest night, are each of them more than 24 hours within the frigid zone.

For, while the sun's complement of declination toward the elevated pole is less than the latitude of the place, the sun will not set; while the complement of declination toward the depressed pole is less

than the latitude of the place, it will not rise; but this must be the case with respect to every place within the frigid zones, in some part of the sun's course toward the tropics.

PROP. XXV. The sun is never vertical to any place in either of the temperate zones.

For the latitude of all places in the temperate zone is greater than any declination of the sun.

PROP. XXVI. The longest day, and the longest night, in any part of the temperate zones, are less than 24 hours; and the days and nights will be longer, the nearer the place is to the polar circles.

For the complement of the sun's declination can never be less than, or equal to, the latitude of any place in the temperate zones; whence the sun will rise and set every day within these zones. But the farther any place is removed from the equator, the nearer the latitude approaches to an equality with the complement of the sun's greatest declination, when the day is 24 hours; that is, at the polar circles.

PROP. XXVII. At different places, the hour of the day differs in proportion to the difference of longitude; 15 degrees of longitude making the difference of one hour in time, 15′ one minute of time, 15″ one second of time; and it is seen at any given place sooner than at places which lie to the west of it, and later than at places which lie to the east of it.

The sun in its daily apparent motion, which is from east to west, must arrive at the meridian of any given place, as London, sooner than it will arrive at the meridian of any place which lies to the west of London, and later than at the meridian of any place to the east of London; that is, since it is noon at any place when the sun is in its meridian, it will be noon at London sooner than at places west, and later than at places east of it.

For example, if any place lies 15 degrees east of London, that is, has 15 degrees of eastern longitude from London taken as the first meridian, the sun will be one hour sooner at its meridian than at the meridian of London; for, since the sun every day appears to make a complete revolution from any meridian to the same, in 24 hours, it will in every hour describes a 24th part of the circle, that is, 15°. And since a minute of a circle is a 60th part of a degree, and a second of a circle a 60th part of a minute, and 15′ the 60th part of 15°, and 15″ the 60th part of 15′, the sun will move at the rate of 15′ in every 60th part of an hour, and 15″ in every 60th part of a minute, that is, in every minute or second of time. Consequently, it will be noon one minute or one second sooner at a place which is 15′ or 15″ east of London, than at London,

PROP. XXVIII. The difference of longitude at two places may be found by observing, at the same time from both places, some instantaneous appearance in the heavens.

If the eclipse of Jupiter's innermost satellite, on the instant of its immersion into the shadow of Jupiter, be observed by two persons at different places, it will be seen by both at the same instant. But if this instant be half an hour, for example, sooner at one place than at the other, because the places differ half an hour in their reckoning of time, their difference of longitude (by Prop. XXVII.) is 7° 30′.

Schol. From tables of eclipses correctly calculated for any place, the longitude of any place may be found by one observer. But such observations can only be made with certainty by land, on account of the motion of a ship at sea. In order to determine accurately the longitude at sea, it is necessary to have a clock which shall not be sensibly affected by difference of climate, difference of gravity at different places, or the motion of the ship. Such a clock, set for the meridian of London, would constantly show the hour of the day at London, which it is easy to compare with the hour of the day where the ship is, found by observations of the sun or stars.

PROP. XXIX. Those who live in opposite semicircles of the same meridian, but in the same circle of latitude, have opposite hours of the day, but the same seasons.

Being both on the same side of the equator and at the same distance from it, when the sun's declina-

tion makes it summer or winter in one of the places, it will be the same at the other; but because they are distant from each 180 degrees of longitude, when it is noon at one place it will be midnight at the other; these are called *Periœci.*

PROP. XXX. Those who live in opposite circles of latitude, but in the same semicircle of the meridian, have opposite seasons of the year, but the same hour of the day.

When the sun has declination toward the north pole, it will be summer to those who live in the northern circle of latitude, and winter to those who live in the southern circle of latitude. But, having the same longitude, their hours of the day will be the same; these are called *Antœci.*

PROP. XXXI. Those who live in opposite circles of latitude and opposite semicircles of the meridian, have both opposite seasons of the year, and opposite hours of the day.

Because they are in opposite latitudes, they will have opposite seasons; and because they are in opposite semicircles of the meridian, they will have noon when it is midnight at the other; these are called *Antipodes.*

Def. XXXVIII. Twelve secondaries to the celestial equator being conceived to be drawn at equal distances from each other, that is, dividing the equator into 24 equal parts, and the meridian of any place being made one of these secondaries, they are called *Hour-Circles* of that place. Compare Prop. XXVII.

PROP. XXXII. If the celestial sphere had an opaque axis, the shadow of the axis would always be opposite to the sun; and when the sun was on one side of any hour-circle, the shadow of the axis would fall upon the opposite side of the same hour-circle.

For all the hour-circles, being secondaries to the equator, pass through the poles, and the celestial axis is in the plane of every hour-circle. And the shadow of any opaque body, being opposite to the sun, is in the plane with the sun. Therefore in whatever hour-circle the sun is, the shadow of the supposed opaque axis would be in the plane of that circle and opposite to the sun; that is, while the sun is in one semicircle of any hour-circle, the shadow of the axis would fall upon the opposite semicircle.

Cor. Hence as the sun performs its apparent course from east to west, the shadow of the supposed axis would move from west to east.

Schol. The gnomon of a sun-dial represents the supposed axis, and hence its shadow is a measure of time.

To construct a Horizontal Dial.

In every sun-dial the gnomon, when fixed, is parallel to the earth's axis. Now when the sun is in the meridian of any place, the 12 o'clock hour-circle is perpendicular to the plane of the horizon, and the arc from the pole to this plane is equal to the latitude of the place; and the one o'clock hour-circle makes an angle at the pole with it, of 15°, and forms the hypothenuse of a right-angled triangle to the above perpendicular, and the base is the arc measuring the angle between 12, and 1 o'clock; therefore we have, by Spherical Trigonometry, Rad : Sin. L :: tan. 15° : tan. of the hour-angle between 12 and 1 o'clock. If instead of 15, we substitute 30, 45, &c. we get the angles between 12 and 2, 3, &c. o'clock; the same may be done for the half hours or other divisions.

Note. The rational and sensible horizons are, in this case, supposed coincident, which, on account of the sun's great distance, will not occasion any sensible error.

To construct a Vertical South-Dial.

In doing this we must conceive a plane passing through the centre of the earth perpendicular both to the horizon and meridian; and on the south side, lines must be drawn from the centre to the points where the hour-circles cut that plane. In finding these points, we say, as Rad : co. s. Lat :: tan. 15° : tan. of the hour-angle between 12 and 1 o'clock. For the arc of the meridian, from the pole to the plane, is equal to the complement of latitude. The other hour-angles &c. must be obtained in the same way as in the last.

On the subject of Dialing, see Ferguson's Lectures.

PROP. XXXIII. The orbit in which the earth revolves about the sun is elliptical.

It is known from observation, that the apparent motion of the sun, that is, the real motion of the earth, in the ecliptic is not uniform. But by the universal law of bodies revolving about a centre, if its orbit were circular, its velocity must be uniform; since (Book II. Prop. LXXII.) it must describe equal areas in equal times. Whereas, if its orbit be an ellipse, and the sun be placed in one of the *foci*, the same law will require (see Book II. Prop. LXVIII.) that its velocity should not be uniform, but that in passing through its greatest distance C, to its least distance A, it should be accelerated, and in passing from the least distance A to the greatest C, it should be retarded. Since then the motion of the earth is in fact thus retarded and accelerated in different parts of its orbit, it is manifest, that its orbit is eliptical. Plate 9, Fig. 8.

Def. XXXIX. The greatest distance of the earth or any other planet from the sun, is called its *Aphelion*; its nearest distance, its *Perihelion*; the longer axis of the ellipse is called the *Linea Apsidum*, the aphelion is also called the *Summa Apsis*, and the perihelion the *Ima Apsis*.

Def. XL. The *Eccentricity* of the orbit of the earth, or any planet, is the distance between the sun and the centre of the elliptical orbit.

PROP. XXXIV. The sun is eight days longer in performing its apparent course through the six northern signs, than through the six southern signs.

Let ABCD be the orbit of the earth, S the sun, and EFGH the ecliptic. While the earth moves in its orbit from B through C to D, the sun appears to move in the ecliptic from F through G to H, passing through the six northern signs; and while the earth passes from D through A to B, the sun appears to move from H to F, through the six southern signs. Now the line HF bisects the circle EFGH, but divides the ellipse ABCD unequally. And, while the sun appears to pass through the northern signs, the earth passes through more than half its orbit; and while the sun appears to pass through the southern signs, the earth passes through less than half its orbit. Therefore, if the velocity of the motion of the earth were uniform, the sun must appear to be longerr in passing through the six northern than the six southern signs. But whilst the earth is passing though the greater part of its orbit BCD, it is farther from the sun, and consequently moves slower than in the lesser part DAB. On both these accounts, the sun's apparent motion is slower in the northern signs than the southern; the difference is found by observation to be about eight days. Plate 9. Fig. 8.

PROP. XXXV. The apparent diameter of the sun is greater in winter than summer.

It is found by observation, that the diameter of the sun in winter is 32 minutes, $35\frac{1}{2}$ seconds; in summer, 31′ 31″. And his mean apparent diameter is 32′ 3″ according to Sir I. Newton, in his theory of the Moon.

Cor. Hence it appears, that the earth, at the winter solstice, or Capricorn, is in its perihelion.

Schol. 1. The difference between summer and winter in the degrees of heat, is owing chiefly to the different heights to which the sun rises above the horizon, and the different lengths of the days. When the sun rises highest, in summer, its rays fall less obliquely, and consequently more of them fall on the earth's surface than in winter; and when the days are long, and the nights short, the earth and air are more heated in the day than they are cooled in the night, and the reverse.

Schol 2. The doctrine of the Sphere having been explained in the preceding propositions, some of the more useful Problems to be performed on the Terrestrial and Celestial Globes are here subjoined.

Problem I. *To find the latitude of any place.* Bring the place to the graduated side of the fixed brass meridian; the degree, under which it is found, is its latitude. All places under the same degree are in the same latitude. Thus the latitude of London is $51\frac{1}{2}$° north, that of the Cape of Good Hope 34° south.

Prob. II. *To find the longitude of any place.* Bring the place to the fixed meridian; the distance of this meridian from the first meridian, measured on the equator, is the longitude of the place. The longitude of Boston in New England is $70\frac{1}{2}$° west, or 4 hours, 42 minutes in time. That of Rome $12\frac{1}{2}$° east, or 50 minutes in time.

Prob III. *To rectify either globe to the latitude of any place, the zenith, and the sun's place.* If the place be in the northern hemisphere, raise the north pole above the horizon; but if the place be in the southern hemisphere, raise the south pole. Then move the meridian up and down in the notch-

es, till the degree of the place's latitude, counted upon the meridian, below the pole, cuts the horizon; and then the globe is adjusted to the latitude of the place.

Having elevated the globe according to the latitude of the place, count the same number of degrees upon the meridian, from the equator toward the elevated pole, and that point will be the zenith or vertex of the place. To this point of the meridian screw the quadrant of altitude, so that its graduated edge may be joined to the said point; then is the globe rectified for the zenith.

Bring the sun's place in the ecliptic to the meridian, and set the hour-index to 12 at noon; and then the globe will be rectified for the sun's place.

Prob. IV. *To determine the difference of time in different places.* Find the longitude of each place, and reduce the difference into time, allowing an hour for every 15 degrees, and proportionally for lesser parts; the difference of time will be found. If the place lies westward of another, it has its noon later than that other; if eastward, sooner. The longitude of Rome is $12\frac{1}{2}°$ east, that of Constantinople 29°, the difference is $17\frac{1}{2}°$, consequently the difference of time between Rome and Constantinople is 1 hour and 10 minutes.

Prob. V. *The latitude and longitude of any place being known, to find the place on the globe.* Bring the degree of the equator which expresses the given longitude to the fixed meridian, then find the given latitude on the meridian; under this point is the place sought.

Prob. VI. *To find the distance between any two places, and their bearing, or relativê situation with respect to the points of the compass.* Rectify the globe to the latitude of one of the places, and bring the place to the fixed meridian. Then fix the quadrant of altitude to the uppermost point of the meridian, and putting its lower end between the horizon and the globe, slide it along till it passes through the other place. The number of degrees on the quadrant between the two places, will give their distance, allowing $69\frac{1}{2}$ English miles for each degree; and the number of degrees upon the horizon between the meridian and the quadrant, will give the bearing of the second place with respect to the first. Thus the bearing of the Lizard Point from the island of Bermudas is nearly E. N. E.

Prob. VII. *To find the right ascension and declination of the sun, or any star.* On the celestial globe find the day of the month under the ecliptic, against which is the sun's place, or find his place by an *ephemeris;* bring that point under the meridian, and the degree which is over the point is the sun's declination, and the degree of the equator then under the meridian will be the sun's right ascension. A star's declination and right ascension are found, by bringing the star on the globe to the meridian, and proceeding as with respect to the sun. The sun's declination, April 19, is 11° 14′ north, and his right ascension 27° 30′.—The right ascension of Sirius is 99°, its declination 16° 20′ south.

Prob. VIII. *To find what stars pass over, or near, the zenith of any place.* Having found the latitude of the place on the terrestrial globe, all those stars on the celestial globe, which pass under the same degree of the meridian with the given latitude, become vertical at that place.

Prob. IX. *To find what stars never rise, or never set, in a given place.* The globe being rectified for the given place, those stars which do not pass under the wooden horizon, never set; those which do not come above it, never rise.

Prob. X. *To represent the appearance of the heavens at any time.* Rectify the globe to the latitude; bring the sun's place in the ecliptic to the meridian, and set the horary index to the upper 12th hour; then turn the globe till the index points to the given hour. The north pole of the globe must be set to the north in the heavens, then will all the stars upon the globe correspond to their places in the heavens, so that an eye at the centre of the globe would refer every star upon its surface to the place of the stars in the heavens. By comparing, therefore, the stars in the heavens with their places on the globe, a person will easily get acquainted with all the stars.

Example. The situation of the stars at London on the 9th of February, at nine o'clock in the evening is as follows; Sirius, or the dog-star, is on the meridian, its altitude 22°; Procyon, the little dog-star, 16° toward the east, its altitude 43°; about 24° above this last, and a little more toward the east, are Castor and Pollux; S. 65° E. and 35° in height is Regulus, or Cor Leonis; exactly in the east, and 22° high, is Deneb in the Lion's tail; 30° from the east toward the north, Arcturus is about 3° above the horizon; directly over Arcturus and 31° above the horizon, is Cor Caroli; in the north-east are the stars in the extremity of the Great Bear's tail.

Reckoning westward, we see the constellation Orion; the middle star of the three in his belt is S. 20° W. its altitude 35°; nine degrees below the belt, and a little more to the west is Rigel, the bright star in his heel; above his belt in a straight line drawn from Rigel, between the middle and most northward in his belt, and 9° above it, is the bright star in his shoulder; S. 49° W. and $45\frac{1}{2}°$ above the horizon is Aldebaran, the southern eye of the Bull; a little to the west of Aldebaran, are the Hyades; the same altitude, and about S. 70° W. are the Pleiades; in the W. by S. is Capella in Auriga, its altitude 73°; in the north-west, and 42° high, is the constellation Cassiopeia; and almost in the north, near the horizon is the constellation Cygnus.

Prob. XI. *The latitude of a place being given, to find the time of the sun's rising and setting, on any given day, at that place.* Having rectified the globe according to the latitude, bring the sun's place in the ecliptic to the graduated edge of the meridian, and set the horary index to the upper 12. Then

turn the globe to bring the sun's place to the eastern part of the horizon, the index will point to the hour at which the sun rises; on the western side, to the time of its setting. On the 5th of June the sun rises at 3 h. and 40 min. and sets at 8 h. and 20 min. The length of the day is 16 h. 40 min. that of the night 7 h. 20 min.

Prob. XII. *To find all the places on the globe to which the sun will be vertical on a given day.* Bring the sun's place to the fixed meridian, and observe the point of the sun's declination; all the places which in turning the globe round pass under that point, have the sun vertical on the given day.

When the sun's declination is equal to the latitude of any place, *then* the sun will be vertical to the inhabitants of that place.

Prob. XIII. *To find the sun's amplitude.* The globe being rectified for the latitude of the place, bring the sun's place to the eastern side of the horizon; the arc of the horizon intercepted between that point and the eastern point, is the sun's amplitude at rising. Thus on the 24th of May, the sun's amplitude at rising is 36° east, and he sets with 36° western amplitude.—The amplitude increases with the latitude of the place.

Prob. XIV. *To find the sun's meridian altitude.* The globe being rectified for the latitude, zenith, and sun's place, the number of degrees contained between the sun's place and the zenith is the distance of the sun from the vertex at noon; the complement of which to 90 degrees, is the sun's altitude. The meridian altitude of the sun on the 17th of May, at London, is 57° 55′. The altitude being given, bring the quadrant of altitude to meet the sun's place and the intersection of the quadrant and horizon will show the *azimuth.* Thus on the 21st of August, at London, when the sun's altitude is 36° in the forenoon, the azimuth is 60° from the south.

Prob. XV. *To find the place of any heavenly body upon the globe, its longitude and latitude being given.* Place the first degree of the quadrant of altitude upon that degree of the ecliptic which expresses the given longitude, and the 90th degree on the pole of the ecliptic; the point of the globe which is under that degree of the quadrant which expresses the given latitude, is the place of the body; for the quadrant represents a secondary of the ecliptic, an arc of which between the body and the ecliptic is its latitude, and the arc of the ecliptic between the secondary and the first degree of Aries its longitude.

Prob. XVI. *To find the place of any heavenly body upon the globe, its right ascension and declination being given.* Bring that point of the equator which expresses the given right ascension to the meridian; the place sought is under that degree in the meridian, north or south, which expresses the given declination.

Prob. XVII. *To find all those places where it is noon, at any given hour of the day, at any given place.* Bring the given place to the brass meridian, and set the index to the uppermost 12; then turn the globe till the index points to the given hour, and it will be noon to all the places under the meridian. When it is 4 h. 50 m. in the afternoon at Paris, it is noon at New Britain, St. Domingo, Terra Firma, Peru, Chili, and Terra del Fuego. As the diurnal motion of the earth is from *west to east*, it is plain that all places which are to the east of any meridian must necessarily pass by the sun before a meridian which is to the west can arrive at it.

Prob. XVIII. *The hour being given at any place, to tell what hour it is in any other part of the world.* Bring the place where the time is given, under the meridian; set the hour index to the given time, and turn the globe till the other place comes under the meridian, and the horary index will point to the hour required. Thus, when it is nine in the morning at London, it is half past four in the afternoon at Canton in China; when it is three in the afternoon at London, it is 18 minutes past ten in the forenoon at Boston in America.

Prob. XIX. *To find the antœci, the periœci, and the antipodes of any place.* Bring the given place to the meridian, then, in the opposite hemisphere, and in the same degree of latitude with the given place will be the *antœci.* The given place remaining under the meridian, set the index to 12, and turn the globe till the other 12 is under the index; then the *periœci* will be under the same degree of latitude with the given place, and the *antipodes* of the given place will now be under the same point of the brazen meridian where the *antœci* stood before.

Prob. XX. *To find the two days on which the sun is in the zenith of any given place between the tropics.* That parallel of declination which passes through the given place, will cut the ecliptic line upon the globe in two points, which denote the sun's place; against which, on the horizon, will be found the days required. The sun is vertical at Barbadoes, April 24, and August 18.

Prob. XXI. *The time and place being given, to find all those places where the sun is rising, setting, culminating; and also where it is daylight, twilight, or dark-night.* Find the place where the sun is vertical at the given hour, rectify for the latitude of that place, and bring it to the meridian. Then all the places that are in the *west* semicircle of the horizon, have the sun *rising*; those in the *east* se-

micircle, have it *setting*; those under the meridian above the horizon, have it *culminating*; and all places above the horizon, have the sun so many degrees above the horizon, as the places themselves are. Those places that are below the horizon, but within 18° of it, have *twilight*; those lower than 18° have dark-night; and to those under the meridian it is midnight.

Prob. XXII. *To find the place of the moon, or any planet, for any given time.* Take the nautical almanac, or White's Ephemeris, and against the given day of the month will be found the degree and minute of the sign which the moon or planet possesses at *noon*. The degree thus found, and marked in the ecliptic on the globe, you may proceed to find the declination, right ascension, latitude, longitude, altitude, amplitude, azimuth, rising, setting, &c.

Prob. XXIII. *To find the planets which are above the horizon at sun-set, upon any given day and latitude.* Find the sun's place for the given day, bring it to the meridian, set the hour index to 12, and elevate the pole for the given latitude, then bring the sun's place to the western semicircle of the horizon, and observe what signs are in that part of the ecliptic above the horizon, then look to the ephemeris for the day, and it will be seen what planets are in those signs, for such will be visible on the evening of that day.

Prob. XXIV. To *find whether Venus be a morning or an evening star.* Rectify the globe for the latitude and sun's place; find the situation of Venus by an ephemeris, and stick there a small black patch; bring the sun's place to the edge of the eastern horizon. If Venus be *in antecedentia*, that is, for instance, in Taurus when the sun is in Gemini, she will be a morning star.

But, if Venus be *in consequentia*, that is, for example, in Gemini when the sun is in Taurus, she will set after the sun, and be an evening star.

Prob. XXV. *To find all the places to which a lunar eclipse will be visible at any instant.* Find the place to which the sun is vertical at the given time, and bring that place to the zenith, and the eclipse will be visible to all the hemisphere below the horizon, because the moon is opposite to the sun.

Schol. 1. Since lunar eclipses continue in general, for a considerable time together, they may be seen in more places than in one hemisphere of the earth; for, by the earth's rotation about its axis, during the time of the eclipse, the moon will rise to several places after its commencement.

Schol. 2. We cannot, by a globe only, determine the places to which a solar eclipse is visible, because that eclipse does not happen to the whole hemisphere next the sun, nor does it happen at the same time to those places where it is visible. Calculations are therefore necessary.

SECTION. III.

Of Twilight.

PROP. XXXVI. The atmosphere by refracting the rays of light increases the apparent altitude of all the heavenly bodies, except when they are in the zenith.

Plate 16. Fig. 2. Let AB be a portion of the earth's surface, *a b* the upper part of the atmosphere over it, HO the horizon of the place A, S the sun or a star seen from A. The rays of light proceeding from S on entering the atmosphere are refracted toward the perpendicular (by Prop. XI. B. 6.), and as the air increases in density in approaching the earth, the light is constantly passing from a rarer to a denser medium, and of course constantly refracted more and more toward a perpendicular, describing a curve, the character of which has not been determined. Now (by Prop. LXXX. B. 6.) the body S will appear at *s* in a line which is a tangent to this curve in the point A, and as the refraction is toward the perpendicular the point *s* or apparent place of the body S will be nearer the zenith or have a greater altitude than the body itself. As in all other cases of refraction the larger the angle of incidence (or the less the altitude) the greater is the refraction or angle of deviation. In the horizon this angle is about 33′, so that the whole of the sun's disc is apparently above the horizon when it is really below it. By this means we enjoy the sight of the sun, in latitude 42½ about 6m. longer every day in clear weather than we should do without this refraction. This elevation diminishes very fast in the first degrees of altitude, and when the body is in the zenith, the rays coincide with the perpendicular and (by Prop. XIV. B. 6.) suffer no refraction.

Tables of refractions have been calculated by various astronomers, as Sir I. Newton, Mr. Thomas Simpson, Dr. Bradley, Mr. Mayer, &c. The following specimen is taken from Dr. Bradley's table, which is esteemed the most correct, and chiefly used by astronomers. For the method of calculating these tables, see Mr. Simpson's Dissert. p. 46. 4to. Gregory's Astron. Vol. i. Pr. 66; and Vince's Astron. Vol. i. 4to. ch. 7.

MEAN ASTRONOMICAL REFRACTIONS IN ALTITUDE.

App Alt.	Refraction.	App. Alt.	Refraction.	App. Alt.	Refraction.	App. Alt.	Refraction.	App. Alt.	Refraction.	App Alt.	Refraction.
°	′ ″	°	′ ″	°	′ ″	°	′ ″	°	″	°	″
0	33 0	11	4 47	23	2 14	35	1 21	48	51	78	12
1	24 29	12	4 23	24	2 7	36	1 18	50	48	80	10
2	18 35	13	4 3	25	2 2	37	1 16	52	44	82	8
3	14 36	14	3 45.	26	1 56	38	1 13	55	40	85	5
4	11 51	15	3 30	27	1 51	39	1 10	58	35	88	2
5	9 54	16	3 17	28	1 47	40	1 8	60	33	89	1
6	8 28	17	3 4	29	1 42	41	1 5	62	30	90	0
7	7 20	18	2 54	30	1 38	42	1 3	65	26		
8	6 29	19	2 45	31	1 35	43	1 1	68	23		
9	5 48	20	2 35	32	1 31	44	0 59	70	21		
10	5 15	21	2 27	33	1 28	45	57	72	18		
		22	2 20	34	1 24		.	75	15		

PROP. XXXVII. The atmosphere above the horizon is enlightend by the rays of the sun, when the sun itself is below the horizon.

Let ADL be the surface of the earth; CBM, the surface of the atmosphere; A, any place upon the earth; PABN, the sensible horizon. When the sun is at G, any point below the horizon, it cannot be directly seen by a spectator at A. But, because rays from the sun at G can pass to the part of the atmosphere above the sensible horizon of the place A, this part of the atmosphere will be illuminated before the sun rises, or after it sets, and will become visible by reflection to the spectator at A; that is, twilight will be produced. Plate 9. Fig. 11.

COR. The atmosphere is the cause of the apparent elevation of the heavenly bodies above their true places.

PROP. XXXVIII. When the evening twilight ends, or the morning twilight begins, a ray of the sun reflected from the highest part of the atmosphere describes, after reflection, a line, which is the plane of the sensible horizon.

As the sun is depressed, the extreme ray of light from the sun gradually recedes from C toward B, till at last it touches the horizon at B, from whence it is reflected in the direction BA, the plane of the horizon. Plate. 9. Fig. 11.

PROP. XXXIX. If the time of the beginning of the morning or end of the evening twilight is known, the height of the atmosphere may be determined.

It is found, that at the beginning of the morning, or the end of the evening twilight, the sun is 18 degrees below the horizon. And the sun's centre being 18° below the horizon, when the first ray of light appears, his upper limb must be depressed 18° — 16′ = 17° 44′, and if the sun were a point, it would be only 17° 44′ below the horizon at the beginning of twilight, or, subtracting the refraction which takes place in passing from S to D. 17° 11′. Hence AED = 17° 11′, and its half AEB = 8° 35½, and if the rays DB, BA, were not refracted, the upper part of the atmosphere would be at B, but the ray SD, tangent to the surface at D, is bent into a curve D b, and at b is reflected and moves in a similar curve from b to A. Draw $b\,c$ tangent to the curve b A in b to cut AB in c, and A a parallel to $c\,b$ to cut EB in a. Hence B $c\,b$ = BA a = the horizontal refraction 33′, since the tangent at b lies in the direction of the ray on its reaching the atmosphere, and it is evident that the real height of the atmosphere falls between HB and H a, to calculate which we have in the triangle AEB all the angles, viz. EAB = 90°, AEB = 8° 35½′, ABE = 81° 24½′, and AE = 1 (radius of the earth), to find EB = 1.01135, and HB = .01135. And in the triangle AE a we have EA a = 89° 27′, E a A = 81° 57½′ = AE a = 8° 35½′, to find E a = 1.00989, and H a = 00989 Hence H b is greater than 0.00989 and less than 0.01135, the radius of the earth being 1. If most of the refraction take place very near the earth at D and A, the point b will fall near a. On the contrary, if it take place very near the upper part of the atmosphere, the Plate 16. Fig. 3.

point b will fall near B. If we suppose, *as usual*, that the curve A b is the arc of a circle, we shall have, after drawing the chord A b, b A $c = \frac{1}{2}$ b c B $=16\frac{1}{3}'$, (as b c B is the external angle of the isosceles triangle A $b c$). Hence EA $b = 89° \ 43\frac{1}{2}'$, AE $b = 8° \ 35\frac{1}{2}'$, E b A $=81° \ 41'$, and, as EA $= 1$, we find the side E b of the triangle EA b to be 1.0106. Hence H $b = 0.0106$, or about 42 statute miles.

Schol. Dr. Keill, in his Lectures on Astronomy, observes, that it entirely owing to the atmosphere that the heavens appear bright in the day time. For without it, only that part of the heavens would be luminous in which the sun is placed; and if we could live without air, and should turn our backs to the sun, the whole heavens would appear as dark as in the night. In this case also we should have no twilight, but a sudden transition from the brightest sunshine to dark night immediately upon the setting of the sun, which would be extremely inconvenient, if not fatal to the eyes of mortals. See Keill's Astron. Lect. xx.

PROP. XL. The twilight is longest in a parallel sphere, and shortest in a right sphere; and in an oblique sphere, the nearer the sphere approaches to a parallel, the longer is the twilight.

In a parallel sphere, the twilight will continue till the sun's declination toward the depressed pole is 18°, but in this sphere his declination is never more than $23\frac{1}{2}$ degrees; whence the twilight will only cease, whilst the sun's declination is increasing from 18° to $23\frac{1}{2}$°, and decreasing again till in its decrease it becomes 18°. The twilight is here caused by the annual motion of the earth. In a right sphere, the sun appears to be carried, by the daily motion of the earth, in circles perpendicular to the horizon; whence it is carried directly downward by the whole daily motion, and will arrive at 18° below the horizon, the soonest possible; whereas, in an oblique sphere its path is oblique to the plane of the horizon, and therefore will be longer before it is descended 18 degrees below the horizon; and the difference of the time of twilight will increase with the degree of obliquity. As the sun sets more obliquely at some parts of the year than others, the twilight varies in its duration.

SECTION. IV.

Of the Equation of Time.

PROP. XLI. The time in which the sun completes one apparent diurnal revolution, is greater than that in which the earth revolves round its axis.

Plate 9. Fig. 2. If the earth turns round its centre T in the direction RCB, and at the beginning of one revolution the sun was seen at Z, in the meridian; from its apparent annual motion it will, after the diurnal revolution of the earth is completed, be seen advancing in its orbit toward E. The earth, therefore. must perform more than one revolution, and the spectator at R, after returning to the station from which he set out, must advance forward to e, before the sun will be again in the meridian.

PROP. XLII. The obliquity of the ecliptic to the equator would cause the daily increments of the sun's right ascension to be unequal, although the sun's motion in the ecliptic were uniform.

Plate. 9. Fig. 13. Let E be the first degree of Aries, EQ an arc of 90° of the equator, EC the same of the ecliptic, and CQ an arc of the solstitial colure between Cancer and the equator. At E the sun has neither longitude nor right ascension; these may therefore be considered as equal, when the sun sets out from Aries. At C, the longitude is equal to the right ascension; for both EC and EQ are by supposition 90 degrees of great circles of the same sphere. But if the sun be any where between the first of Aries and the first of Cancer, as at S, the longitude will be *greater* than the right ascension ER. For SR being an arc to the secondary of the equator passing through the sun, ES is the longitude, and ER the right ascension; but ES is greater than ER, because the angle at R is a right angle, but the angle at S an acute angle. Now, if the sun be supposed to move uniformly in the ecliptic, or to describe equal arcs in equal times, the daily increments of longitude will be equal to one another; and consequently, since at the two extremes E and C the longitude and right ascension are equal, and the longitude is supposed to increase uniformly, if the right ascension also increased uniformly, they would at all times be equal. But at S, R, or any other points in the same secondary, between the first of Aries and Cancer, the longitude is *greater* than the right ascension; the daily increments of right ascension are therefore unequal.

The longitude and right ascension are equal when the sun is at C and at E, the former being 90° the latter 180° from Aries both on the ecliptic and equator. But between C and E, the longitude is *less* than the right ascension; because ES, opposite to the right angle R, is greater than ER opposite to the acute angle S, and consequently the point S is nearer Aries than the point R. But, the sun being supposed to move uniformly, or to increase its longitude equally every day, if the right ascension also increased equally every day, since the longitude and right ascension are equal at C and E, they would be always equal. But at S, or any where between Cancer and Libra, the longitude is less than the right ascension; consequently, the daily increments of right ascension are not equal. In like manner it may be shown, that in the third quarter the sun's longitude is greater than its right ascension, and in the fourth, less. Plate 9. Fig. 14.

PROP. XLIII. If the plane of the ecliptic coincided with that of the equator, the daily increments of the sun's right ascension would nevertheless be unequal.

Because (by Prop. XXXIV.) the apparent annual motion of the sun is not uniform, it would in some days describe a longer arc than in others; that is, since its right ascension and longitude would in this case be the same, the daily increments of its ascension would be unequal.

DEF. XLI. A *Natural Day* is the time the sun takes in passing from the meridian of any place, till it comes round to the same meridian again.

PROP. XLIV. Any place upon the earth's surface describes more than a circle round the earth's axis in a natural day; and the arc which it describes more than a circle in any day, is the sun's increment of right ascension for that day.

While the earth revolves round its axis, any place upon the earth's surface describes a circle; but (by Prop. XLIII.) while the sun completes its apparent diurnal revolution, any place on the earth's surface will move through one circle and an arc of a second; therefore any such place describes more than a circle (Def. XLI.) in a natural day.

And since both a meridian, and a secondary of the equator, passing through the poles, are perpendicular to the equator (Def. VI. and X.), if the sun at S be in SR, the meridian of any given place, it is also a secondary of the equator passing through that place. In like manner, if the sun be at T, and TV be after a natural day the situation of the meridian of the given place, the sun will be in TV, which will be both the meridian of the place, and a secondary of the equator. Whence, RV being part of the equator, since ER was the sun's right ascension when it was at S, and EV is its right ascension when it is arrived at T, RV must be the increment of the sun's right ascension for the natural day in which it is advanced from S to T. And, because SR, TV, are both perpendicular to the equator, and any place, in one diurnal revolution of the earth, describes a circle parallel to the equator, RV taken in this circle will always be the same arc with RV in the equator, and therefore will be equal to the sun's daily increment of right ascension. Plate 9. Fig. 13.

PROP. XLV. The natural days are not equal to one another.

For any natural day is the time in which the earth performs one revolution round its axis, and such a portion of a second, as is equal to the sun's increment of right ascension for that day; but the sun's daily increments of right ascension are unequal (by Prop. XLII. and XLIII.); therefore the additional portion of the second revolution will sometimes be greater and sometimes less, and consequently the times in which the natural days are completed will be unequal.

DEF. XLII. The *Equation* is the difference between *mean time* and *apparent time.*

PROP. XLVI. If the sun were to move uniformly round the equator in the same time in which it appears to describe the ecliptic, its apparent daily motion would be a measure of mean time.

For the natural days in that case being liable to no variation, either from the obliquity of the sun's orbit, or the irregularity of its motion, must be equal.

PROP. XLVII. The portion of time which passes between the arrival of the sun in the ecliptic to the meridian of any place, and its supposed arrival at the same meridian, if it were to move uniformly in the equator, is the equation.

For (by the last Prop.) it would be noon by mean time at any place, when the sun, if it moved

uniformly in the equator, was arrived at the meridian of that place; and it is noon at the same place by apparent time, when the sun in the ecliptic arrives at the same meridian; therefore the difference between these two arrivals is the difference between mean time and apparent time, or the equation.

PROP. XLVIII. In the time which passes between the arrival of the sun in the ecliptic to the meridian of any place, and its supposed arrival at the same meridian, if it were to move uniformly in the equator, an arc of the equator passes under the meridian, which is equal to the difference between the right ascension of the sun, as it moves in the ecliptic, and the right ascension which the sun would have, if it moved uniformly in the equator.

Plate 9. Fig 6.

Let the sun be at S; and let EC be the ecliptic, EQ the equator, and E the first of Aries; then if a secondary of the equator passes through the sun, SR, being at right angles to EQ, is an arc of that secondary, and (by Def. XXXII.) ER is the sun's right ascension, and the point R is the point in which the right ascension ends; which being in the secondary of which SR is a part, that is, the secondary passing through the sun, arrives at the meridian at the same time with the sun. If the sun were to move uniformly in the equator, and were arrived at P, EP would be its right ascension, and consequently P would be the point in which its right ascension would end, which point P must arrive at the meridian at the same time with the sun, because the sun is supposed to be in that point. Therefore RP the distance of the two points R and P, is an arc of the equator (passing under the meridian in the time specified in the Proposition), which is equal to the difference between the real and supposed right ascensions of the sun, when he arrives at the meridian by his real motion in the ecliptic, and when he arrives at the same meridian by an uniform motion in the equator.

PROP. XLIX. The right ascension of the sun, if it were to move unifomly in the equator, would at any time be equal to the longitude which it would have at that time if it were to move uniformly in the ecliptic, or to its mean longitude.

For on this supposition, the sun, describing the equator with an uniform velocity in the same time in which it actually describes the ecliptic, its velocity would be the same with the mean velocity in the ecliptic. Consequently, the distance of the sun from the first of Aries in the equator would at any time be the same with its distance from the same point in the ecliptic, if it were to move uniformly therein with its mean velocity; that is, its right ascension in the equator would always be equal to its mean longitude in the ecliptic.

PROP. L. An arc of the equator, equal to the difference between the sun's right ascension and its middle longitude, at any given time and place, converted into time, is the equation.

It has been shown (Prop. XLVIII.) that in the portion of time which passes between the arrival of the sun in the ecliptic to the meridian of any place, and its supposed arrival at the meridian, if it were to move uniformly in the equator, an arc of the equator passes under the meridian, which is equal to the difference of the right ascension of the sun as it moves in the ecliptic, and the right ascension which it would have if it moved uniformly in the equator. And it has been proved (Prop. XLVII.) that this portion of time is the equation, and (Prop. XLIX.) that the right ascension which the sun, at any given time and place, would have if it moved uniformly in the equator, is equal to its mean longitude in the ecliptic. Therefore, in the equation, an arc of the equator passes under the meridian, equal to the difference of the right ascension of the sun in the ecliptic, and its mean longitude. Consequently, if this arc be converted into time, that is, if for 15° be taken an hour, for 15′ one minute of time, for 15″ one second of time, the equation of time will be found.

PROP. LI. If the sun's mean longitude be greater than its right ascension, mean time *follows* apparent time; if its mean longitude be less than its right ascension, mean time *precedes* apparent.

Plate 9. Fig. 13.

If the right ascension of the sun, as before supposed in the equator, EP, that is, (by Prop. XLIX.) its mean longitude, be greater than the sun's real right ascension ER, the supposed place of the sun in the equator P, will be to the east of the point R, where the sun's real right ascension ends. Therefore when this point R, at apparent noon, is come to the meridian, the point P will not be arrived at the

meridian; and mean noon will be later than apparent noon. Therefore, when the sun's middle longitude is greater than its right ascension, mean time *follows* apparent. In like manner the reverse may be proved.

COR. Hence in the *former* case the equation is to be *subtracted* from the apparent time found by the dial, and in the *latter*, to be *added* to it, in order to obtain the mean time.

SCHOL. It appears from the foregoing Propositions, that the difference between mean and apparent time, depends upon two causes; (1) the obliquity of the ecliptic with respect to the equator; and (2) the unequal motion of the earth in an elliptical orbit. The obliquity of the ecliptic to the equator would make the sun and clocks agree on four days of the year, viz. when the sun enters Aries, Cancer, Libra, and Capricorn. But the other cause which arises from his unequal motion in his orbit would make the sun and clocks agree only twice a year, that is, when he is in perigee and apogee; consequently, when those two points fall in the beginnings of Cancer and Capricorn, or of Aries and Libra, they will concur in making the sun and clocks agree in those points. But the apogee, at present, is in the tenth degree of Cancer, and the perigee in the tenth degree of Capricorn; and, therefore, the sun and clocks cannot be equal at the beginnings of these signs, nor indeed, at any time of the year, except when the equation resulting from one of the causes is just equal to that arising from the other, one being positive and the other negative, which happens about the 15th of April, the 15th of June, the 31st of August, and the 24th of December; at all other times the sun is too fast or too slow for equal time by a certain number of minutes and seconds, which at the greatest is 16′ 14″, and happens about the first of November; every other day throughout the year having a certain quantity of this difference belonging to it, which, however, is not exactly the same in every year, but only every fourth year, for which reason it is necessary, where great accuracy is required, to have four tables of this equation, viz. one for each of the four years in the period of leap years. The following concise table, adapted to the second year after leap year, will always be found within about a minute of the truth, and therefore sufficiently accurate for common clocks and watches.

TABLE FOR THE EQUATION OF TIME.

Months.	Days.	Equation in Minutes.	Months.	Days.	Equation in Minutes.	Months.	Days.	Equation in Minutes.	Months.	Days.	Equation in Minutes.
Jan.	1	4+	Apr.	1	4+	Aug.	9	5+	Oct.	27	16—
	3	5		4	3		15	4	Nov.	15	15
	5	6		7	2		20	3		20	14
	7	7		11	1		24	2		24	13
	9	8		15	0		28	1		27	12
	12	9		*	——		31	0		30	11
	15	10		19	1—		*	——	Dec.	2	10
	18	11		24	2	Sept.	3	1—		5	9
	21	12		30	3		6	2		7	8
	25	13	May	13	4		9	3		9	7
	31	14		29	3		12	4		11	6
Feb.	10	15	June	5	2		15	5		13	5
	21	14		10	1		18	6		16	4
	27	13		15	0		21	7		18	3
Mar.	4	12		*	——		24	8		20	2
	8	11		20	1+		27	9		22	1
	12	10		25	2		30	10		24	0
	15	9		29	3	Oct.	3	11		*	——
	19	8	July	5	4		6	12		26	1+
	22	7		11	5		10	13		28	2
	25	6		28	6		14	14		30	3
	28	5					19	15			

Those columns that are marked +, show that the clock or watch is faster than the sun; and those marked —, that it is slower. See Ferguson's Astronomy, ch. 13. Phil. Trans. vol. 54.

CHAPTER III.

Of the Inferior Planets, Mercury and Venus.

Def. XLIII. The *Elongation* of any planet is its apparent distance from the sun.

Def. XLIV. The *Nodes* of the orbit of a planet are the two points, in which the orbit cuts the plane of the ecliptic; and a right line, drawn from one node to the other, is the *Line of the Nodes.*

Def. XLV. The *Limits* of the orbit of a planet are two points in the middle between the two nodes.

Def. XLVI. An inferior planet is in its *inferior* conjunction, when it is nearer the earth than the sun is, and in its *superior* conjunction, when it is farther than the sun is from the earth, and both in the same secondary.

Plate 9. Fig. 12. Let A be the place of the earth in its orbit ABO, E the place of Venus in its orbit EHG, S the sun, and FD an arc of a circle in the heavens. Venus will be in its inferior conjunction when it is at E, and in its superior when it is at G, and both are in the same secondary.

PROP. LII. An inferior planet is at its greatest elongation, when a line drawn from the earth through the planet is a tangent to the orbit of the planet.

Plate. 9. Fig. 12. When the planet is at E, being in conjunction with the sun, it has no elongation. As it moves from E toward X its elongation increases, till at X, when AF is a tangent to the orbit of Venus, its apparent place is F, and its elongation FQ, which is the greatest elongation it can have; for in passing from X to G, its elongation decreases, till at G it becomes nothing. This will be true in elliptical as well as circular orbits.

Cor. If the orbits of these planets were circular, the distance of each, from the sun would be to the earth's distance, as the sine of its greatest elongation to the radius.

The orbits are not circles, but ellipses, having the sun in one focus; for on this supposition their computed places are always found to agree with their observed places.

PROP. LIII. The inferior planets are never in opposition to the sun.

For in opposition the earth is between the sun and the planets, which can never happen when the orbit of the planet is nearer to the sun than that of the earth.

Def. XLVII. A planet is in *Quadrature,* when it is 90 degrees in the celestial sphere distant from the sun.

PROP. LIV. The inferior planets are never in quadrature.

Plate 9. Fig. 12. The greatest angle of elongation is that contained by AQ, drawn from the earth through the sun, and AF a tangent to the orbit of the planet. Now if QAF were a right angle, AF would be (El. III. 18.) a tangent to the earth's orbit; but AF is a tangent to an orbit less than that of the earth; it therefore makes an angle with AQ less than a right angle; that is, QF, the greatest elongation, is less than 90 degrees.

Cor. Hence the inferior planets never appear far from the sun.

PROP. LV. While Venus is moving from the superior conjunction to the inferior, it sets after the sun; while it is moving from the inferior conjunction to the superior, it rises before the sun.

Plate 9. Fig. 12. Whilst Venus is moving from G, its superior conjunction, through P, to E, its inferior conjunction, being in the eastern part of its orbit, the sun will be westward of Venus; therefore Venus, if far enough from the sun, will be seen in the west after the sun is gone down, whence it is then called the evening star. But on the western side of its orbit, the sun being eastward of it, Venus will set before the sun, and consequently rise before it, whence it is then called the morning star.

PROP. LVI. The greatest elongation of an inferior planet on one side of the sun is not always equal to that on the other.

For since the planet moves in an elliptical orbit, at the time of its greatest elongation on one side it may be in its aphelion; and at its greatest elongation on the other side, it may be in some part nearer the sun; hence its real distance from the sun at its elongations being unequal, its apparent distances will be so likewise.

COR. The greatest elongation of Venus being found greater than that of Mercury, the latter must be nearer the sun than the former.

PROP. LVII. The apparent velocity of the inferior planets is greatest at the conjunctions.

Since the plane of the orbit of Venus is oblique to that of the earth, those parts of this orbit, which are viewed by a spectator on the earth directly, would appear longer than other equal parts viewed obliquely; whence its motions, if uniform, must appear unequal. If the orbit EPGH of Venus be seen obliquely by an eye placed at A, the parts about E and G, or near the conjunctions, will be seen directly, for AE is perpendicular to a tangent at E; but the parts about X and P will be seen obliquely; whence the Proposition is manifest. Plate 9. Fig. 12.

SCHOL. The time when an inferior planet will come again into a given situation with respect to the sun and the earth may be thus found. Whilst Venus performs one revolution, the earth, whose periodical time is longer than that of Venus, will not have completed its revolution. Before Venus and the earth can be again in the inferior conjunction, Venus must, therefore, besides its entire revolution, describe an arc equal to that which the earth has passed over; consequently, the number of degrees passed over by each, or their angular motions, in the same time, will be reciprocally as their periodical times; that is, as the periodical time of the earth is to the periodical time of Venus, so is the angular motion of Venus (which is equal to four right angles added to the angular motion of the earth between two inferior conjunctions) to the angular motion of the earth in the same time; whence (El. V. 17.) as the difference between the periodical times of the earth and Venus, is to the periodical time of Venus, so are four right angles, or 360°, to the number of degrees over which the earth passes in her orbit from one inferior conjunction to another. This is only true upon the supposition that the planets moved in circular orbits, in which case the following general rule would apply to the finding of the time from conjunction to conjunction, or from opposition to opposition of any two planets. "Multiply their periodic times together, and divide the product by their difference, and you have the time sought." For let P $=$ the periodic time of the earth, $p =$ that of the planet (suppose an inferior), $t =$ time required; then $P : 1 \text{ day} :: 360° : \frac{360°}{P}$, the angle described by the earth in 1 day; for the same reason $\frac{360°}{p}$ is the angle described by the planet in 1 day; hence $\frac{360°}{p} - \frac{360°}{P}$ is the daily angular velocity of the planet *from* the earth. Now if they set out from conjunction, they will return into conjunction again, after the planet has gained 360°; hence $\frac{360°}{p} - \frac{360°}{P} : 360° :: 1 \text{ day} : t = \frac{Pp}{P-p}$. For a superior planet $t = \frac{Pp}{p-P}$.

DEF. XLVIII. The apparent motion of a planet, if seen from the earth, is called its *Geocentric Motion;* if seen from the sun, its *Heliocentric Motion.*

PROP. LVIII. When the inferior planets are passing from their stationary point, through their superior conjunction, to their stationary point on the other side, their geocentric motion is direct, or they appear to move from west to east.

When Venus is at X, it appears to a spectator at A to be in the point F of the concave sphere of the heavens; when it has moved forward in its orbit to H, G, N, P, it will appear successively in the several points F, Q, R, D. But the motion from F to D is from west to east; for whilst the sun and earth are on the same side of the planet, it must appear to move in the same direction, whether it be viewed from the earth or the sun, because the spectator at either station views the concave side of the planet's orbit; but from the sun it is always seen to move from west to east; therefore its apparent geocentric motion in this situation is direct, or *in consequentia.* Plate 9. Fig. 12.

PROP. LIX. While the inferior planets are moving from the stationary point on one side, to the stationary point on the other, through their inferior conjunction, their geocentric motion is retrograde, or from east to west.

While the planet is in this situation, the convex side of its orbit is toward a spectator on the earth, but its concave side toward a spectator at the sun; hence the former will see the planet move in a direction contrary to that in which it will appear to the latter to move. Thus, when the planet is at P, it will appear in the heavens at D, and as it passes through E to X, it appears to move from D through R, Q, P, to F; but the motion from D to F is from east to west; therefore the apparent motion of the planet in this part of its orbit is retrogade, or *in antecedentia*.

PROP. LX. When the inferior planets are at their greatest elongation, they have the same apparent motion as the sun.

For if at any time a planet be moving apparently slower or faster than the sun, its elongation must evidently be increased or diminished, and of course cannot then be the greatest.

Schol. As seen from the sun the motion of these, and indeed of all the planets, appears to be constantly direct; the appearance of the stations and retrogradations is entirely owing to the situation and motion of the earth. And from these causes the apparent motion of Mercury and Venus appears to us to be in looped curves. In order to represent the apparent paths of these planets, Ferguson removed from an Orrery the balls representing the Sun, Mercury, and Venus, and put in their places black-lead pencils, with the points upward, and fixed a circular piece of pasteboard, so that it remained constantly in the same position during the motion of the machine, with its centre resting on the terrestrial ball. Then, the pasteboard being gently pressed on the pencils, and the pencils representing Mercury and Venus, together with the terrestrial ball, being made to revolve about the pencil in the sun's place, by turning the winch of the machine, the three pencils described on the pasteboard Plate 14. Fig. 1. a diagram, of which Plate 14, Fig. 1. is a copy, reduced to a less size. The solar pencil marked the dotted circle of months, which represents the sun's apparent motion in the ecliptic, and is the same every year. The pencil in Mercury's place traced the curve, having 23 loops and crossing itself the same number of times, which represents that planet's apparent motion for 7 years. The pencil in the place of Venus described the curve having 5 loops, which represents the apparent path of Venus for 8 years; after which time this planet moves again very nearly in the same apparent path. The ecliptic encloses these curves, in the figure, and dotted lines are drawn from the earth to the ecliptic, to show the apparent or geocentric motion of Mercury for one year, which is easily traced from A, where it commences, through B, C, &c. At B Mercury is stationary, and the dotted line, drawn through B, shows its place in the ecliptic; from B to C it is retrograde; at C stationary; from C it proceeds directly. And universally, on each side of the loops the planets appear to be stationary; in the part of the loops nearest to the earth, retrograde; and in the rest of the path, direct.

PROP. LXI. The heliocentric latitude of an inferior planet is the greatest when the planet is in one of its limits.

For the planet is then (Def. XLV.) at its greatest distance from the ecliptic, and, therefore, will have the greatest latitude, as seen from the sun.

PROP. LXII. The geocentric latitude of an inferior planet is directly as its heliocentric latitude, and inversely as its distance from the earth.

The apparent length of a line drawn from the planet to the plane of the ecliptic, that is, its geocentric latitude, is (by Book VI. Prop. LXIX. LXX.) directly as its real length, and inversely as the distance of the spectator's eye. But the real length of a line drawn from the planet to the plane of the ecliptic, is its heliocentric latitude; and the spectator's eye is at the earth; whence the proposition is manifest.

Cor. When Venus is in its inferior conjunction, its heliocentric latitude is less than its geocentric; for it is then farther from the sun than from the earth. The contrary takes place with respect to Mercury.

PROP. LXIII. The sun enlightens only one half of a planet, and only one half of a planet is visible at once.

This is sufficiently manifest from the spherical form of the planets, and the rectilinear motion of light.

Schol. Venus and Mercury, in passing from the inferior to the superior conjunction, are observed to have all the phases of the moon from new to full.

Def. XLIX. The hemisphere of a planet which is toward the earth is called its *Disk*, because it appears like a plane circle.

PROP. LXIV. The inferior planets are invisible in their inferior conjunction; their whole disk is illuminated, when they are in their superior conjunction; and they are more or less illuminated, as they are nearer or farther from their superior conjunction.

When Venus, or Mercury, is in its superior conjunction, the whole of its enlightened hemisphere is toward the earth, and its entire disk is visible; as it passes toward its inferior conjunction, its enlightened hemisphere turns, by degrees from the earth, till, at the inferior conjunction, it is wholly turned from the earth, and the planet becomes invisible.

PROP. LXV. If an inferior planet is in one of its nodes at the time of its inferior conjunction, it will appear as a spot in the disk of the sun.

When the planet is in the nodes, it will be in the plane of the ecliptic; and if at the same time it be in its inferior conjunction, it will neither appear above nor below the sun, as it does when in conjunction in other parts of its orbit, but on the sun's disk.

CHAPTER IV.

Of the Superior Planets, Mars, Jupiter, Saturn, and the Herschel.

Def. L. The superior planets are Mars, Jupiter, Saturn, and the Herschel.

PROP. LXVI. The superior planets are sometimes in conjunction with the sun, sometimes in opposition, and sometimes in quadrature.

This is known from observation.

Let S be the sun; QPO a part of the orbit of Jupiter; P the planet; *a d g*, or *n k g*, the earth's orbit. When the earth is at *d*, the sun at S, and the planet at P, the planet is in conjunction; when the earth is at *k*, the sun at S, and the planet at P, the planet is in opposition; when the earth is at *n* or *g*, and the planet at P, the planet will be in quadrature. Plate 10. Fig 1.

Cor. Therefore the superior planets move in orbits farther distant from the sun, than the orbit of the earth.

PROP. LXVII. The apparent diameter of a superior planet is greatest when the planet is in opposition.

For, when the planet is in conjunction, its distance from the earth is greater than when it is in opposition, by the diameter of the earth's orbit.

PROP. LXVIII. If a superior planet were at rest, its apparent geocentric motion in any given time would be proportional to the angle subtended by the arc described by the earth in that time, and seen from the planet.

When the earth is at *a*, the planet P appears in the right line *a* PA, and among the stars in the heavens at A; when the earth is at *b*, the planet appears at B. Therefore while the earth moves from *a* to *b*, the planet appears to move from A to B. But this arc AB is proportional to the angle APB, that is, to the opposite angle *a* P *b*, which is the angle which the arc *a b* would subtend to an eye placed at the planet P. Plate 10. Fig. 1.

PROP. LXIX. The geocentric velocity of a superior planet is greatest at its conjunction and opposition.

Arcs of a given length near the points *d* and *k*, when the planet is seen from the earth in conjunction or opposition, would appear greater than arcs in any other part of the earth's orbit, viewed from the planet P, because the former are seen directly, the latter obliquely; consequently, these arcs would subtend greater angles; whence the apparent velocity of the planet, as viewed from the earth, is greater at the conjunction or opposition, than at any other time.

PROP. LXX. When a line drawn from a superior planet to the earth is a tangent to the earth's orbit, the planet has the same apparent motion, as it has viewed from the sun.

At the time described, the earth viewed from the planet is at its greatest elongation from the sun (by Prop. LII.), and therefore has the same apparent motion with it (by Prop. LX.). But the apparent motion of the earth or sun viewed from the planet is equal to that of the planet viewed from either the earth or sun, which consequently must be the same.

PROP. LXXI. When a superior planet is passing from one of its stationary situations to the other through the conjunction, its geocentric motion is direct; when through the opposition, retrograde.

While the earth is moving from a through d to g, the sun and planet being both on the same side of the earth, the motion of the earth will produce an apparent motion in the sun and planet the same way, and both will appear to move from A toward G. But while the earth moves from g to n through k, the sun and planet being on contrary sides of the earth, the motion of the earth in its orbit will produce an apparent motion of the sun and planet in contrary directions; that is, whilst the sun appears to move from west to east, the planet will appear to move from east to west in the direction G, H, I, &c.

PROP. LXXII. Mars sometimes appears round, sometimes gibbous; Jupiter, Saturn, and the Herschel, always appear round.

When Mars is either in opposition or conjunction, his whole illuminated hemisphere is toward the earth, but when he is in quadrature, some part of his illuminated disk is turned from the earth. The same must happen in the revolutions of Jupiter, Saturn, and the Herschel, about the sun; but their great distance renders the difference between the perfect and imperfect illumination of their disk imperceptible.

Schol. The following particulars respecting Mars are given by Dr. Herschel after long and accurate observations.

The axis of Mars is inclined to the ecliptic 59° 42′.

The node of the axis is in 17° 47′ of Pisces.

The obliquity of the ecliptic on the globe of Mars is 28° 42′.

The point Aries on the *martial* ecliptic answer to our 19° 28′ of Sagittarius.

The figure of Mars is that of an oblate spheroid, whose equatorial diameter is to the polar one as 1355 to 1272, or as 16 to 15 nearly.

The equatorial diameter of Mars, reduced to the mean distance of the earth from the sun, is 9′ 8″.

And that planet has a considerable, but moderate atmosphere, so that its inhabitants, probably, enjoy a situation, in many respects, similar to ours. Phil. Trans. vol. lxxiv. Part 2.

As the most important particulars respecting the stationary positions of the planets, being erroneous, have been expunged from the preceding propositions in this edition, it is thought proper to introduce the following.

Prob. *To determine the position of two planets when they appear mutually stationary, supposing them to move in circular orbits and in the plane of the ecliptic.*

Plate 16. Fig. 4. Let ☉ be the sun, I the inferior, and S the superior planet. Now as no two planets can ever be moving in the same right line, it is very obvious that whenever two appear mutually stationary, (or, which is the same thing, either appears stationary from the other,) while that appearance continues, a right line passing through their centres must be so moved as to continue parallel to itself. Let IS be that right line, and v V a right line passing through the same two planets after a certain portion of time, taken so small that the parts of the orbits described in the interval may not differ sensibly from right lines; if the planets appear mutually stationary the line v V must be parallel to IS, and the lines I i and S s drawn perpendicular to them must be equal. Now taking ISV from the right angles IS s and ☉ SV respectively, VS s = ☉ SI or the elongation of I from the sun, and in the same manner it may be shown that v I i = the supplement of the elongation of S from the sun. But I v and SV express the cotemporaneous motion of I and S respectively, and they are also the secants of i I v and s SV respectively, I i or S s being radius. Therefore as the velocity of the inferior is to the velocity of the superior, so is the secant of the elongation of the superior, to the secant of the elongation of the inferior;

and since secants are inversely as the cosines of the same angles, as the velocity of the inferior is to the velocity of the superior, so is the cosine of the elongation of the inferior to the cosine of the elongation of the superior. Now from the triangle ☉IS it is plain that, at all times, the sine of the elongation of I is to that of S, as the distance of I from the sun is to that of S. Therefore as the relative distances are given, and the velocities easily computed from the distances and periods, which are likewise given, the problem is reduced to this, viz. The ratio of the sines and the ratio of the cosines of two angles being given, to find those two angles. Put $d =$ the distance, $v =$ the velocity, $p =$ the periodical time, and $o =$ the circumference of the orbit of the inferior planet, and D, V, P, & O, for the same particulars of the superior planet respectively. Let $a\,c\,b$ and $a\,c\,e$ (fig. 5.) be the two angles, $a\,b$, $d\,e$, their sines, and $a\,c$, $d\,c$, their cosines respectively, radius being 1; then $d : \mathrm{D} :: a\,b : d\,e$, and $v : \mathrm{V} :: a\,c : d\,c$. Put $a\,b = x$, then $d\,e = \frac{\mathrm{D}\,x}{d}$, $a\,c = \sqrt{1 - x^2}$, and $c\,d = \sqrt{1 - \frac{\mathrm{D}^2\,x^2}{d^2}}$; now by substitution $v : \mathrm{V} :: \sqrt{1-x^2} : \sqrt{1 - \frac{\mathrm{D}^2\,x^2}{d^2}}$; and $v^2 : \mathrm{V}^2 :: 1 - x^2 : 1 - \frac{\mathrm{D}^2\,x^2}{d^2}$. Now to express the velocities in terms of the distances, we find (B. 2. p. 7.) that in uniform motions, the times are as the spaces divided by the velocities, that is, $\mathrm{P} : p :: \frac{\mathrm{O}}{\mathrm{V}} : \frac{o}{v}$, and since circles are as their radii, $\mathrm{P} : p :: \frac{\mathrm{D}}{\mathrm{V}} :: \frac{d}{v}$, and $\mathrm{P}^2 : p^2 :: \frac{\mathrm{D}^2}{\mathrm{V}^2} : \frac{d^2}{v^2}$; but in the solar system $\mathrm{P}^2 : p^2 :: \mathrm{D}^3 : d^3$, therefore $\mathrm{D}^3 : d^3 :: \frac{\mathrm{D}^2}{\mathrm{V}^2} : \frac{d^2}{v^2}$ or $\mathrm{D} : d :: \frac{1}{\mathrm{V}^2} : \frac{1}{v^2} :: v^2 : \mathrm{V}^2$, and taking the first of these couplets for the last in the last analogy containing x, we have $\mathrm{D} : d :: 1 - x^2 : 1 - \frac{\mathrm{D}^2\,x^2}{d^2}$, which by reduction gives x or $a\,b = \frac{d}{\sqrt{\mathrm{D}^2 + \mathrm{D}\,d + d^2}} =$ sine of elongation of the inferior planet, and $d\,e = \frac{\mathrm{D}}{\sqrt{\mathrm{D}^2 + \mathrm{D}\,d + d^2}} =$ sine of supplement of elongation of the superior planet. That is, in words, add together the squares of the distances of any two planets from the sun and their product, divide the distance of either by the square root of the sum, and the quotient will be the natural sine, radius being 1, of the elongation of that planet from the sun, when it appears stationary seen from the other.

This relation between the distances, velocities, and angles does not strictly subsist for any assignable time, but as it is gradually approached, and gradually destroyed, it apparently continues for a considerable time, especially with reference to the superior planets.

CHAPTER V.

Of the Moon.

SECTION I.

Of the Variations in the Appearance of the Moon.

DEF. LI. When the moon is at its greatest distance from the earth in its orbit, which is elliptical, or at its higher apsis, it is said to be in its *Apogee*; when at its least distance, or lower apsis, in its *Perigee*.

DEF. LII. When the moon is in conjunction with the sun, it is *New Moon*; when in opposition, it is *Full Moon*; its conjunction and opposition are called by the common name of its *Syzygies*.

DEF. LIII. A *Periodical Month* is the time in which the moon describes its orbit; a *Synodical Month* is the time which passes between one new moon and the next.

PROP. LXXIII. A synodical month is longer than a periodical month.

Because the moon moves in the same direction with the sun, while the moon performs one revolution in its orbit, the sun, by its apparent annual motion, is advanced in the ecliptic; consequently the moon must pass beyond the point in which it has completed its revolution before it comes again into conjunction with the sun.

Plate 10 Fig. 8. Let S be the sun, BA a part of the earth's orbit, *m d*, MD, the diameter of the moon's orbit when the earth is at B, or A. While the earth is at A, if the moon be at D, it will be in conjunction, and if the earth continued in the same place, after one revolution in its orbit, it would be again in conjunction; but if, during the revolution of the moon, the earth is removed to B, the moon at the end of the revolution will be at *d*, a point which is not between the earth and sun; it must therefore move on from *d* to *e* before it will be in conjunction.

PROP. LXXIV. The moon, at its conjunction, is invisible; at its opposition, its whole disk is enlightened; at its quadrature, it is half enlightened; between the conjunction and quadrature, it is horned; and between the quadrature and opposition, it is gibbous.

Plate 10 Fig. 2. Let QTL be a part of the earth's orbit, S the sun, T the earth, ACEG the moon's orbit. When the moon is at E, or in conjunction, because it is between the earth and sun, its illuminated hemisphere will be wholly turned from the earth, consequently its disk will be dark. At A, being in opposition, its illuminated hemisphere will be wholly toward the earth, and its whole disk will be visible. At C, or G, the apparent distance of the moon from the sun will be 90 degrees; for a right line from C to G will make a right angle with the line TS, in which the sun appears; whence the moon at each of them is half way between the opposition and conjunction, that is, in the middle state between the perfect illumination and the entire darkness of its disk; consequently, its disk is half enlightened. In passing from C to E, and from E to G, its disk will be less than half illuminated, or it will appear horned; in passing from G to A, and from A to C, its disk will be more than half illuminated, or it will appear gibbous.

Cor. The earth must be a satellite to the moon, and subject to the same changes, but more than 13 times larger than the moon appears to us. At new moon to us the earth appears full to her.

Def. LIV. A circle supposed to be drawn upon the surface of the moon, separating the illuminated from the dark hemisphere, is called the *Circle of Illumination;* a circle which separates its visible from its invisible hemisphere, is called the *Circle of the Disk.*

PROP. LXXV. If the centres of the sun, the earth, and the moon, are joined by straight lines forming a triangle, the external angle of this triangle at the moon is equal to the angle contained between the circle of illumination and the circle of the disk.

Plate 10. Fig. 3. Let S be the sun, T the earth, *m r n q* the moon, and O its centre. Let STO be the supposed triangle. Draw the line *r q* perpendicular to SO, and *n m* perpendicular to TOP. The angle SOP will be equal to the angle *m* O *q*, contained between *r q*, which represents the circle of illumination, and *n m*, which represents the circle of the disk. Because the angles SO *q*, PO *m*, are by construction right angles, they are equal; taking SO *m* from each, the remaining angles POS, *m* O *q*, are equal. Consequently the remaining angles POS, *m* O *q*, are equal.

Plate 10. Fig. 3. Schol. This Proposition serves to explain all the different phases of the moon. For example; when the moon is moved from O to A, the line SO coincides with SA, and TO with TA; therefore TO, OS, lie in the same line, and the external angle POS is nothing; whence the two circles of illumination and of the disk coincide; and because the disk is then turned from the sun, it is wholly dark. When the moon is in quadrature, the line SO will be a tangent to the moon's orbit; whence SOP will be a right angle, and the two circles will be at right angles to each other, and the disk will appear half illuminated. If the angle POS be less or greater than a right angle, the circle of illumination will make an angle with that of the disk less or greater than a right angle; whence the illuminated part will appear horned or gibbous. Lastly, when the moon is in opposition, the lines SO, ST, become one and the same line; whence the circles coincide, and the whole disk is illuminated.

PROP. LXXVI. The horns of the moon, after the conjunction, are turned toward the east; before it, toward the west.

The sun, after the conjunction, setting before, that is, to the west of the moon, illuminates the west side of the moon's disk; whence its horns, which are toward the dark part of the disk, are toward the east. The reverse takes place, before the conjunction, when the moon is seen in the east, before the sun rises.

PROP. LXXVII. When the moon is horned, its obscure part is visible by the reflection of the rays of the sun from the earth.

When the moon is at D or F, near the conjunction, the enlightened hemisphere of the earth will be toward the moon, and reflect the rays of the sun upon it. Plate 10. Fig. 2.

PROP. LXXVIII. The moon always has nearly the same side toward the earth.

This is proved by observation.

COR. 1. Hence, if the moon revolves about its axis, its periodical time must be equal to that of its revolution in its orbit round the earth. This is also found to be the case with the fifth satellite of Saturn as it regards its primary.

COR. 2. Though the year is of the same absolute length, both to the earth and moon, yet the number of days in each is very different; the former having $365\frac{1}{4}$ natural days, but the latter only about $12\frac{7}{18}$, every day and night in the moon being as long as $29\frac{1}{2}$ on the earth.

SCHOL. This proposition would be true without any limitation, if the angular velocity of the moon about the earth were equal to the angular velocity about her axis, for then the same face must be always exactly turned toward the earth. But the angular velocity of the moon about the earth is *not* uniform, while that about its axis is uniform, and therefore the same face is not always turned to the earth. Since, however, the opposite face of the moon is never seen, the time of the moon's rotation about her axis must be equal to the *mean* time in her orbit. See Prop. LXXX. and LXXXI.

PROP. LXXIX. The moon appears to have two librations, one upon a line perpendicular to its axis, called its libration in latitude; the other upon its axis, called its libration in longitude.

This appears from observation, some small portions of the surface of the moon being visible in some parts, and invisibe in other parts, of its orbit; that is, in consequence of her libration in latitude we sometimes see one pole, and sometimes the other. And by the libration in longitude, more of the western limb is sometimes seen, and at others, more of the eastern. The inclination of her axis to her orbit is 6° 49′.

COR. The axis of the moon being almost perpendicular to the ecliptic, there will be but a small difference in its seasons.

SCHOL. The moon's axis being so nearly perpendicular to the ecliptic, that the small obliquity of her orbit cannot cause the sun sensibly to decline from the equator; hence we learn, that the inhabitants of the moon must devise some method, different from ours, for ascertaining the length of their year. We know the length of our years by the return of the equinoxes; but these are not discoverable in the moon, where the days and nights are equal.

The Lunarians may, perhaps, measure their year, by observing when either of the poles of our earth begins to be enlightened, and the other to disappear, which happens at our equinoxes; they being conveniently situated for observing large tracts of land about the poles of the earth, which, at present, are unknown to the most experienced of our navigators.

PROP. LXXX. The librations of the moon may be explained on the supposition that the moon has a revolution round its axis.

Let IH represent the plane of the moon's orbit, E the earth, and CMD the moon; and let AB be the axis round which the moon revolves, and A be called its north pole, and B its south pole. CMD will in this situation be the visible hemisphere, and CD the plane of the disk. By the libration in latitude the line AB appears to vibrate on the line IH, so that sometimes the point A is visible, and sometimes the point B. This variation attends the moon's revolution in its orbit; in one half of the orbit the pole A is always visible, and in the other half the pole B. It must therefore arise from the inclination of the axis of the moon to the plane of its orbit. When the moon is at I, if the axis AB be inclined to IH, the plane of the orbit, making with it the angle AIH to a spectator at E, the visible hemisphere will be CND, and the pole A will be within the disk; but when the moon is at H, the Plate 10 Fig. 4,

visible hemisphere will be CMD; and the axis AB being always parallel to itself, the pole B will be within the disk.

Plate 10. Fig. 5. Again, let the moon be at A, the earth at T. If a circle, whose plane is perpendicular to T *c*, pass through the line *d c s*, this circle will be that of the disk, and *d b p s* will be the visible hemisphere. But the moon has an apparent motion, by which the disk is changed, so that at one time *r b* is the circle of the disk, *r p b* the visible hemisphere, at another *f p* and *f c p*. In the former case, *s r* becomes visible, and *d b* invisible; in the latter case, *d f* becomes visible, and *p s* invisible. This libration in longitude arises from the elliptical form of the moon's orbit. If the moon has a rotation round its axis, it has been shown that its revolution will be completed in the time of one revolution in its orbit; and because the motion round the axis is uniform, one quarter of a revolution will be completed in one fourth part of the periodical time. But, the moon's orbit being supposed elliptical, and the earth placed at T one of the foci, the moon will move slower at A its apogee, than at P its perigee, and its velocity will continually increase in moving from A to P, and decrease in moving from P to A. If therefore when the moon is at A, *d b p s* is the visible hemisphere, after it has moved from A to B, through that quarter of its orbit, in which it moves with its less velocity, and consequently takes up more than a quarter of its periodical time, *d s* will not be perpendicular to T*c*, but the point *s* will have turned from west to east more than a quarter round the axis C; hence the point *s* will not be visible when the moon is at B, and *f p* instead of *d s* will be the circle of the disk. In passing from B to P, its excess of velocity will make up for its defect in passing from A to B; and at B it will have completed half its orbit in half its periodical time; but in half its periodical time, it will have revolved half round its axis, therefore at P, *d s* will again be perpendicular to T *c*, and *d x s* will again be the visible hemisphere. The reverse will take place in passing from P to A through D, when *b r* will be the circle of the disk, and *s* will be within it.

PROP. LXXXI. The moon revolves about its axis in the same time in which it revolves about the earth.

late 10 Fig. 4. Without such a revolution, the phenomena of its librations could not happen. If the point A were visible in one part of the moon's orbit, it would be always visible, without a rotation about an axis, oblique to the plane of the orbit, to produce an apparent motion of the point A, or the libration of latitude. If *d s* were perpendicular to *c* T, when the moon is at A, it would be so in every other part of the orbit; and therefore *d x s* would always be the illuminated hemisphere, if there were not a revolution about the axis to produce, in the manner above explained, the libration of longitude. These librations therefore prove the existence of this revolution; and it has been shown, that if there be such a revolution, its periodical time is the same with that of the moon in its orbit.

Fig. 5.

Fig. 2. Cor. Hence it is evident, that one half of the moon is never in darkness; the earth (Prop. LXXIV. Cor.) constantly affording it a strong light, during the absence of the sun; but the other half has a fortnight's light and darkness by turns.

PROP. LXXXII. The orbit of the moon is an ellipse.

It is only on this supposition that the libration of longitude can be explained; from this phenomenon, therefore, the elliptical form of the orbit may be inferred.

Plate 14 Fig. 2. Schol. The moon revolves about the earth in an ellipse, and with the earth revolves about the sun. In Fig. 2, Plate 14, the broad curve line ABCDE represents as much of the earth's annual orbit, as it describes in 32 days, which are marked with numeral figures. The small circles at *a*, *b*, *c*, *d*, *e*, show the moon's orbit in proportion to that of the earth, and the narrow curve *a b c d e f* represents the moon's path in the heavens, or with respect to the sun, for 32 days, reckoning from a new moon at *a*.

The earth being supposed to move from A, and the new moon at the same time from *a*; 7 days afterward the earth will be at B, and the moon in quadrature at *b*; when the earth is at C, the moon is full at *c*; when the earth is at D, the moon is in quadrature at *d*; and when the earth is at E, the moon is again new, or in conjunction at *e*.

From this figure it appears, that the moon's path is always concave toward the sun. For, if we suppose the concavity of the earth's orbit to be measured by the length of a perpendicular C *g*, falling from the place of the earth at C, when the moon is full, on the right line *b g d*, connecting the places of the earth at the end of the first and third quarters of the moon, the length of the perpendicular will be about 640,000 miles; when the moon is new, it is about 240,000 miles nearer to the sun than the earth is; therefore the length of a perpendicular from its place at that time on the same right line will be about 400,000 miles, and shows the concavity of that part of the path.

PROP. LXXXIII. The moon's surface is irregular.

If the surface were every where regular, the limit between the enlightened and dark parts of the disk being the circle of light, that is, a perfect great circle of a sphere, would be exactly defined when the moon is horned, half enlightened, or gibbous, contrary to observation.

Schol. The face of the moon, as seen through a telescope, appears diversified with hills and vallies. This is proved by viewing her at any other time than when she is full; for then there is no regular line bounding light and darkness; but the edge or border of the moon appears jagged; and even in the dark part near the borders of the lucid surface there are seen some small spaces enlightened by the sun's beams.

Besides, it is moreover evident, that the spots in the moon taken for mountains and vallies, are really such from their shadows. For in all situations of the moon, the elevated parts are constantly found to cast a triangular shadow in a direction from the sun; and the cavities are always dark on the side next to the sun; and illuminated on the opposite side. Hence astronomers are enabled to find the height of the lunar mountains. Dr. Keil, in his Astronomical Lectures, has calculated the height of St. Catharine's hill to be nine miles. Since, however, the loftiest mountains upon the earth are but about three miles in height, it has been long considered as very improbable that those of a planet so much inferior in size to the earth should exceed in such vast proportion the highest of our mountains.

By the observations of Dr. Herschel, made in November, 1779, and the four following months, we learn, that the altitude of the lunar mountains has been very much exaggerated. His observations were made with great caution, by means of a Newtonian reflector, 6 feet 8 inches long, and with a magnifying power of 222 times, determined by experiment; and the method which he made use of to ascertain the altitude of those mountains, which, during that time, he had an opportunity of examining, seems liable to no objection. The rock situated near *Lacus Niger*, was found to be about one mile in height, but none of the other mountains, which he measured, proved to be more than half that altitude; and Dr. Herschel concludes, that with a very few exceptions, the generality of the lunar mountains do not exceed half a mile in their perpendicular elevation. See Keil's Astron. Lect. x. Phil. Trans. Vol. lxx.

To Dr. Herschel also we are indebted for an account of several burning volcanoes, which he saw at different times in the moon. In the 77th vol. of the Phil. Trans. he says, April 19, 10 hours 36 minutes sidereal time, "I perceive three volcanoes in different places of the dark part of the new moon. Two of them are nearly extinct; or, otherwise, in a state of going to break out, which perhaps may be decided next lunation. The third shows an actual eruption of fire, or luminous matter." On the next night, Dr. Herschel saw the volcano burn with greater violence than on the preceding evening. He considered the eruption to resemble a small piece of burning charcoal when it is covered by a thin coat of white ashes, which frequently adhere to it, when it has been some time ignited; and it had a degree of brightness about as strong as that with which such a coal would be seen to glow in faint daylight. It is not yet determined whether there is an atmosphere belonging to the moon. The question is fully discussed in the Encyclo. Brit. Vol. II. p. 457, &c. See also Phil. Trans. for the year 1792.

SECTION II.

Of Eclipses.

Def. LV. An *Eclipse of the moon* happens when the earth, passing between the sun and moon, casts its shadow on the moon.

PROP. LXXXIV. The moon can only be eclipsed at the full, or in opposition.

For it is only when the moon is in opposition that it can come within the shadow of the earth, which must always be on that side of the earth which is from the sun.

PROP. LXXXV. The moon can only be eclipsed when, at the full, it is in or near one of its nodes.

The earth being in the plane of the ecliptic, the centre of its shadow is always in that plane; if therefore the moon be in its nodes, that is, in the plane of the ecliptic, the shadow of the earth will fall upon it; also, since this shadow is of considerable breadth, it is partly above and partly below the the plane of the ecliptic; if therefore the moon in opposition be so near one of its nodes, that its latitude is less than half the breadth of the shadow, is will be eclipsed. But, because the plane of the moon's orbit makes an angle of more than 5 degrees with the plane of the ecliptic, it will frequently have too much latitude at its opposition to come within the shadow of the earth.

PROP. LXXXVI. The sun is larger than the earth, and the shadow of the earth is a cone, the base of which is on the surface of the earth.

If the earth were larger than, or equal to, the sun, it is manifest, that its shadow would either perpetually enlarge, or be always of the same dimension; but in this case, the superior planets would

sometimes come within it, and be eclipsed, which never happens. Therefore the sun is larger than the earth, and produces a shadow from the earth of a conical form, which does not extend to the orbit of Mars.

PROP. LXXXVII. The moon is eclipsed by a section of the earth's shadow.

Plate 10. Fig. 7. Let CDE be the earth, CME the cone of its shadow, and FH a part of the moon's orbit passing through the shadow; it is manifest that as the moon describes this part of its orbit, it passes through the circular section FGHL.

Def. LVI. The moon's *Horizontal Parallax* is the angle which a semidiameter of the the earth would subtend, if it were viewed directly from the moon.

LEMMA. *Half the angle of the cone of the earth's shadow may be taken as equal to the angle of the sun's apparent semidiameter.*

Plate 10. Fig. 6. Let AFBG be the sun, HED the earth, HMD or BMA the angle of the cone of the earth's shadow, and CMD half this angle. SA, a semidiameter of the sun, drawn from its centre to the point of contact of the tangent AM, is perpendicular (El. III. 16.) to AM, and is therefore seen directly from D; and it subtends the angle SDA; it must therefore appear large in proportion to the magnitude of this angle. But in the triangle SDM, the external angle SDA is equal to the two angles CSD, CMD; of which, CSD, the angle in which the semidiameter of the earth is viewed from the sun, is so small, that without any sensible error it may be reckoned as nothing, and SDA be said to be equal to SMD.

PROP. LXXXVIII. The semidiameter of the section of the earth's shadow, which eclipses the moon, is equal to the difference between the horizontal parallax of the moon, and the sun's apparent semidiameter.

Plate 10. Fig. 7. CT, being a semidiameter of the earth drawn from the point of contact of the tangent CM, is perpendicular to CM. CT will therefore be seen directly from the point F in the moon's orbit, subtending the angle CFT, which is (Def. LVI.) the moon's horizontal parallax. FMG is the semiangle of the cone of the earth's shadow, equal to the angle of the sun's apparent semidiameter, because MC produced would be a tangent to the circle of the sun's disk. FG is the semidiameter of the section FGHL of the earth's shadow through which the moon passes in an eclipse; and FTG the angle which this semidiameter will subtend when it falls upon the moon, and is viewed from the earth. Now the angle CFT is equal to the two angles FMG, FTG (El. I. 32.), consequently, FTG is equal to CFT — FMG; but by the preceding Lemma, FMT may be taken for the angle of the sun's apparent semidiameter; and CFT is the moon's horizontal parallax; whence the Proposition is manifest.

Cor. Hence the section of the earth's shadow, by which the moon is eclipsed, is broader than the moon; for the semidiameter of the shadow, by this Prop. is $61' - 16' = 45'$, and the diameter of the moon is about $31'$.

Plate 10. Fig. 10. Def. LVII. An eclipse of the moon is *partial*, when only a part of its disk is within the shadow of the earth; it is *total*, when all its disk is within the shadow; and it is *central*, when the centre of the earth's shadow falls upon the centre of the moon's disk.

PROP. LXXXIX. The moon, at the full, will not be eclipsed, if its latitude is equal to, or greater than, the sum of its own semidiameter, added to the semidiameter of the earth's shadow.

Plate 10. Fig. 10. Because a circle or ellipse appears as a right line when the eye is in the same plane with it, let OO represent the plane of the ecliptic, RR the plane of the moon's orbit, and N the node. At the full moon, if the earth's shadow be at A, the moon F must be in the same part of the heavens, because it is in opposition. But because only one half of the shadow of the earth, or about $45'$ is on the same side of the ecliptic with the moon, and only one half of the moon's breadth, or about $16'$, is on the side of its orbit toward the earth's shadow; if the centre of the moon be more than $61'$ from the centre of the shadow, the moon will pass clear of the shadow, and will not be eclipsed.

Fig. 9. Let GE be an arc of the moon's orbit, AB an arc of the ecliptic, and C *c* an arc in a secondary of the ecliptic equal to the moon's latitude. If this arc be equal to, or greater than, C *t* and *c l*, the sum of the semidiameters of the earth's shadow, and of the moon, it is manifest that the shadow cannot pass over any part of the moon's disk.

PROP. XC. The moon, at the full, will be partially eclipsed, if its latitude is less than the sum, but greater than the difference, of its own semidiameter and that of the earth's shadow.

If Cc, the latitude of the moon, be supposed less than $C\,t$, $c\,l$, the sum of the semidiameters of the shadow and the moon, l, the lower edge of the moon, will be below t, the upper edge of the shadow; whence the side of the moon, toward the ecliptic will be eclipsed. But, because $C\,c$, the moon's latitude, added to $c\,o$, its semidiameter, is greater than $C\,t$, the semidiameter of the earth's shadow, the upper edge of the moon o cannot come within the shadow; whence the eclipse will be partial. And because in this case the moon's latitude, together with its semidiameter, is greater than the semidiameter of the shadow, the moon's latitude is greater than the difference of the semidiameters of the shadow and the moon; or, because $C\,c + c\,o$ is greater than $C\,t$, $C\,c$ is greater than $C\,t - c\,o$. Plate 10. Fig. 9.

PROP. XCI. The moon will, at the full, be totally eclipsed, if its latitude be less than the difference between its own semidiameter, and that of the earth's shadow.

If $C\,c + c\,o$ be less than $C\,t$, that is, if $C\,c$ be less than $C\,t - c\,l$, the upper edge of the moon may come within the shadow of the earth. Fig. 9.

PROP. XCII. The moon is centrally eclipsed, only when, in opposition, it is in one of its nodes.

The node N, being the common intersection of the moon's orbit, and the plane of the ecliptic, is in both. Therefore when the moon is in the node, its centre is in the plane of the ecliptic, in which is the centre of the earth's shadow, and consequently at the full these centres coincide.

SCHOL. 1. Both the moon and the shadow moving from west to east, the moon would always be in eclipse while it was at, or near, its nodes, if it moved with the same velocity as the earth; but because it moves much faster than the shadow of the earth, it soon passes from its opposition out of the shadow.

SCHOL. 2. The semiangle of the cone of the earth's shadow being known, the length of the shadow may be found.

The semiangle CMT being equal to the sun's apparent semidiameter, or about 16′, and CT a semidiameter of the earth being perpendicular to the tangent CM, if TM be radius, CT is the sine of the angle CMT. Therefore, as the sine of an angle of 16′ is to radius, so is CT to TM, the height of the cone; that is, as 1 to 217. Whence the length of the shadow is about 217 semidiameters of the earth. Plate 10. Fig. 7.

PROP. XCIII. The moon in a total eclipse, is not wholly invisible.

This is known by observation; and the phenomenon is produced by the reflection of rays of light falling upon the earth's atmosphere, toward the shadow, and consequently toward the moon in the shadow.

DEF. LVIII. An *Eclipse of the Sun* happens when the moon, passing between the sun and the earth, intercepts the sun's light.

PROP. XCIV. The sun can only be eclipsed at the new moon.

For it is only when the moon is in conjunction, that it can pass directly between the sun and the earth.

PROP. XCV. The sun can only be eclipsed, when the moon, at its conjunction, is in or near one of its nodes.

For unless the moon is in or near one of its nodes, it cannot appear in or near the same plane with the sun; without which it cannot appear to us to pass over the disk of the sun. At every other part of its orbit, it will have so much northern or southern latitude, as to appear either above or below the sun. If the moon be *in* one of its nodes, having no latitude, it will cover the whole disk of the sun, and produce a *total eclipse*, except when its apparent diameter is less than that of the sun; if it be *near* one of its nodes, having a small degree of latitude, it will only pass over a part of the sun's disk, or the eclipse will be partial.

PROP. XCVI. In a total eclipse of the sun, the shadow of the moon falls upon that part of the earth where the eclipse is seen.

Plate 10. Fig. 11. Let SL be the sun, TR the moon, VM a part of the surface of the earth. A spectator placed any where between V and M, will not see any part of the sun, because the moon will intercept all the rays of light which come to him directly from the sun; and it is manifest, that, in this situation, the moon, being an opaque body, will cast its shadow upon VM, the part of the earth where the eclipse is total.

PROP. XCVII. The shadow of the moon is a cone terminated in a point.

Because (by Prop. LXXXVIII. Cor.) the diameter of the moon is less than the diameter of the earth's shadow, where it eclipses the moon, and this diameter (by Prop. LXXXVI.) is less than the diameter of the earth, the diameter of the moon is much less than that of the sun; consequently, its shadow will be a cone, terminated in a point. The tangents LAR, SAT, terminate in A; and only the rays that would pass within these tangents are intercepted by the moon; therefore RTA is the form of the moon's shadow.

LEMMA. *Half the angle of the cone of the moon's shadow is equal to the angle of the apparent semidiameter of the sun.*

Plate 10. Fig. 6. Let FBGA be the sun; HED the moon; the cone HMD the moon's shadow; CMD the semiangle of this cone; SA the semidiameter of the sun, and SDA the angle which this semidiameter would subtend, if viewed from D. It may be proved, as in Prop. LXXXVIII. *Lemma*, that CMD is equal to SDA; for the distance of the moon from the sun is so nearly equal to that of the earth from the sun, that the apparent semidiameter of the sun, as seen from the earth or moon, may be considered as equal.

PROP. XCVIII. A semidiameter of the moon's shadow, where it falls upon the earth, is equal to the difference between the apparent semidiameters of the moon and sun.

Plate 10. Fig. 7. Let CDE represent the moon, CME the cone of its shadow; FG the semidiameter of the moon's shadow where it falls upon the earth in a solar eclipse; CT the semidiameter of the moon, CFT, its angle viewed from F, and FTG the angle of the apparent semidiameter of the moon's shadow viewed from T.

In the triangle TFM, the external angle TFC (El. I. 32.) is equal to the two angles FTG, FMG Therefore FTG is equal to the difference between TFC and FMG; and FMG is (by the preceding Lemma) equal to the angle of the sun's apparent semidiameter, and TFC is the angle of the moon's apparent semidiameter; whence the Proposition is manifest.

PROP. XCIX. In a partial eclipse of the sun, a *penumbra*, or imperfect shadow of the moon, falls upon that part of the earth where the partial eclipse is seen.

Plate 10. Fig. 11. A spectator at N, or P, might see the whole sun; for a ray passing from the most remote side of the sun, S or L, would not be intercepted by the moon. But at any point in NM, VP, the spaces between the moon's shadow and the points N, P, the spectator would only see a part of the sun; thus at G, or D, he would see that half of the sun which lies without the tangents DRC, GTC; consequently, in all places between the points N, M, and P, V, there will be less light from the sun, than if it were not at all eclipsed. This deficiency of light is called the moon's *penumbra*.

PROP. C. The moon's penumbra is an increasing cone, and its darkness increases toward the shadow of the moon.

Plate 10. Fig. 11. While a spectator is in the space between N and M, or P and V, he is in the penumbra; but at the points N, P, passes out of it. Therefore the *tangents* NS, PL, are the limits of the penumbra. If tangents be supposed drawn round the spherical surfaces of the sun and moon, they will form two cones, having their common vertex at F, and increasing, the one toward the points L, S, the other toward the points N, P. And as the spectator moves from N toward M, or from P toward V, a greater and greater portion of the sun continually becomes invisible to him; whence the penumbra increases in darkness toward M and V.

PROP. CI. The semiangle of the moon's penumbra is equal to the angle of the sun's apparent semidiameter.

Let SL be the sun, and TR the moon. By Prop. C. the tangents SFN, LFP, terminate the cone of the penumbra. CE, drawn from the centre of the sun to that of the moon, bisects TFR, the angle of this cone; whence EFT is its semiangle. LC being the semidiameter of the sun, LTC is the angle under which this semidiameter would appear from the moon T. I say TFE is equal to LTC. For, in the triangle TCF, the external angle EFT is equal to the two internal and opposite angles FTC, FCT; that is, to the two angles LTC, TCE. Therefore EFT is equal to LTC, TCE. But TCE, being the angle which the moon's semidiameter would subtend, if viewed from the sun, is so small that it may be neglected; whence EFT may be considered as equal to LTC. Plate 10. Fig. 11.

PROP. CII. The semidiameter of the moon's penumbra, in that part through which the earth passes in an eclipse of the sun, is equal to the sum of the apparent semidiameters of the sun and moon.

Let CD be the moon, CDAB its penumbra, and CMB the angle of the cone of the penumbra; and let AEBF be the section of the penumbra through which the earth passes in an eclipse of the sun, AB its diameter, and AT its semidiameter. MLT, drawn from the vertex through the centre of the moon, will bisect the angle CMD. Therefore (by Prop. CI.) CML, the semiangle of the penumbra, is equal to the sun's apparent semidiameter. And CL is the moon's apparent semidiameter, as seen from the earth A, subtending the angle CAL; and AT, the semidiameter of the penumbra, seen from L, subtends the angle ALT. Now the angle ALT is equal to the two angles CML, CAL; whence the truth of the Proposition is manifest. Plate 10. Fig. 12.

DEF. LIX. The disk of the earth is that hemisphere of the earth, which is seen, as a circle, from the moon.

PROP. CIII. At the new moon the whole disk of the earth is enlightened.

For, the moon being then between the sun and the earth, the earth viewed from the moon will appear in opposition, and consequently its enlightened hemisphere will be toward the moon.

PROP. CIV. The semidiameter of the earth's disk is equal to the moon's horizontal parallax.

The moon's horizontal parallax is the apparent semidiameter of the earth, as viewed from the moon, that is, it is equal to the semidiameter of the disk, since the disk is a hemisphere of the earth viewed from the moon.

PROP. CV. If the latitude of the moon, when new, is equal to, or greater than, the sum of the semidiameter of the penumbra and the moon's horizontal parallax, there will be no eclipse of the sun; if less, there will be an eclipse either partial, or total.

Let ACB be a part of the ecliptic; let GNE be the plane of the moon's orbit; let N be the node, DL the earth's disk, and $o\,l$ a section of the moon's penumbra. Because the centre of the moon's penumbra is always in a right line passing through the centres of the sun and the moon, the distance of c, the centre of the moon's penumbra, from the plane of the ecliptic, must always be the same with the distance of the centre of the moon from the same plane; that is, with the latitude of the moon. And because the centre of the earth, or the earth's disk C, is in the ecliptic, C c, the distance between the centres of the penumbra and of the earth's disk, is always equal to the latitude of the moon. Now if C c, or the latitude of the moon, be equal to, or greater than $c\,l$, the semidiameter of the penumbra, together with t C, the semidiameter of the earth's disk, or the moon's horizontal parallax, then no part of the penumbra will fall upon the disk, that is, there will be no eclipse. If C c the latitude of the moon, be less than $t\,C + l\,c$, the edge of the penumbra will be nearer the ecliptic than the edge of the disk, and there will be a partial eclipse. And if C c be less than C t, the shadow of the moon will pass over some part of the disk of the earth, and where this happens, the eclipse will be total. Plate 10. Fig. 9.

PROP. CVI. If the moon, when new, is in one of its nodes, the eclipse of the sun will be central.

For then the centres of the earth, sun, and moon, being all in the plane of the ecliptic, the centre of the moon will pass between the sun's centre, and that of the earth.

Schol. 1. The penumbra of the moon in a central eclipse will not cover the whole disk of the earth.

The semidiameter of the moon's penumbra, being equal to the sum of the apparent semidiameters of the sun and moon, that is, about 16′, 23″ + 15′ 37″, or 32′ at the medium, its diameter is about 64′; whereas the diameter of the earth's disk is about 120′; whence the penumbra cannot cover the whole disk.

Schol. 2. The height of the shadow of the moon is about $60\frac{1}{2}$ semidiameters of the earth. The semiangles of the earth's shadow, and of the moon's shadow, being each equal to the sun's apparent semidiameter, the angles are equal to one another, and these cones are similar. Therefore as the semidiameter of the base of the earth's shadow (that is, of the earth) is to the semidiameter of the base of the moon's shadow (that is, of the moon), so is the height of the earth's shadow to the height of the moon's shadow. Now the semidiameter of the earth is to that of the moon, nearly as 100 to 28, and the height of the earth's shadow is about 217 semidiameters of the earth; whence the height of the moon's shadow is equal to about $60\frac{1}{2}$ semidiameters of the earth; for 100 : 28 : : 217 : $60\frac{1}{2}$ nearly.

Def. LX An eclipse of the sun is said to be *annular*, when, at the time of the eclipse, a ring of the sun appears round the edges of the moon.

PROP. CVII. A central eclipse of the sun will be an annular one, if the distance of the moon from the earth at the time of the eclipse be greater than its mean distance.

Plate 10. Fig. 11. SL being the sun, TR the moon, TAR the moon's shadow, and EA the height of this shadow, which is about $60\frac{1}{2}$ semidiameters of the earth; if at the time of a central eclipse PN is a part of the surface of the earth, those who live in the parts PV, MN, being in the penumbra, will (by Prop. XCIX.) see a partial eclipse; and those who live between V and M, being in the shadow, will (by Prop. XCVI.) see a total eclipse. But if the distance of the moon from the earth be equal to EA, or $60\frac{1}{2}$ semidiameters of the earth (which is the moon's mean distance), A will be the only point from which the eclipse will appear total. And if the moon's distance be greater than EA, as EO, the shadow not reaching the earth, there will be no total eclipse. Consequently, though a spectator at O would see a central eclipse (because the centres C, E, are in the same line with the point of vision S), yet the eclipse would not be total, because the spectator is not in the shadow of the moon. Hence it must appear annular; for let ORX be a tangent to the moon drawn from the eye at O; and it will fall upon the sun at X, and the part XL of the sun will be visible; in like manner parts of the sun equal to XL will be visible all round the moon, forming a ring.

Cor. Hence it appears, that in an annular eclipse it is the penumbra of the moon, which falls upon the earth.

Def. LXI. The *Lunar Ecliptic Limit* is the least distance that the moon can be at from one of its nodes, without being eclipsed at the time of opposition; the *Solar Ecliptic Limit* is the least distance the moon can be at from one of its nodes, without eclipsing the sun at the time of conjunction.

PROP. CVIII. The solar ecliptic limit is greater than the lunar.

Plate 10. Fig. 9. By Prop. CV. it is found, that 92′ is the least latitude the moon can have, when new, without eclipsing the sun.* If, therefore, N c be the distance of the moon from the node, when its latitude, or c C, is 92′, in the triangle C c N, the angle c CN being a right angle, because c C is perpendicular to the plane of the ecliptic; the angle c NC being about 5° 30′, the inclination of the moon's orbit to the plane of the ecliptic; and the side c C being 92′, N c will be found by trigonometry to be about 16°. In the same manner, supposing C c, the latitude of the moon, to be 61′, according to Prop. LXXXIX. the length of the side N c, or the lunar ecliptic limit, will be found to be about 12°.* Whence the truth of the Proposition is manifest.

Cor. There are more eclipses of the sun, in a course of years, than of the moon; for the sun will be eclipsed, if, when the moon is new, it is within 16° of one of the nodes, but the moon only when at

*The moon's latitude may sometimes be several minutes less than that mentioned in the text, without producing an eclipse.

the full it is within 12° of one of the nodes; the sun may be eclipsed while the moon is in 64 degrees of its orbit, but the moon only while it is in 48 degrees of its orbit.

SCHOL. Every eclipse of the moon will be visible, if the moon be above the horizon at the time of the eclipse; because that part of the moon on which the shadow of the earth falls, must appear obscured wherever the disk of the moon is visible. But the sun may be eclipsed, and yet the eclipse be invisible in places to which the sun is above the horizon, because there can be no eclipse of the sun except in those parts of the earth which are within the shadow or penumbra of the moon, and neither of these is large enough to cover the whole disk. Hence, in any given place, more eclipses of the moon than of the sun will be seen in a course of years; for though there are more eclipses of the sun than of the moon, many of the former are not visible at any one place while the sun is above the horizon; but all the latter are visible at the same place while the moon is above the horizon.*

PROP. CIX. When the moon is near the first of Aries, and is moving toward the tropic of Cancer, the time of its rising will vary but little for several days together.

If the moon were to move in the equator, its motion in its orbit, by which it describes a revolution, in respect of the sun, in 29 days 12 hours, would carry it every day eastward from the sun about 12° 11′; whence its time of rising would vary daily about 50 minutes. But, because the moon's orbit is oblique to the equator, nearly coinciding with the ecliptic, different parts of it make different angles with the horizon, as they rise or set, those parts which rise with the smallest angles setting with the greatest, and the reverse. Now the less this angle is, the greater portion of the orbit rises, in the same time. Consequently, when the moon is in those parts which rise or set with the smallest angles, it rises or sets with the least difference of time, and the reverse. But in northern latitudes, the smallest angle of the ecliptic and horizon is made when Aries rises and Libra sets, and the greatest when Libra rises and Aries sets; and therefore, when the moon rises in Aries, it rises with the least difference of time. Now the moon is in opposition in or near Aries, when the sun is in or near Libra, that is, in the autumnal months; when, the moon rising in Aries, whilst the sun is setting in Libra, the time of its rising is observed to vary only two hours in 6 days in the latitude of London. This is called the harvest moon.

SCHOL. This circumstance takes place every month; but as it does not happen at the time of full moon, there is no notice taken of it. When the moon's right ascension is equal to six signs, that is, when she is in or about the beginning of Libra, there is the greatest difference of the times of rising, viz. about an hour and 15 minutes. Those signs which rise with the least angle set with the greatest, and the contrary; therefore, when there is the least difference in the times of rising, there is the greatest in setting, and *vice versâ.*

The following table shows the daily mean difference of the moon's rising and setting, on the parallel of London, for 28 days; in which time the moon finishes her period round the ecliptic, and gets 9 degrees into the same sign from the beginning of which she set out.

Days.	Signs.	Degrees.	Rising Diff. H. M.	Setting Diff. H. M.	Days.	Signs.	Degrees.	Rising Diff. H. M.	Setting Diff. H. M.
1	♋	13	1 5	0 50	15	♑	17	0 46	1 5
2		26	1 10	0 43	16	♒	1	0 40	1 8
3	♌	10	1 14	0 37	17		14	0 35	1 12
4		23	1 17	0 32	18		27	0 30	1 15
5	♍	6	1 16	0 28	19	♓	10	0 25	1 16
6		19	1 15	0 24	20		23	0 20	1 17
7	♎	2	1 15	0 20	21	♈	7	0 17	1 16
8		15	1 15	0 18	22		20	0 17	1 15
9		28	1 15	0 17	23	♉	3	0 20	1 15
10	♏	12	1 15	0 22	24		16	0 24	1 15
11		26	1 14	0 30	25		29	0 30	1 14
12	♐	8	1 13	0 39	26	♊	13	0 40	1 13
13		21	1 10	0 47	27		26	0 56	1 7
14	♑	4	1 4	0 56	28	♋	9	1 00	1 58

* For the calculation and projection of lunar and solar eclipses, see the Problems immediately preceding the Tables at the end of this book.

Exp. Let small patches be placed on the ecliptic of a globe, as far from one another as the moon moves from any point of the ecliptic in 24 hours, that is, about $13\frac{1}{4}$ degrees; then, while the globe is turned round, observe the rising and setting of the patches in the horizon; the hour index will show the difference of time at which the moon rises or sets in different parts of its orbit.

CHAPTER VI.

Of the Satellites of Jupiter, Saturn, and the Herschel Planet.

PROP. CX. Any satellite is at its greatest elongation from its primary, when a line drawn from the earth through the satellite is a tangent to the orbit of the satellite.

Plate 10. Fig. 14. Let FIE be a part of the orbit of the primary planet, AXBT the earth's orbit, S the sun, KGNL the orbit of a satellite. If the earth is at X, and the satellite at L or N, so that a line XL or XN, drawn from the earth, is a tangent to the orbit KGNL, it may be shown, as before concerning the planets, that L or N is the greatest elongation of the satellite.

PROP. CXI. Any satellite appears in inferior conjunction with its primary, when the satellite is between the earth and the primary, and in superior conjunction, when the primary is between the satellite and the earth.

If the earth be at X, and the planet at I, the outermost satellite will be in conjunction with its primary when they both appear in the same line MIV, and in its inferior conjunction at M, and its superior at V.

PROP. CXII. The apparent motion of any satellite, as it passes from its greatest elongation on one side of its primary through the superior conjunction, to its greatest elongation on the other side, is direct.

As the satellite passes from L, its greatest elongation on one side through V, its superior conjunction to N, its geocentric motion is from west to east, or *in consequentia*, as was shown concerning the planets.

PROP. CXIII. The apparent motion of any satellite, as it passes from its greatest elongation on one side of its primary through the inferior conjunction, to its greatest elongation on the other side, is retrograde.

As the satellite passes from N through M to L, its geocentric motion will be from east to west, or *in antecedentia*, as was proved concerning the planets.

Cor. The satellites are sometimes to the west and sometimes to the east of their primaries.

PROP. CXIV. The greatest elongations of a satellite on each side are equal.

For by observation it is found, that the angles LXI, NXI, are equal with respect to all the satellites.

Cor. Hence it appears that the orbits of the satellites are circular, or nearly so, having their primaries at the centre of their orbits.

Plate 15. Fig. 1. Schol. The satellites move round the primary in orbits, that are nearly circular, but round the sun in curves of a different kind. Let ABCDE, &c. to T, (Plate 15, Fig. 1.) be as much of Jupiter's orbit as is described by that planet in 18 days. Then the curves *a*, *b*, *c*, *d*, (according to Ferguson) represent the paths of the four satellites.

If we suppose Jupiter to move from A, the first satellite from *a*, the second from *b*, the third from *c*, and the fourth from *d*; or each of the satellites from the point of conjunction with the sun, as seen from the primary; then, at the end of one day, Jupiter will be at B, and the satellites at 1, in their respective courses. At the end of the second day Jupiter will be at C, and the satellites at 2 in the curves they respectively describe, and so on; the capital letters showing Jupiter's place in its

path at the end of each day, the figure under each of them the number of the day, and the like figures on the paths of the satellites, their places at the same time. The first satellite appears to be stationary at + near C, as seen from the sun; retrograde from + to 2, at 2 stationary again; thence direct till beyond 3; twice stationary and once retrograde between 3 and 4. This satellite intersects its own path every $42\frac{1}{2}$ hours, making loops as in the diagram at 2, 3, 5, &c. soon after every conjunction. The second crosses its own path every 3 days 13 hours, as at 4, 7, 11, &c. making only 5 loops, and as many conjunctions while the first makes 10. The third satellite at the end of every 7 days 4 hours, makes an angle in conjunction with the sun, as at 7, 14. The fourth satellite is always progressive, making neither loops nor angles; its first conjunction is at *e*, at the end of 16 days 18 hours. The first, second, and third are, according to the figure, nearly in the same relative situation, every seventh day.

The path of Saturn's first satellite about the sun is looped, but not those of the second, third, fourth, and fifth.

PROP. CXV. The satellites of Jupiter, Saturn, and the Herschel, are eclipsed by their respective primaries.

The planet I, being an opaque body, casts a shadow IV, opposite to the sun. Therefore, when one of its satellites in describing the arc VO comes to V, it will be eclipsed by falling into this shadow. If the earth is at A in its orbit, a spectator from the earth will lose sight of the satellite, when it is thus eclipsed at V; and then, as it emerges from the shadow, it becomes again visible, till, at O, it passes behind its primary. If the earth be at X, the satellite will be eclipsed, and in occultation at the same time. Plate 10. Fig 14.

PROP. CXVI. When one of the satellites passes between the sun and its primary, it eclipses the sun.

A satellite at M will be between the sun and its primary, and occasion an eclipse of the sun on that part of the primary where the shadow of the satellite passes, which shadow will appear as a dark spot on the disk of the planet to an inhabitant of the earth.

SCHOL. Dr. Herschel has discovered, that the fifth satellite of Saturn is, in its rotation, subject to the same law that our moon obeys, that is, it turns round its axis in the same time in which it revolves about the planet. Hence the doctor thought it natural to conclude, that all the secondary planets might be governed by the same laws to which those are subject. This theory he thinks considerably confirmed by certain observations, which he made on the satellites of Jupiter, and which he communicated to the Royal Society, June 1, 1797.

The following table will give the periodical times and distances of Jupiter's satellites, and the angles under which their orbits are seen from the earth, as its mean distance from Jupiter.

Satellites.		Days.	h.	min.		Dist. in miles.		Angles of orbit.	
1	—	1	18	27.6	—	266,000	—	3′	55″
2	—	3	13	13.7	—	423,000	—	6	14
3	—	7	3	42.6	—	676,000	—	9	58
4	—	16	16	31.8	—	1.189,000	—	17	30

These satellites of Jupiter are of great use in astronomy. (1.) In determining the distance of Jupiter from the earth. (2.) They afford a method of demonstrating that the motion of light is progressive, and not instantaneous, as was once supposed. See Prop. CXVII. And (3.) The most considerable advantage is derived from the eclipses of the satellites of Jupiter, in ascertaining the longitude of different places on the earth.

EXP. Suppose two observers of an eclipse, the one at London, the other at the Cape of Good Hope; the eclipse will appear to both at the same moment of time; but being situated under different meridians, they count different hours, according to which their difference of longitude is found. Thus, if at the Cape of Good Hope, an emersion of a satellite is observed at 10h. 46′ 45″ apparent time, and the same is seen at Greenwich at 9h. 33′ 12″, the difference of which times is 1h. 13′ 33″, the longitude of the cape *east* of Greenwich in time, or 18° 23′ 15″.

Note. The third satellite is the largest of all; the first and fourth are nearly of the same size; the second is the smallest.

OF THE SATELLITES OF SATURN.

Satellites.		Periodical times.					Distance in miles.
1	—	1 day. 21h.	18′	27″	—		170,000
2	—	2	17	41	22	—	217,000
3	—	4	12	25	12	—	303,000
4	—	15	22	41	13	—	704,000
5	—	79	7	48	—	—	2.050,000
6	—	1	8	53	8	—	135,000
7	—	—	22	37	22	—	107,000

Plate 14. Fig. 3. Fig. 3. Plate 14, is a view of the proportional magnitudes of the orbits of the moon, Jupiter's four satellites, and the five first of Saturn. M m represents the moon's orbit, the earth being supposed to be at E; J Jupiter, 1, 2, 3, 4, the orbits of the four satellites; Sat. Saturn, 1, 2, 3, 4, 5, the orbits of the five first satellites.

The 6th and 7th satellites were discovered by Dr. Herschel, in the years 1787 and 1788. To prevent mistakes, he calls them the 6th and 7th, though nearer to the planet than the other five. Dr. Herschel observes, that Saturn has probably a considerable atmosphere. It turns on an axis perpendicular to the ring, in 10 h. 16′ 0.44 ′, and is flattened at the poles, so that the equatorial diameter is to the polar as 11 to 10. Phil. Trans. Vol. 80, Part I. and II. and Vol. 84.

OF THE SATELLITES OF HERSCHEL.

Dr. Herschel has at different times discovered six satellites belonging to the new planet, two of which he described in the 77th and 78th Vols. of the Phil. Trans. where he says, "I confess that this scene appeared to me with additional beauty, as the little secondary planets seemed to give a dignity to the primary one, which raises it into a more conspicuous situation among the great bodies in our system."

The following is the arrangement of the six satellites.

Satellites,		When discovered.		Periodical times.			
1	—	Jan. 18, 1790	—	5d.	21h.	25′	0″
2	—	Jan. 11, 1787	—	8	17	1	19
3	—	Mar. 26, 1794	—	10	23	4	0
4	—	Jan. 11, 1787	—	13	11	5	1½
5	—	Feb. 9, 1790	—	38	1	49	0
6	—	Feb. 28, 1794	—	107	16	40	0

"It will be hardly necessary," says Dr. Herschel, "to add, that the accuracy of these periods depends entirely upon the truth of the assumed distances; some considerable difference, therefore, may be expected, when observations shall furnish us with proper *data* for more accurate determinations." See Phil. Trans. 1798.

PROP. CXVII. A ray of light is about 8 minutes in coming from the sun to the earth.

Plate 10. Fig. 13. Let A be the sun, BECD the earth's orbit, F the planet Jupiter, and HNG the orbit of its inner satellite. Let FGH represent the shadow of Jupiter. While the satellite is between H and G, it is eclipsed; when it comes to H, it emerges, and becomes visible to a spectator at B. From comparing the times of the apparent entrance and emersion of the satellite, with tables calculated for the mean distances of the earth from the satellite, the visible emersion at the least distance is found to happen about 8 minutes sooner, and at the greatest distance about 8 minutes later, than by the tables; consequently, the ray of light is about 16 minutes in passing through the earth's orbit, or 8 minutes in coming from the sun to the earth.

Cor. The diameter of the earth's orbit being 190,000,000 miles, the velocity of light will be $\frac{190,000,000}{16 \times 60} = 197,916\frac{2}{3}$ miles in a second.

PROP. CXVIII. Jupiter is surrounded by cloudy substances, subject to frequent changes in their situation and appearance, called his Belts. Saturn is encompassed with a Ring, whose greatest apparent diameter is to that of the planet as 9 to 4.

Plate 10. Fig. 15. These are known from observation. The Belts of Jupiter are sometimes of a regular form; sometimes interrupted and broken; and sometimes not at all to be seen. The plane of Saturn's ring is

inclined to the plane of the ecliptic at an angle of 31 degrees; which appears like two arms to the planet, and is only visible when the sun and the earth are both on the same side of its plane. On account of its inclination, it always appears oblique to the eye, and therefore elliptical; whence the part behind Saturn is invisible, and the part before cannot be distinguished from the planet. The ring, being opaque, can only be visible when the sun's rays are reflected from its broad surface to the earth, that is, when the sun and the earth are both on the same side of the plane of the ring.

The latter discoveries of Dr. Herschel have shown, that what was supposed to be a single broad flat ring of Saturn, is divided into two parts, lying exactly in the same plane, and revolving about an axis perpendicular to that plane, in 10 h. 32′ 15″. The dimensions of these concentric rings, and the space between them, he states to be as in the following table.

	Miles.
Inner diameter of the smaller ring - - -	146,345
Outside diameter of ditto - - - -	184,393
Inner diameter of the larger ring - - - -	190,248
Outside diameter of ditto - - - - - - -	204,883
Breadth of the inner ring - - - - - - -	20,000
Breadth of the outer ring - - - - - - -	7,200
Breadth of the vacant space - - - - - - -	2,830

CHAPTER VII.

Of Comets.

PROP. CXIX. Comets are opaque and solid bodies.

A comet, at a given distance from the earth, shines much brighter when it is on the same side of the earth with the sun, than when it is on the contrary side; from whence it appears that it owes its brightness to the sun. The resistance of the comet of 1680 to the action of the great heat, to which it was probably exposed in its near approach to the sun, furnishes evidence in favour of its being a fixed and solid body.

PROP. CXX. The comets describe very eccentric ellipses about the sun, placed in one of their *foci*.

They are observed to approach toward, and afterward recede from, the sun, and to describe paths in the heavens, which agree with elliptic orbits; it is therefore most probable, that, agreeably to the general analogy of nature, they move in such orbits, and have the sun in one of the *foci* of the ellipse. The calculations framed upon this supposition, by which the returns of comets have been foretold, having, as far as observations have been made, been found to agree with the phenomena, strongly confirm the truth of the Proposition.

SCHOL. 1. Comets are often accompanied with a luminous train, called the tail, which is conjectured to be smoke rising from the body in a line opposite to the sun. The body of the comet is supposed to be surrounded by an atmosphere; the sun is also supposed to be surrounded by an ether, or a subtle fluid, extending to a great distance from the sun, which may be considered as the solar atmosphere. From the heat which the comet has acquired by approaching toward the sun, and by the reflection of the sun's rays from the solid body and atmosphere of the comet, the parts of the solar atmosphere where the comet passes are more heated, and consequently more rarefied or specifically lighter than elsewhere. The parts thus rarefied will be put into motion; and since there will be a constant succession of fresh portions of the sun's atmosphere within that of the comet, there will be a perpetual stream of this rarefied matter. This stream will impel the particles of the comet's atmosphere, and make them move along with it, thus producing the smoke which, reflecting the sun's rays, forms the visible tail. And this stream of rarefied solar atmosphere will move those parts of this atmosphere which have the least specific gravity, that is, directly from the sun.

SCHOL. 2. Of all the comets, the periods of only three are known with any degree of certainty. The first of these comets appeared in the years 1531, 1607, and 1682; and is expected to appear every 75th year. The second of them appeared in 1532 and 1661, and was expected to return in 1789, and every 129th year afterward. The third, having last appeared in 1680, and its period being no less

than 575 years, cannot return until the year 2225. This comet, at its greatest distance, is about 11 thousand two hundred millions of miles from the sun; and at its least distance from the sun's centre, which is 49,000 miles, is within less than a third part of the sun's semidiameter from his surface. In that part of its orbit which is nearest the sun, it moves at the rate of 880,000 miles in an hour.

Schol. 3. Dr. Halley, who saw the comet which appeared in 1682, observes, "that there are many things which make me believe, that the comet which Apian saw in the year 1531, was the same with that which Kepler and Longomontanus more accurately described in the year 1607; and which I myself have seen return, and observed in the year 1682. All the elements agree, and nothing seems to contradict this opinion besides the inequality of the periodic revolutions; which inequality is not so great, but that it may be owing to physical causes. For the motion of Saturn is so disturbed by the rest of the planets, especially Jupiter, that the periodic time of that planet is uncertain for some whole days together. How much more, therefore, will a comet be subject to such like errors which rises almost four times higher than Saturn, and whose velocity, though increased but very little, would be sufficent to change its orbit from an ellipse to a parabola. And I am the more confirmed in my opinion of its being the same; for, in the year 1456, in the summer time, a comet was seen passing retrograde between the earth and sun, much after the same manner; which, though nobody made observations upon it, yet, from its period, and manner of transit, I cannot think different from those I have just now mentioned. And since looking over the history of comets, I find, at an equal interval of time, a comet to have been seen about Easter in the year 1305, which is another double period of 151 years before the former. Hence I think I may venture to fortell that it will return again in the year 1758."

Dr. Halley computed the effect of Jupiter upon this comet in 1682, and found that it would increase its periodic time above a year; in consequence of which, he predicted its return at the end of the year 1758, or the beginning of 1759. M. Clairaut computed the effects of both Saturn and Jupiter, and found that the former would retard its return in the last period 100 days, and the latter 511 days; and he determined the time when the comet would come to its perihelion to be on April 15, 1759; observing, that he might err a month from neglecting small quantities in the computation. The comet did pass the perihelion on March 13, within 33 days of the time computed. Now, if Dr. Halley meant the time of its passing the perihelion, and we add 100 days for the action of Saturn, which he did not consider, it will bring it very near to the time in which it passed the perihelion, and prove his computaion of the effect of Jupiter to have been very accurate. But if he meant the time when the comet would first appear, his prediction was accurate, for it was seen on December 14, 1758. Dr. Halley, therefore, had the glory first to fortell the return of a comet, and the event answered, in a remarkable manner, his prediction. He farther observed, that the action of Jupiter, in the descent of the comet toward its perihelion in 1682, would tend to increase the inclination of its orbit; and accordingly the inclination in 1682 was found to be 22′ greater than in 1607.

Dr. Halley suspected the comet in 1680, to have been the same which appeared in 1106, 531, and 44 years before Christ. He also conjectured, that the comet observed by Apian, in 1532, was the same as that observed by Hevelius in 1661; if so, its period was 129 years, and it ought to have returned in 1789, but it did not appear. M. Mechain having collected all the observations in 1532, and calculated the orbit again, found that it differed materially from that calculated by Dr. Halley, which renders it extremely doubtful whether this was the comet which appeared in 1661; and this doubt is increased by its not appearing in 1789.

Schol. 4. From the beginning of our era to this time, it is probable, according to the best accounts, that there have appeared about 500 comets. Before that time above 100 others are recorded to have been seen, but it is probable that not above half of them were comets. And when we consider that many others may not have been perceived, from being too near the sun;—from appearing in moonlight;—from being in the other hemisphere;—from being too small to be perceived; or which may not have been recorded, we might imagine the whole number to be considerably greater; it is, however, highly probable, that of the comets which are recorded to have been seen, the same may have appeared several times, and therefore the number may be less than is stated. Miss Caroline Herschel, the sister of Dr. Herschel, has discovered several comets within the last 15 years, accounts of which are in the different volumes of the Philosophical Transactions.

On the subject of Comets, see Mr. Vince's very excellent "*Complete System of Astronomy*," Vol. I. Quarto.

CHAPTER VIII.

Of the Sun.

PROP. CXXI. The spots, which appear upon the sun's disk, adhere to its surface.

If one of these spots appears upon the eastern limb or edge of the sun's disk, it moves from thence toward the western edge, and arrives at the western edge in about $13\frac{1}{2}$ days. Here the spot disappears; and in about $13\frac{1}{2}$ days more, it is seen again upon the eastern edge; and so continues to go round, completing its apparent revolution in 27 days; during one half of which time we see it on the disk of the sun, and during the other half it disappears; which could not happen, if the spots did not adhere to the surface of the sun. Let A be the centre of the sun's disk, D its eastern, and C its western edge; HEG the orbit of an opaque body moving round it, and B the eye of the spectator at the earth. If two lines BD and BC are supposed to be drawn from the spectator's eye B, so as to touch the sun at D and C, then DBC, the angle contained between these lines, is the angle under which the sun's diameter appears to a spectator on the earth. EG is the only part of the supposed body's orbit that is within this angle DBC; and consequently, if the body was in any other part of its orbit, except EG, it would not appear upon the sun's disk. But EG is less than half its orbit; and the body would not take up half the time of a revolution to describe EG. Therefore such a body would not be seen upon the sun's disk, as the spots are, for half the time of a revolution. But if the orbit HEG is not greater than LDFC, or is close to the sun; that is, if the spot adheres to the sun's surface, then half its orbit DFC will be within the angle DBC, and, consequently, the spot will appear upon the sun's disk during one half of its revolution; but during the other half of its revolution, while it describes CLD, it will disappear, because then it will be behind the sun, and so will be concealed from the earth; which agrees with the phenomena. Plate 10. Fig. 16.

PROP. CXXII. The sun is a spherical body, which revolves upon its axis from west to east.

The spots, which appear in the sun's disk, adhere to its surface, (by Prop. CXXI.) and those spots revolve; therefore the sun revolves round its axis.

Whatever side of the sun is turned toward the earth in this rotation, it always appears to be a flat, bright circle; but all the sides of it could not appear in this manner unless it was a sphere; therefore the sun is a spherical body.

SCHOL. A real revolution of a spot, and consequently of the sun round its axis, is completed in 25 days, two days less than its apparent revolution, in consequence of the earth's motion in its orbit in the same direction in which the spot moves.

PROP. CXXIII. The axis of the sun is inclined to the plane of the ecliptic.

Each spot upon the sun must describe a circle round the sun, either coinciding with its equator, or parallel to it. If therefore the sun's axis were perpendicular to the plane of the ecliptic, the plane of the sun's equator would be in the plane of the ecliptic; and a spectator on the earth, whose eye is in the ecliptic, would see the spots describing right lines, either in the sun's equator, or parallel to it; but the spots are sometimes seen to describe lines oblique to the plane of the ecliptic; therefore the axis of the sun is inclined to the plane of the ecliptic. This inclination is observed to be an angle of about $82\frac{1}{2}$ degrees. When the sun is in that part of the ecliptic where its equator crosses the plane of the ecliptic, the spots appear to describe right lines parallel to the sun's equator.

SCHOL. The following particulars respecting the sun are related by Sir I. Newton.

1. That the density of the sun's heat, which is proportional to his light, is 7 times as great in Mercury as with us, and that water there would be all carried off in the shape of steam; for he found by experiments with the thermometer, that a heat 7 times greater than that of the sun's beams in summer will serve to make water boil.

2. That the quantity of matter in the sun is to that in Jupiter nearly as 1100 to 1, and that the distance of that planet from the sun is in the same ratio to the sun's semidiameter; consequently, that the centre of gravity of the sun and Jupiter is nearly in the superficies of the sun.

3. That the quantity of matter in the sun is to that in Saturn as 2360 to 1, and that the distance of Saturn from the sun is in a ratio but little less, than that of the sun's semidiameter. And hence the common centre of gravity of Saturn and the sun is a little within the sun.

4. By the same method of calculation it will be found, that the common centre of gravity of all the planets cannot be more than the length of the solar diameter distant from the centre of the sun.

5. The sun's diameter is equal to 100 diameters of the earth, and therefore its magnitude must exceed that of the earth one million of times.

6. If 360 degrees (the whole ecliptic) be divided by the quantity of the solar year, it will give 59′ 8″, which therefore is the medium quantity of the sun's apparent daily motion; hence his horary motion is equal to 2′ 27″. By this method the tables of the sun's mean motion are constructed, as found in astronomical books.

CHAP. IX.

Of the Parallaxes, Distances, and Magnitudes of the Heavenly Bodies.

Def. LXII. The *Parallax* of the heavenly bodies is the change of their apparent situation with respect to each other, as the spectator views them from different stations.

Def. LXIII. The *Diurnal Parallax* is the distance between the apparent place of a heavenly body, as viewed from the surface of the earth, and its apparent place, as viewed from the centre of the earth.

Plate 10. Fig. 17. Let DAB be the earth, C its centre, A the station of a spectator on the surface of the earth; and F, G, H, different places of the moon, or any other heavenly body; TO, NM, LI, are its different parallaxes, and THO, or AHC; MGN, or AGC, &c. angles of parallax.

Schol. If a spectator in his first station at A, sees a planet at G, its apparent place in the heavens will be N; if now, by the diurnal rotation of the earth, he comes into the station P, the planet will appear at M, which is the place in which it would have appeared if viewed from C the centre; thus, in all cases, the parallax which arises from the diurnal motion, is the same which would arise from a change of station from the surface to the centre; for in either case, the change of the spectator's line of view is the same. Hence appears the propriety of the above definition of the *diurnal parallax.*

PROP. CXXIV. The parallax of any planet is always proportional to the angle which a semidiameter of the earth, drawn from the station of the spectator upon the surface to the centre, would subtend, if viewed from the planet.

Plate 10. Fig. 17. If the planet be at H, and the spectator at A, AHT will be his line of view; on changing the station of the spectator from A to C, the line of view will become CHO; whence TO will be the parallax. But TO subtends and is proportional to THO, or (El. l. 15.) AHC, the angle which the earth's semidiameter would subtend, if viewed from the planet H.

PROP. CXXV. The parallax of a planet depresses its apparent place, by the parallactic arc.

Plate 10. Fig. 17. If the planet be viewed from C, its apparent place is O; if from A, its apparent place is T, farther from Z the vertex than O, by the parallactic arc TO.

Cor. When the altitude of a body is observed, it must be corrected by parallax and refraction, adding the former, and subtracting the latter, in order to get the true altitude, or the altitude above the rational horizon at the centre of the earth. See the table of refractions, Schol. 1 Prop. XXXIX.

PROP. CXXVI. The diurnal parallax of any planet, at a given distance from the earth, is greatest when the planet is in the horizon, and decreases as the altitude of the planet increases.

The parallax (by Prop. CXXIV.) is proportional to the angle which AC would subtend, if seen from the planet H; but this given line, viewed from the given distance of the planet, would continually diminish in its apparent magnitude (by Book VI. Prop. LXXIII.) as the degree of obliquity at which it is viewed increases; that is, as the planet advances from H toward E; therefore the parallax is greatest in the horizon, and decreases as the planet approaches the vertex. The parallactic angle AGC is less than AHC, and AFC less than AGC.

PROP. CXXVII. To find the parallax of the moon, or any planet.

Plate 10. Fig. 18. Let HMO be an arc of the horizon; APVM an arc in the meridian; P the elevated pole; V the

vertex; E the apparent place of the planet, as seen from the surface of the earth, and S its place, as seen from the centre; then ES is the diurnal parallax in the vertical circle VE. Before the planet comes to the meridian, observe its altitude, at E, above the horizon, whence the complement of its altitude, VE, will be known; at the same time observe its distance from the meridian, or its azimuth, EVM. After the planet has passed the meridian, observe when it has the same altitude as at the first observation; that is, when eV is equal to EV. Now, if E is the apparent place of the planet when at the time of the first observation it is viewed from the earth's surface, and S would be its place, at that time, if viewed from the centre; and if e is its apparent place when viewed at the second observation, from the surface, and s would be its place, at that time, if viewed from the centre; the parallax ES is equal to the parallax es, since the altitude was the same at both observations, and consequently SV is equal to sV. So that if PS is the secondary of the equator which passed through the planet at the first observation, and Ps the secondary which passed through it at the second observation, the planet between the times of the first and second observation, must have described the arc Ss in a circle of daily motion. From the time which has passed between the two observations, the arc Ss (by Prop. XXVII.) may be found, and consequently the angle SPs. Now, because the angle EVM is known, PVS, its supplement to two right angles, is known; and, because at the two observations the planet was at equal altitudes, that is, at equal distances from the meridian, the meridian bisects the angle SPs, which is known; whence its half VPS is found.* Also, if the latitude of the place be known, PV, the distance of the elevated pole from the vertex, or the complement of its distance from the horizon, that is, (by Prop. III.) the complement of latitude, is known. Therefore in the spherical triangle PVS, two angles and one side are known; whence the length of SV may be determined. Take SV from EV, which is already known, and SE, the planet's parallax, will be found. The moon's mean parallax has been found to be 57′ 11″.

Or thus; observe when the planet, whose parallax is to be found, and any fixed star in conjunction with it, cross the meridian at the same instant; observe the same planet and star after three hours, and remark how much sooner the planet reaches a line placed perpendicularly in the telescope than the star. As 24 hours is to this difference of time, so will 360 degrees be to the arc which subtends the angle of the parallax; whence the arc and angle will be known.

PROP. CXXVIII. Any parallax of a planet being given, to find any other parallax.

The paraliactic angle AFC being given, it is required to find the angle AHC. Having measured the angle ZAL, let the angle ZAH, the apparent distance of the planet from the zenith, be also measured. Then, in the triangle CAF, the sine of the angle CAF is to the sine of the angle CFA, as the side CF is to the side AC. Again, in the triangle CAH, the sine of the angle CAH is to the sine of the angle CHA, as CH is to AC. But CH is equal to CF; therefore the sine of the angle CAF, is to the sine of the angle CFA, as the sine of the angle CAH is to the sine of the angle CHA; but the three first terms are known, therefore the fourth, namely, the angle CHA, may be found. Plate 10. Fig. 17.

PROP. CXXIX. At a given altitude of different planets, their diurnal parallaxes are inversely as their distances from the centre of the earth.

Let one planet be at f, where its altitude is fAp, and another at h, having an equal altitude hAp. If the planet f is viewed from A at the surface of the earth, the line of view is Afr, and r is its apparent place in the heavens; viewed from C, its apparent place would be t; whence, its parallax (by Prop. CXXV.) is rt. In the same manner it may be shown, that rs, which is less than rt, is the parallax of the planet h. But (by Prop. CXXIV.) the parallax of each planet is proportional to the angle which AC would subtend, if viewed from the planet; and since AC is given, and also the degree of obliquity at which it is viewed, the apparent length of AC, or the angle which AC would subtend, at either planet, would be (by Book VI. Prop. LXIX.) inversely as the planet's distance from C. Therefore the parallaxes of these planets are inversely as their distances from the centre of the earth. Plate 10. Fig. 17.

PROP. CXXX. The diurnal parallax of a planet in a vertical circle generally produces a parallax of declination, and also, if the planet is not in the meridian, of right ascension.

Let HQ be the horizon; EC an arc of the equator which cuts the horizon at C; P the pole of the equator; Z the zenith; ZV a vertical circle; F the apparent place of a planet in the vertical circle ZV, Plate 11. Fig. 1.

* The moon's declination sometimes varies 12′ or 15′ an hour, which would render the consecutive distances from the meridian at equal altitudes materially unequal, especially in high latitudes, and thus render this method of finding the moon's parallax totally useless.

as viewed from the surface of the earth, and I its apparent place, as viewed from the centre; then (by Def. LXIII.) FI is the diurnal parallax in a vertical circle. When the apparent place is F, PFA is a secondary of the equator passing through it, and when it is I, PIB is the secondary which passes through it. Therefore AF is the declination of the planet when it appears at F, and BI its declination when it appears at I; the difference of which, DI, is the change of the apparent declination arising from the different station of the spectator at the surface or centre of the earth. When the apparent place is F, the distance of A from the first of Aries is the right ascension; when it is I, the distance of B from the first of Aries is the right ascension; for PFA and PIB are secondaries of the equator passing through the planet. The difference of right ascension, therefore, produced by the parallax FI is AB. If the planet is in the meridian PZH, and if L be its apparent place, as viewed from the surface, and N, as viewed from the centre of the earth, LN will be its diurnal parallax; LE its declination, as viewed from the surface; NE its declination as viewed from the centre; and NL its parallax of declination. But, because PZH is a secondary of the equator, in whatever part of this vertical circle the planet appears, its right ascension will be the distance of the point E from the first of Aries; that is, the diurnal parallax, in this case, makes no parallax of right ascension.

PROP. CXXXI. The diurnal parallax of a planet in a vertical circle generally produces a parallax of latitude, and also, if the vertical circle be not a secondary of the ecliptic, of longitude.

Plate 11. Fig. 1. Let HQ be the horizon; P the pole of the ecliptic; EC an arc of the ecliptic, which cuts the horizon at C; and ZV a vertical circle; and this Proposition may be proved in the same manner as the last.

PROP. CXXXII. The semidiameter of the earth is to the distance of any planet from the centre of the earth, as the sine of the planet's parallax is to the sine of its apparent distance from the vertex.

Plate 10. Fig. 17. If a planet is at F, and the spectator at A, where the line of view is AFL, the planet will appear at L, and ZAL will be the angle of its apparent distance from the vertex Z. Let the parallax IL, or the angle AFC proportional (by Prop. CXXIV.) to IL, be found. In the plane triangle ACF, (the sides being to one another as the sines of the opposite angles) AC the semidiameter of the earth, is to FC, the distance of the planet from the centre of the earth, as the sine of the angle AFC, the angle of the parallax, is to the sine of the angle FAC, or of its supplement to two right angles ZAL, the angle of the planet's apparent distance from the vertex.

Cor. When the horizontal parallax is taken, the semidiameter of the earth AC, is to HC, the distance of the planet, as the sine of the horizontal parallax AHC is to the sine of HAC or radius.

PROP. CXXXIII. To measure the distance of the moon from the earth.

Plate 10. Fig. 17. Let H be the moon in the sensible horizon observed by a spectator at A, and C the centre of the earth. In the triangle AHC, let the angle AHC, the moon's horizontal parallax, be found, (by Prop. CXXVII.) The angle HAC is a right angle, and AC, the semidiameter of the earth, is known to be 3956 miles. Hence AC, the sine of AHC, 57′ 11″, is to 3985, as AH, taken as radius, to the number of miles in AH, the moon's distance from the earth; the moon's mean distance is thus found to be 238,200 English miles.

Schol. According to Mr. de la Lande, the horizontal semidiameter of the moon is to its horizontal parallax for the mean radius of the earth, as 15′ is to 54′ 57″.4, or very nearly as 3 to 11; hence the semidiameter of the moon is $\frac{3}{11}$ of the radius of the earth. And as the magnitudes of spherical bodies are as the cubes of their radii, the magnitude of the moon is to that of the earth, as 3^3 to 11^3, that is, as 1 : 49.

PROP. CXXXIV. To determine the relative distances of the inferior planets from the sun.

Plate 9. Fig. 12. Let S be the sun, EHG the orbit of Venus, and LCM the orbit of Mercury. Let AXF be a tangent to the orbit of Venus, and let the elongation of Venus, that is, the angle XAS, be found by observation. Then as radius to the sine of the angle XAS, so is AS to XS or ES. In like manner, if the elongation of Mercury, or the angle CAS, be observed; as radius to the sine of CAS, so is

AS to CS or LS. If AS the sun's distance from the earth, be supposed to be divided into 1000 equal parts, the distance of Mercury will in this manner be found to be 387, and that of Venus 723.

PROP. CXXXV. To determine the relative distances of the superior planets from the sun.

Let S be the sun, *n k g* the orbit of the earth, OPQ the orbit of Mars, and NKG a part of a great circle in the heavens, in which the planet appears to have a retrograde motion; let P be the place of Mars. Whilst the earth is passing in its orbit from *k* to *n*, Mars will appear to move from K to N. The angle of retrogradation KPN is then known by observation. To this the vertical angle *n* PS is equal. In the triangle *n* SP, the angle at *n* is a right angle; the angle *n* PS is the angle of retrogradation which is known, whence the other angle *n* SP is known, and the ratio of the sides of the triangle to each other is known; whence the ratio of S *n* to SP is found. If the mean distance of the earth from the sun be called 1000, that of Mars will be found to be 1523, that of Jupiter 5203, and that of Saturn 9539. Plate 10. Fig. 1.

PROP. CXXXVI. To find the parallax of the sun by the transit of Venus.

In order to explain the general principles of this operation, it must be understood, that the periodical times of Venus and the earth, and the proportions and positions of their orbits, are accurately known from previous observations. From these elements are computed the horary motion of both planets, the latitude of Venus, the direction and length of its path across the sun's disk, and the duration of the transit as viewed from the earth's centre. Then let S be the sun, B′E′*e* a part of the orbit of Venus, its apparent motion being retrograde, or from left to right, (see Prop. LIX.) and OW a part of the earth's orbit; and let the transit be observed from some place on the earth's surface, where the sun, for the greater advantage, is on the meridian, about the middle of the transit; and, the earth being at O, let *b* represent the situation of that place at the beginning of the transit, when Venus at B′ is seen just entering on the sun's disk at B. If that place should continue stationary with respect to the earth's centre, Venus must reach *e′* in its orbit at the end of the transit when it is apparently passing off from the sun's disk at E, but during this time the place is carried from *b* to the eastward by the earth's diurnal motion to the situation *e*, where it is at the end of the transit, so that the planet passes off from the sun's disk when it has only reached E′ in its orbit, the duration of the transit, as computed for the earth's centre, being shortened by the motion of the place of observation in the ratio of B′ *e′* to B′E′. Now as one hour is to this difference between the computed and observed durations, so is the heliocentric horary motion of Venus from the earth (or the difference between the heliocentric horary motions of Venus and the earth) to the arc E′ *e′* or the angle *e* E *b* subtended at the sun by the line *e b*. This line may be computed from the latitude of the place of observation and the observed duration of the transit, and by comparing it with the semidiameter of the earth, allowing for the obliquity of the angles at *b* and *e*, we may obtain the angle subtended at the sun by the semidiameter of the earth, or the sun's horizontal parallax. Thus an angle so small as to be scarcely measurable, if it were directly accessible, is determined with great accuracy by observing the time in which an arc, subtending it, is described by a motion so slow, as to afford an interval of time amply sufficient for determination. In fact, it is found that if the line *e b* should be equal to the earth's semidiameter, and perpendicular to *e* E, Venus, with only the excess of its heliocentric horary motion above that of the earth, would be more than five minutes in passing over the arc E′ *e′* subtending the angle *b* E *e*, which would then be exactly equal to the sun's horizontal parallax. Plate 11. Fig. 2.

But lest there should be some error in the computed length of the transit, it ought to be observed at some place near the meridian opposite to the former, and at such a distance from the enlightened pole that the beginning may be observed before sunset and the end after sunrise, so that during the transit the observer may be carried in a direction, with respect to the sun and planet, opposite to that in which the former was carried. Let W now denote the earth during the transit, and *b* the place of the observer at the beginning when the planet at B′ is apparently just e tering on the sun's disk at B; if the observer, as before, should continue at *b*, Venus would perform the transit, describing the computed chord of the sun's disk, while moving from B′ to *e′* in its orbit, but during this time the observer is carried from *b* to *e*, by the diurnal motion of the earth, so that Venus must proceed to E′ in its orbit before the observer at *e* can see it pass off from the sun's disk at E, the length of the transit being increased by the motion of the observer in the ratio of B′ *e′* to B′E′. From this difference between the computed and observed durations the parallax is ascertained in the same manner as before. If now there be any error in the computed duration of the transit, the results of these two operations will be found unequal, since any change in the computed duration increases one, while it diminishes the other, so that the mean between the two results (and in fact the mean of many has

been taken) must in all probability give very correctly the sun's horizontal parallax on the day of the transit. Whence the sun's horizontal parallax, at the time of his mean distance from the earth, may be found; for (by Prop. XXIX.) as the sun's mean distance from the earth is to his proportional distance at the transit, so is his horizontal parallax at that time to his mean horizontal parallax.

In this manner the sun's mean horizontal parallax has been found, from comparing the transits of Venus in 1761 and 1769, to be 8.65″ or about $8\frac{2}{3}$ seconds. See Phil. Trans. Vol. LXII. p. 611, and Ferguson's Astronomy, Chap. XXIII.

Schol. The transits of Venus happen but very seldom; the first that seems to have been noticed was in the year 1639, by Mr. Horrox and his friend Mr. Crabtree. With a view of engaging the attention of future astronomers to the above method of determining the sun's parallax, and thereby his real distance from the earth, Dr. Halley communicated a paper to the Royal Society in the year 1691, containing an account of the several years in which such a transit would happen. He particularly mentioned those which would be seen in 1761 and 1769, presuming that on those periods this important problem would be solved with great accuracy. No other transit will happen until the year 1874.

Except such transits as these, Venus exhibits the same appearance to us regularly every eight years; her conjunctions, elongations, and times of rising and setting, being nearly the same, on the same days as before.

PROP. CXXXVII. To find the distance of the sun from the earth.

Plate 10. Fig. 17. In the triangle AHC, suppose H to be the sun. As the sine of $8\frac{2}{3}$ seconds, the horizontal parallax of the sun AHC, is to radius, so is the semidiameter of the earth AC, which is found by mensuration to be 3956 English miles, to the number of semidiameters of the earth contained in the distance of the sun from the earth. Hence the sun's distance from the earth is found to be about 95,173,000 English miles; for by log. we have 5.621914 (sine of 8″.65) : 10.000000 : : 3.600428 (log. of 3956) : 95,173,000 miles.

PROP. CXXXVIII. To measure the distance of Mercury or Venus from the sun.

Plate 11. Fig. 3. Let S be the sun, E the earth, and M Mercury or Venus. Measure the angle SEM, and observe accurately the time when this measure is taken. When mercury has made one revolution, and arrives at the same point M, the earth will be in some other part of its orbit, as R; measure at that time the angle SRM, and observe the time when the measure is taken.

By these two observations the time in which the earth passes from E to R is known; hence, as 1 year is to the time employed in passing from E to R, so are 360 degrees to the arc ER; whence the arc ER, and the angle ESR, are found. In the triangle ESR the sides SE, SR, (the distance of the sun from the earth) being known, and also the contained angle RSE, let the angles at the base SER, SRE, and the base RE be found. Then from the known angle SER take away the angle SEM, which is also known, there will remain MER; and from the known angle SRE take away the known angle SRM, there will remain MRE. The two angles MER, MRE, being thus found, the third angle RME is also known; and the side RE is known. Wherefore, the sine of the angle RME is to the side RE, as the sine of the angle MRE is to the side ME, or as the sine of the angle MER is to the side MR. In the triangle SRM, the sides RS, RM, being thus found, the sum of the two sides RS, RM, is to their difference, as the tangent of half the sum of the angles at the base RSM, RMS, is to the tangent of half their difference. To half the sum add half the difference, and the greater angle at M is found; and from half the sum take away half the difference, and the less angle at S is found. Whence, the sine of the angle at M is to the side RS, or the sine of the angle at S is to the side RM, as the sine of the angle at R is to the base SM, which is the distance required.

PROP. CXXXIX. To measure the distance of Mars from the sun.

Plate 11 Fig. 4. Let S be the sun, E the earth, and M Mars. Measure the angle SEM; when Mars has made one revolution, observe the place of the earth in its orbit R, and measure the angle SRM. Having found as before the arc ER, and the angle ESR, in the triangle ESR, in which the two sides SE, SR, and the contained angle ESR, are known, let the angles at the base SER, SRE, and the base RE, be found. If from the angle SEM (which has been observed) be taken SER, there remains REM; and if from the angle SRM (which has been observed) be taken SRE, there remains ERM. Whence, in the triangle RME, the angles at R and E being found, the third angle is known; and the sine of the angle at M is to the side RE, as the sine of the angle at E is to the side RM. Wherefore, in the triangle SRM, the two sides of which, RS, RM, and the contained angle at R, are known; the two angles at the base, S, M, and lastly the base SM, which is the distance required, may be found.

PROP. CXL. To measure the distance of Jupiter or Saturn from the sun by their satellites.

Let S be the sun, E the earth, and I Jupiter. First, observe the instant in which the satellite R disappears behind Jupiter, and the instant in which it again appears; then, dividing the intermediate time into two equal parts, this will give the instant in which the earth E, Jupiter I, and the satellite R, are in one right line EID. Next, observe the instant in which the satellite disappears behind the shadow of Jupiter, and the instant in which it again appears; and divide the time between these instants into two equal parts to find the instant in which the satellite is in the midst of the shadow, that is, in which the sun, Jupiter, and the satellite form a right line SIR. Hence, the time taken up in passing from D to R is known; whence, the time of the entire revolution of the satellite is to 360 degrees as the time employed in passing from D to R is to the arc DR. Thus the arc DR, and the angles RID, EIS are found. Lastly, having taken an observation of the angle IES, the other angle ESI is found; and the side ES, the earth's distance from the sun, is known; whence, the sine of the angle EIS is to the side ES, as the sine of the angle IES is to the side IS, the distance required. Plate 11. Fig. 5.

PROP. CXLI. To measure the distance of any planet from the sun.

Because the real distances of the planets from the sun are as their proportional distances; as the proportional distance of the earth from the sun is to the proportional distance of any other planet from the sun, so is the real distance of the earth from the sun in miles, to the real distance of any other planet from the sun in miles.

Hence are found the distances of the planets from the sun in English miles. Mercury, 36,841,468; Venus, 68,891,486; Mars, 145,014,148; Jupiter, 494,990,976; Saturn, 907,956,130; and the Herschel, 1,800,000,000.

PROP. CXLII. The horizontal parallax of any planet being given, to find its distance from the earth.

Let H be the planet, whose horizontal parallax AHC is known. The semidiameter of the earth AC being known, in the triangle CAH the sine of the angle AHC is to the side AC, as the sine of the angle HAC is to the side HC, the distance sought. Plate 10. Fig. 17.

PROP. CXLIII. The distance of any planet being given, to measure its real magnitude.

Let A be the earth and C the centre of any planet; and let the distance CA be known. Suppose two right lines, AB, AD, drawn tangents to the planet, the angles CBA, CDA, are right angles; therefore the square of AC is equal to the two squares of AB and BC together. The same square of AC will also be equal to the two squares of AD and CD. And since the square of the radius CB is equal to the square of the radius CD (on account of the spherical figure of the planets), the square of the tangent AB is equal to the square of the tangent AD, and the tangent AB to the tangent AD. Hence the two triangles ABC, ADC, are equal, and consequently the angles BAC, CAD, are equal. The angle BAD being measured by a micrometer, its half BAC is known; whence, in the triangle ABC, the sine of the angle at B, which is a right angle, is to the side AC, as the sine of the angle at A is to the side BC. The radius, and consequently the diameter of the planet, being thus found, because spheres are as the cubes of their diameters, its magnitude is known by finding the cube of its diameter. Plate 11. Fig. 6.

SCHOL. In Plate 15. Fig. 4, we have a view of the *proportional* magnitudes of the planets Mercury, Venus, the Earth and Moon, Mars, Jupiter, and Saturn, according to Ferguson; with the addition of Herschel. In proportion to these figures of the planets, the sun's diameter is about two feet. Plate 15 Fig. 4.

PROP. CXLIV. To find the periodical time of a planet.

Because, whilst any planet is performing its revolution, the earth is carried forward in its path, the planet, after one greatest elongation, must not only complete a revolution, but likewise the whole angular space which the earth described in that time, before it arrives again at the same elongation. Thus, before Venus can return to the same elongation, besides performing an entire revolution in its orbit (equal to 4 right angles), it must pass through as much more angular space, as the earth has done in the mean time. Hence, as the angular motion of Venus is to the angular motion of the earth in the time between the greatest elongation and its return, so is the periodical time of the earth to the periodical time of Venus. In this manner the periodical times of all the planets may be found.

Or, observe when a planet is in any point of its orbit, and, after any number of revolutions, observe when it comes to the same point again; then divide that interval of time by the number of revolutions, and you get the time of one revolution. The observations of ancient astronomers are here very useful; for as they have put down the places of the planets from their own observation, by comparing them with the places observed now, we take in a very great number of revolutions, and, therefore, if we divide the interval of time by the number of revolutions, should a small error be made in the whole time, it will affect so much less the time of one revolution. The periodical times of the planets will be found in the table at the end of the chapter.

Cor. Because the squares of the periodical times of the planets were found by Kepler to be as the cubes of their distances, the periodical times of any two planets being known, and the comparative or real distance of one of them from the sun being given, the distance of the other may from this proportion be found.

PROP. CXLV. To find the mean velocities of the planets.

The periodical time of a planet being known, and also its diameter, and consequently its circumference (for the diameter of a circle is to its circumference nearly as 113 to 355), its mean velocity, or the velocity with which it would move if its motion were uniform, may be thus found; as the whole periodical time of the planet is to an hour, so is the whole circumference of its orbit to the angular space passed over in an hour. Thus it is found, that the mean horary velocity of the earth is 68216.9 English miles. In like manner, the horary velocity of the other planets may easily be found.

Cor. By comparing this proposition with Cor. Prop. CXVII. the velocity of light will be found to be to the velocity of the earth in its orbit as 10632 to 1.

PROP. CXLVI. The planets revolve round their axes.

It is found by observation, that the earth revolves about its axis in 23h. 56′ 4″ mean solar time; Saturn in 12h. 13′$\frac{1}{4}$; Jupiter in 9h. 56′; Mars in 24h. 40′; Venus in 23h. 20′. The sun is found to revolve on his axis in 25d. 10h.

The time of Saturn's rotation is computed from the ratio of its diameters, which Dr. Herschel makes to be about 11 to 10. The time of the rotation of the other planets is not known; nor has it yet been determined whether they do revolve about their axes.

Schol. The following Table contains a synopsis of the distances, magnitudes, periods, &c. of the several planets, according to the latest observations.

TABLE OF PLANETARY MOTIONS, DISTANCES, &c.

Names of the Planets.	Mean diameters in English miles.	Mean distances from the Sun, in round numbers of miles.	Correct mean distance, that of the Earth being 100000.	Mean apparent diameters as seen from the Earth.	Mean diameters as seen from the Sun.	Densities, that of water being 1.	Proportional quantities of matter.	Diurnal rotations round their own axes.	Inclinations of axes to orbits.	Inclination of orbits at the beginning of 1801.
The Sun	883246			32′ 1″.5		$1\frac{2}{15}$	333928	25ᵈ 14ʰ 8ᵐ 0ˢ	82° 44′ 0″	
Mercury	3224	37,000,000	38710	10	16″	$9\frac{1}{6}$	0.1654	24 5 28		7° 0′ 1″
Venus	7687	68,000,000	72333	58	30	$5\frac{11}{15}$	0.8899	0 23 20 54		2 23 32
The Earth	7911.73	95,000,000	100000		17.2	$4\frac{1}{2}$	1	1 0 0 0	66 32	0 0 0
The Moon	2180	95,000,000	100000	31 8	4.6	$5\frac{1}{2}$	0.025	29 17 44 3	88 17	5 9 3
Mars	4189	144,000,000	152369	27	10	$3\frac{2}{7}$	0.0875	0 24 39 22	59 22	1 51 3.6
Vesta	238	225,000,000	237300	0.5						7 8 46
Juno	1425	252,000,000	265700	3				27 hours probably.		13 3 28
Ceres	163 1024	263,000,000	276500	1 6.4		2				10 37 34
Pallas	80 2099	265,000,000	279100	0.5 6.5		2				34 37 7.6
Jupiter	89170	490,000,000	520279	39	37	$1\frac{1}{24}$	312.1	0 9 55 37	90 nearly	1 18 51
Saturn	79042	900,000,000	954072	18	16	$0\frac{13}{32}$	97.76	0 10 16 2	60 probably	2 29 34.8
Herschel	35112	1,800,000,000	1918352	3.54	4	$0\frac{99}{100}$	16.84			46 26

Names of the Planets.	Tropical revolutions.	Sidereal revolutions.	Place of Aphelion in January 1800.	Motions of the Aphelion in 100 years.	Longitude of ascending node in 1801.	Motions of nodes in 100 years.	Eccentricities; the mean distance of the earth being 100000.	Greatest equations of the centres.
The Sun								
Mercury	87ᵈ 23ʰ 14ᵐ 32.7ˢ	87ᵈ 23ʰ 15ᵐ 43.6ˢ	8ˢ 14°20′50″	1° 33′ 45″	45° 57′ 31″	1° 12′ 10″	7955.4	23°40′ 0″
Venus	224 16 41 27.5	224 16 49 10.6	10 7 59 1	1 21 0	74 52 38.6	0 51 40	498	0 47 20
The Earth	365 5 48 49	365 6 9 12	9 8 40 12	0 19 35	0 0 0		1681.395	1 55 30.9
The Moon								
Mars	686 22 18 27.4	686 23 30 35.6	5 2 24 4	1 51 40	48 14 38	0 46 40	14183.7	10 40 40
Vesta	3 years 60 days 4h.		2 9 42 53		103 0 6		9322	
Juno	4 years 128 days		7 29 49 33		171 6 37		25096	
Ceres	1681ᵈ 12ʰ 9ᵐ		4 25 57 15 in 1802.		80 55 2		8141	9 20 8
Pallas		1703 16 48	10 1 7 0 in 1802.		172 32 35		24630	28 25 0
Jupiter	4330 14 39 2	4332 14 27 10.8	6 11 8 20 in 1800.	1 34 33	98 25 34	0 59 30	25013.3	5 30 38
Saturn	10746 19 16 15.5	10759 1 51 11.2	8 29 4 11 in 1800.	1 50 7	111 55 46	0 55 30	53640.42	6 26 42
Herschel	30637 4 0 0	30737 18 0 0	11 16 30 31 in 1800.	1 29 2	72 51 14	1 44 35	90804	5 27 16

BOOK VII. PART II.

OF THE CAUSES OF THE CELESTIAL MOTIONS AND OF OTHER PHENOMENA.

CHAPTER I.

Of the Cause of the Revolutions of the Heavenly Bodies in their Orbits.

PROP. CXLVII. The moon is retained in its orbit by a force which impels it toward the centre of the earth.

Since (by Book II. Prop. I.) the moon, or any other planet, being put into motion, would continually move on uniformly in a right line, there must be some force which draws it from its rectilineal path. Whatever this force is, since it is found by observation that the moon by a radius drawn to the earth's centre describes equal areas in equal times, it follows (from Book II. Prop. LXXIII.) that it is impelled by that force toward the earth's centre. The earth indeed is not at rest; but because both the moon and earth revolve round the sun, the motion of the moon with respect to the earth is the same as if the earth were at rest.

PROP. CXLVIII. The force which retains the moon in its orbit is, at different distances from the earth, inversely as the squares of those distances.

The moon's orbit being an ellipse, which has the earth in one of its foci, the force which retains it in its orbit must (by Book II. Prop. LXXXI.) in different parts of the orbit be inversely as the squares of the distances from the earth.

PROP. CXLIX. The moon is retained in its orbit by a force which carries it toward the earth with the same velocity with which a body, acted upon by gravitation at the distance of the moon, would fall toward the earth.

Plate 11. Fig. 7. Let AER be the earth, PLV a part of the moon's orbit, LC an arc which the moon describes in its orbit in one minute of time. Since the moon describes its whole orbit in 27 days 7 hours 43 minutes, that is, in 39343 minutes, the length of the arc LC, which the moon describes in one minute, is the $\frac{1}{39343}$ part of 360°, or 33″. If the moon, setting out from L, were not impelled toward the earth, it would move in the right line LB. Since therefore it moves in the arc LC, there must be a force impelling it toward the earth's centre which draws it from the tangent LB, so that, at the end of 1 minute, when it is arrived at C, it will have departed from the tangent as far as BC, or LD; or, because the moon describes the diagonal LC in 1 minute, it would in the same time, by the projectile force, describe LB, and by the centripetal, LD. LD is then the space through which the centripetal force makes the moon fall toward the earth in 1 minute; and LD is the versed sine of the arc LC, which is an arc of 33″. Therefore the force which impels the moon, would make it fall, in 1 minute of time, through the versed sine of an arc of 33″. Because AER, the earth's circumference, is found to measure 123249600 Paris feet, its semidiameter AT will be about 19615800 such feet. Since therefore the mean distance of the moon from the earth is found to be 60 semidiameters of the earth, AT multiplied by 60 will give the length of LT, a semidiameter of the moon's orbit, namely, 1176948000 feet. And as the radius is to the versed sine of 33″, so is LT to LD, or nearly as 1176948000 to $15\frac{1}{12}$ Paris feet, which is nearly equal to $16\frac{1}{2}$ English feet, or 1 pole. Therefore, if the moon were to fall toward the earth, the centripetal force which impels it toward the earth would make it fall 1 pole in the first minute of its descent. But because (by Prop. CXLVIII.) this centripetal force is inversely as the squares of the distances, a body which is at the distance of the moon, or 60 semidiameters of the earth, will be

attracted by a force as much less than that at the surface, as the square of 60, or 3600, is greater than the square of 1, or 1; that is, the force at the surface being 1, it will be to the force at the distance of the moon, as 1 to $\frac{1}{3600}$, and the velocities will have the same ratio. But a body at the surface of the earth falls through 1 pole in a second of time; that is, (by Book II. Prop. XXVI.) through 3600 poles in a minute. Therefore, at the distance of the moon the body would fall through $\frac{1}{3600}$ part of this length, that is, through 1 pole in a minute. But it has been shown that the moon, by its centripetal force, falls toward the earth 1 pole in a minute; therefore the moon is retained in its orbit by a force which moves it with the same velocity with which a body, acted upon by gravitation, and removed to the distance of the moon from the earth, would fall toward the earth.

PROP. CL. The moon is retained in its orbit by the force of gravitation.

The force which retains the moon in its orbit agrees with gravitation in its direction (by Prop. CXLVII.) and in its degree of force (by Prop. CXLIX.); it may therefore be concluded to be the force of gravitation.

PROP. CLI. The primary planets are retained in their orbits by a force which impels them toward the sun.

It is found by observation, that each of them, as they revolve in their respective orbits, describes by a radius drawn to the sun, equal areas in equal times. Therefore (by Book II. Prop. LXXIII.) the force which retains them in their orbit, impels them toward the sun.

PROP. CLII. The forces which retain the primary planets in their respective orbits are, at different distances from the sun, inversely as the squares of those distances.

It appears from observation, that the squares of the periodical times of the planets are as the cubes of their mean distances from the sun. For example, Saturn's periodical time being found to be to Jupiter's about as 30 to 12, and the distance of the former from the sun, to that of the latter nearly as 9 to 5; the squares of the times are 900 and 144; which are to one another nearly as 729 to 125, the cubes of the distances. This proportion takes place in all the primary planets; hence (by Book II. Prop. LXXIX.) the force by which they are retained in their orbits is inversely as the squares of the distances.

Schol. It has been remarked in the last chapter, that Kepler found by observation that the squares of the periodical times of all the primary planets are as the cubes of their mean distances from the sun. Astronomers have since found, that the same law holds good in the secondaries with respect to their primaries. To Sir I. Newton we are indebted for an investigation of this law on physical principles. He has demonstrated, that, in the present state of nature, such a law was inevitable.

PROP. CLIII. The primary planets are retained in their orbits by the force of gravitation.

The moon having been shown from the direction, and the law of its centripetal force, to be retained in its orbit by gravitation; since the primary planets are impelled toward the sun (by Prop. CLI.) as the moon is toward the earth, and since their centripetal force acts with respect to the sun by the same law, by which the force which retains the moon in its orbit acts with respect to the earth, namely, that this force is inversely as the square of the distance of the planet from the sun; it may be concluded, as in the case of the moon, that they are retained in their orbits by the force of gravitation.

This follows likewise from their moving in elliptical orbits, since it has been proved (Book II. Prop. LXXXI.) that bodies revolving in such orbits have their centripetal forces inversely as the squares of their distances from the centre about which they revolve.

PROP. CLIV. The satellites of Jupiter and Saturn are retained in their respective orbits by the force of gravity.

They are observed to describe equal areas round the respective primaries in equal times, and consequently (by Book II. Prop. LXXIII.) are impelled toward them; and the forces which retain them in their orbits are at different distances inversely as the squares of those distances (by Book II. Prop. LXXXI.), because it has been observed that the squares of their periodical times are as the cubes of their

distances from their respective centres. Therefore the force which retains the satellites of Jupiter and Saturn in their orbits acts in the same manner, and by the same law, as the force which retains the moon in its orbit acts with respect to the earth. But all effects of the same sort are, without proof of the contrary, to be considered as produced by the same cause. Therefore the power which retains the satellites in their orbits is gravitation.

PROP. CLV. The sun and any planet revolve round a common centre of gravity, which remains at rest.

Plate 11. Fig. *a*. Let S be the sun, and P any planet, mutually attracting each other. If neither of the two bodies revolved in any orbit, they would move toward each other, and would meet at C, their common centre of gravity; and during the approach of these two bodies, C, their common centre of gravity, would be at rest, (by Book II. Prop. LI.) But if the body P have a projectile force given to it in the direction P *t*, and if this projectile force and its gravitation toward S make it describe an orbit round S, (by Book II. Prop. LXVIII.) such a projectile force will prevent the body P from approaching to S, though it gravitates toward S. But if S has not as great a projectile force given to it at the same time in the opposite direction S *s*, then because S continues to gravitate toward P, and there is no force which can prevent its approaching to P, it follows that S will approach to P, or, as P revolves round S, the mutual gravitation of these two bodies will diminish the distance SP. Now it appears (from Book II. Prop. LI.) that C, the common centre of gravity, always divides this distance SP in the inverse ratio of S to P, or that SC is always as much less than PC, as the quantity of matter in S is greater than the quantity of matter in P; consequently, since the quantity of matter in S and in P is always the same, SC and PC have always the same ratio to one another. But as S approaches to P, SC decreases. Therefore PC must decrease in the same ratio. But PC can decrease no otherwise than either by the approach of P to C, or by the approach of C to P. But the projectile force prevents P from approaching to C. Therefore C must approach to P. Thus it appears that, if P has a projectile force given to it, and is made to revolve, unless S has an equal projectile force given to it at the same time, the mutual gravitation of these two bodies toward each other will put C, which is their common centre of gravity, in motion; contrary to Book II. Prop. LI. Cor. Therefore as the planet P begins to move in the direction P *t*, the sun S will likewise begin to move in the direction S *s*; and C, their common centre of gravity, will continue at rest. And as they tend mutually toward each other, or toward C, their common centre of gravity, their motions will not continue to be performed in right lines, but the planet P will revolve round C in an orbit, of which PR is a part, and the sun S will revolve round C in an orbit, of which SQ is a part.

PROP. CLVI. The sun and any planet, whilst they mutually gravitate toward each other, describe similar figures round their common centre of gravity, and round each other.

Plate 11. Fig. *a*. Let S be the sun, P the earth, or any other planet, and C their common centre of gravity, about which (by the last Prop.) they revolve. To a spectator at P, who imagines the planet to be at rest, the sun will appear to revolve about P, and the reverse at S. Because the common centre of gravity of the sun S, and any planet P, is always in a right line drawn from the sun to the planet, if the planet moves through any small space from P to *p*, the line *p* C continued must pass through the sun; and consequently the sun must have moved from S to *s*. Thus P *p*, S *s*, are arcs described by the planet and sun in their respective orbits in the same time, and PC *p*, SC *s*, are areas described in the same time by the radii CS, CP. And because the vertical angles at C are equal, and SC is to PC as *s* C to *p* C (for SP, *s p*, are both divided in C in the inverse ratio of the quantities of matter in P and S) the areas PC *p*, SC *s*, are similar. In like manner, any other parts of the two orbits described in the same time may be shown to be similar; consequently, the whole orbits are similar.

Again, when P has completed its revolution round C, or $\frac{1}{2}$ or $\frac{1}{4}$ of its orbit, it will appear to a spectator at S, to whom S seems at rest, to have completed its orbit, or $\frac{1}{2}$ or $\frac{1}{4}$ of its orbit round S. And universally, the angular motion of the planet P about C, in any given time, will be equal to its apparent angular motion about S, considered as at rest by a spectator at S. If therefore the planet P in any given time has moved from P to *p*, in which (by the last Prop.) the sun S has moved from S to *s*, the angle PC *p*, which is the measure of the planet's angular motion about C, will be equal to the apparent angular motion round S. Let S *t* be taken equal to *s p*, and make the angle PS *t* equal to the angle PC *p*; P will, by a radius drawn to S, apparently describe the area PS *t*, whilst by a radius drawn to C, it is describing the area PC *p*. Now, because (as was before shown) SC is to PC, as *s* C to *p* C, (El. V. 18.) SC + CP, or SP, is to PC, as *s* C + C *p*, or *s p* is to *p* C; and *s p* is equal to S *t*; therefore PS is to PC as S *t* is to *p* C. Consequently, the two figures PC *p*, PS *t*, are similar. In like manner it may be shown, that any other part, described in any given time, of the orbit the planet

appears to move in round the sun considered as at rest, will be similar to other parts, described in the same time, of the orbit in which the planet moves round the common centre of gravity of the sun and the planet; therefore the whole orbits are similar. And since the orbits which the sun and the planet describe about their common centre have been proved to be similar, it follows, that the orbit which any planet appears to describe round the sun, considered as at rest, is similar to the orbit which the sun in the mean time describes round the common centre of gravity.

In like manner it might be proved, that the orbit which the sun S appears to describe round the planet P, considered as at rest, is similar to either of the orbits which the planet and sun describe about the centre of gravity.

Cor. If the sun's apparent motion, seen from the earth, be an ellipse, having the earth in one of its foci, the earth's apparent motion, seen from the sun, will be in a similar ellipse, having the sun in one of its foci; and if the sun and earth mutually gravitate toward each other, they describe similar elliptic orbits about their common centre.

PROP. CLVII. The common centre of gravity of the sun and all the planets is at rest, and is the centre of the solar system.

Since, from the mutual gravitation of the sun and any one planet, they will revolve about their common centre (by Prop. CLV.), the same must hold good with respect to the sun and all the planets. Consequently, there must be some one point in the solar system which is its centre of gravity, and is at rest.

CHAPTER II.

Of the Lunar Irregularities.

PROP. CLVIII. The nearer the moon is to its syzygies, the greater is its velocity; and the nearer it is to its quadratures, the slower it moves.

Let S represent the sun, T the earth, and LMNO the orbit of the moon; let the moon be in one of its quadratures at L, and let the lines LS and TS be drawn. It is obvious, that the tendency which the moon has toward the sun is along the line LS, and that which the earth has, is along the line TS; let then the former of these be resolved into two others, the one along LA parallel and equal to TS, the other from L to T, along the line LT. The former of these tendencies being parallel and equal to that by which the earth tends along the line TS, alters not the situation of the two bodies L and T with respect to each other, that is, it disturbs not the motion of the body L; but the other along LT increases its tendency toward T. And this increase will be to the tendency the moon has to A, which is the same which the earth has to S, as the distance LT to LA, or TS; that is, the gravity of the moon toward the earth in the quadratures is augmented by the action of the sun, and that augmentation is to the tendency which the earth has to the sun, as the length of the line LT, or the distance of the moon from the earth, to TS, the distance of the earth from the sun. Plate 11. Fig. 8.

Hence the greater the moon's distance is from the earth, the distance of the sun remaining the same, the greater will this increase of the moon's gravity toward the earth be. But if the distance of the moon from the earth remain the same, and the distance of the sun be augmented, this additional increase will be the less in the ratio of the cube of that distance. For, if TS be increased while LT remains the same, LT will be so much the less with respect to TS, that is, the increase will be diminished in the ratio of the sun's distance; but when TS, the distance of the sun, is increased, the absolute force of the sun, and with it the abovementioned increase, will be diminished also in proportion to the square of that distance; consequently, taking in both considerations, it will upon the whole be diminished in the ratio of the cube of that distance.

Let now the moon be in one of its syzygies at M, then will the tendency it has to the sun, more than that which the earth has, which is farther off at T, be to that which the earth has, as the difference of the squares of SM and ST is to the square of SM; but the difference between the squares of SM and ST has nearly the ratio to the square of SM, which twice MT, that is, MO, has to SM; because the difference between the squares of two numbers whose difference is very small with respect to either of them (as the difference between SM and ST is with respect to the distance of S) has little more than double the ratio to the square of the less number, that the difference between the numbers themselves has to the less number. The tendency therefore which the moon, when at M, has to the sun, more than that which the earth has, is to that which the earth has, nearly as MO, or twice

TL, to SM, or, because of the sun's great distance, as twice LT to TS. Her tendency therefore to the earth is now diminished in that ratio; but, as was shown above, it was augmented in the quadratures in the ratio only of LT to TS. The diminution here is therefore nearly double of the augmentation there. And whereas that augmentation, when the distance of the sun remains the same, was shown to increase with the distance of the moon; but when the distance of the moon remains the same, to decrease with the cube of the sun's distance; this diminution, being always nearly double of that, will do the same.

When the moon is in the other syzygy at O, it is attracted toward the sun less than the earth is by the difference of the squares of SO and ST; which, in effect, is the same thing as if the earth were not attracted at all toward S, and the moon were attracted the contrary way; so that its tendency to the earth is here also diminished, as well as when it was at M, and almost in the same degree; for, on account of the sun's great distance, the difference between the squares of SO and ST is nearly the same as between ST and SM.

Or thus; the annual course of the moon round the sun being performed in the same time that the earth's is, it ought to be retained in that course by the same force that the earth is; whereas, when the moon comes to M, the action of the sun upon it is greater than it is upon the earth, by the difference of the squares of SM and ST; and when the moon is at O, it is less than it is upon the earth by the difference between the squares of ST and SO; so that in the former case the moon is drawn too much toward the sun, and in the latter too little; and therefore in both cases its tendency toward the earth is diminished, and almost in the same degree; because, as was observed above, the difference of the abovementioned squares is nearly the same in either case.

Next, let the moon be in the point of her orbit between the quadrature and the syzygy. Then being nearer the sun than the earth is, she will be attracted with a stronger force; let it be expressed by lS produced to D, till lD be of such a length, that TS being put to express the action of the sun upon the earth, lD may express the stronger force of the sun upon the moon; and let lD be resolved into two others, one of which let be la, equal and parallel to TS, then will the other be aD, or its equal and parallel lG. This lG is the only disturbing force upon the moon at L, the other La being parallel to TS, affects the moon just as the sun does the earth; and thus alters not their situations with respect to

Plate 11. Fig. 9. each other. Let then this figure with the line LG be transferred to fig. 9. This force LG may be resolved into LI and LH, the one a tangent to the orbit of the moon, and the other a perpendicular thereto; the former accelerates the motion of the moon when going from the quadrature at O to the syzygy at B; and will retard it when going from B to R. The other, when H falls upon TL produ ed, as in this figure, diminishes the tendency of the moon toward the earth; and when it falls between L and T, it augments it.

Thus the nearer the moon is to its syzygies, the greater will be its velocity; and the nearer it is to the quadratures, the slower it will move; because one of the forces into which LG is resolvable, accelerates its motion from the quadrature to the syzygies; and retards it as much from thence to the quadratures.

Cor. Hence the moon in her monthly revolution is, by the action of the sun, alternately accelerated and retarded.

PROP. CLIX. **The moon describes equal areas in equal times only at the syzygies and quadratures, and deviates from this law the farthest in the octants.**

Plate 11. Fig. 9. The disturbing force being resolved into two others, one of them, at the quadratures, or syzygies, will be found to point from or toward T the centre of the earth directly, and therefore will not hinder the moon from describing equal areas in equal times; the other likewise, in those places, will be found to tend toward the centre of the sun, and therefore neither of them will prevent the moon there from describing equal areas in equal times, that is, will not at the quadratures disturb the moon's motion at all.

But when the moon is in the octants, as at L, this force being resolved into two others, one of them, as LH, will point directly to or from the centre of the earth, and therefore will increase or diminish the moon's tendency toward the earth, but not hinder it from describing equal areas in equal times. But the other, as LI, or HG, points neither toward the centre of the earth, nor sun, and therefore, in the octants, prevents its describing equal areas in equal times. But this being the mid-way between the quadrature and the syzygy, in both which places this disturbing force doth not prevent the moon from describing equal areas in equal times, it follows, that at the octants, this disturbing force will be greatest of all.

Schol. Hence it has always been found more difficult to obtain the moon's place in the octants, so as to agree with observation, than at the quadratures or syzygies

PROP. CLX. The orbit of the moon is more curved in the quadratures, and less in the syzygies, than it would be if it were only attracted by the earth.

For its motion (by Prop. CLVIII.) being accelerated during its progress from the quadratures to the syzygies, in the syzygies its motion will be quicker than it ought otherwise to be, and therefore its centripetal force less than it would otherwise be. It will therefore at the syzygies describe the portion of a larger curve, which consequently will be less curved than a smaller. On the other hand, while the moon passes from the syzygies to the quadratures, its motion is continually retarded, and therefore, at the quadratures, its motion will be slower than it would otherwise be. At the quadratures, therefore, the moon will describe the portion of a lesser curve, which therefore will be more curved than a larger curve.

PROP. CLXI. When the earth is in its perihelion, the periodical time of the moon will be the greatest; when the earth is in its aphelion, the periodical time of the moon will be the least.

Since the irregularities explained in the three preceding Propositions proceed from the action of the sun, it follows, that where the action of the sun is greatest, the irregularities arising from it will be greatest too. But the nearer the earth is to the sun, the greater will be the action of the sun upon the moon; and the more the moon tends toward the sun, the less will it tend toward the earth. When Plate 11.
therefore the earth is at the perihelion A, and consequently at its least distance from the sun S, the action of the sun upon the moon will be greatest, and destroy more of its tendency toward the earth than at any other distance, as SE, SC, SD, &c. Therefore when the earth is at the perihelion A, the moon will describe a larger orbit about the earth, than when the earth is at any other distance from the sun, and consequently her periodical time will then be the longest. Fig. 10.

But the earth is at its perihelion in the winter, and, consequently, then the moon will describe the outermost circle about the earth, and her periodical time will be the longest; which agrees with observation. For the same reason, when the earth is at its aphelion B, the tendency of the moon toward the earth will be greatest, and consequently her periodical time the least. And in this case, which will be in the summer, it will describe the innermost circle about the earth.

PROP. CLXII. The line of the moon's *apsis* goes forward when the moon is in syzygy, and backward when it is in quadrature; but it goes farther forward than backward each time, so that at length it performs a revolution according to the order of the signs.

Since the moon describes an elliptical orbit CEDF about the earth S, placed in one of its *foci*, and since its centripetal force toward the earth, by means of the action of the sun (by Prop. CLVIII.) is Plate 11. Fig. 10.
continually increasing, or decreasing, but not equably; that is, sometimes less, and sometimes more, than in the inverse duplicate ratio of the distance of the moon from the earth, therefore the line of the moon's *apsis* AB will be continually going backward or forward; that is, the axis AB will not always lie in that situation, but go backward into the situation CD, or forward into the situation EF. Since however, taking one whole revolution of the moon about the earth, the action of the sun more diminishes the tendency of the moon toward the earth than it augments it, therefore the motion of the *apses* forward exceeds their motion backward. Upon the whole, therefore, the *apses* of the moon's orbit go forward, or according to the order of the signs. Their revolution is completed in about 9 years.

PROP. CLXIII. The eccentricity of the moon's orbit is varied in every revolution of the moon, and is greatest when the moon is in syzygy, and least when it is in quadrature; and the orbit is most of all eccentrical when the line of the *apsis* is in the syzygies, and least of all eccentrical when this line is in the quadratures.

Because the moon describes an eccentrical orbit CEDB about the earth S, and the action of the Plate 11.
sun upon it sometimes increases its tendency toward the earth, and sometimes diminishes it, that is, Fig. 10.
makes its gravity toward the earth increase or decrease too fast; if, while the moon ascends from its lower *apsis* A, its gravity toward the earth decrease too fast, instead of describing the path DBF, and coming to the higher *apsis* at B, it will run out into a curve beyond DBF; that is, the orbit will become more eccentric, or farther from a circle. On the other hand, if the moon is passing from her higher *apsis* B, to her lower A, and its gravity toward the earth, by the action of the sun, increase too fast, it will approach nearer to the earth than the curve CAE, and describe a curve within CAE, or a portion of an orbit less eccentric, or nearer to a circle, than CEDF. And if we compare several revolu-

tions of the moon together, we shall find that when the line of the *apsis* is in the syzygies, the eccentricity will be the greatest of all, because in that situation, the difference between the tendency which the moon has to the earth in one of the *apses*, and that which it has in the opposite one, is the greatest of all; whereas, when the line of the *apsis* is in the quadrature, this difference is the least, and therefore the lunar eccentricity will be so too.

Cor. When the gravity of the moon toward the earth decreases too fast, the eccentricity of her orbit will increase; and when her gravity toward the earth increases too fast, the eccentricity of her orbit will decrease; and the orbit itself will approach nearer to a circle.

PROP. CLXIV. The line of the nodes moves backward, but not uniformly; when it is in the syzygies it stands still, and moves fastest in the quadratures.

Plate 11. Fig. 11. When the line of the nodes is in the syzygies, as CD, the plane of the moon's orbit passes through the centre of the sun S, as well as through that of the earth E; whence, the disturbing force acting in the direction of the line of the nodes, and consequently in the plane of the lunar orbit, the moon is not drawn out of the plane of its orbit by the sun. But when the line of the nodes is in any other situation, and the moon not in one of the nodes, it is continually drawn out of the plane of its own orbit, on that side on which the sun lies. For instance, if the plane of its orbit CGDF produced passes above the sun, the sun draws it downward; if, on the contrary, the plane of its orbit produced passes below the sun, it draws it upward. Hence it follows, that when the line of the nodes is not in the syzygies, and the moon, having passed either of the nodes, has got out of the plane of the ecliptic ACBD, on either side of it, the action of the sun occasions the moon to return back to the plane of the ecliptic sooner than it otherwise would do. But where the moon enters that plane, there is the next node; so that each node does, as it were, come toward the moon; and the nearer the line of the nodes is to the quadratures, the greater is this effect, because, in that case, the sun is the farthest of all from the plane of the lunar orbit produced. So that the line of the nodes goes backward the fastest of all, when it is in the quadratures; and not at all in the syzygies.

PROP. CLXV. The inclination of the lunar orbit is liable to change, and is greatest when the nodes are in the syzygies, and least when they are in the quadratures.

Plate 11 Fig. 12. When the nodes are in the quadratures A, B, and the moon in its orbit AGBF has passed A, and is approaching the syzygy which is next to the sun, the action of the sun upon the moon prevents its ascending so high, that is, departing so far from the plane of the ecliptic ADBC; whence the inclination of its orbit to the ecliptic will become less, and it will come to conjunction with the sun at H, making an angle with the ecliptic HAD, less than GAD. As the moon goes on to the next quadrature B, the action of the sun upon the moon, in its descent toward the node, hastens its descent, and thus, bringing it down to the ecliptic at K sooner than it would otherwise arrive there, increases the inclination of the plane of its orbit as much as it was diminished in ascending from A to H. And for the same reason, while the moon passes from B to the opposite syzygy F, the action of the sun decreases the inclination of its orbit, and increases it again on its passage from thence to A, the next quadrature.

When the nodes are in the syzygies, C, D, the plane of the moon's orbit produced, passes through the centre of the sun; and consequently, not being affected by the action of the sun, its inclination is neither increased nor diminished.

But while the nodes are passing from the syzygies C, D, to the quadratures A, B, the inclination of the moon's orbit is diminished in every revolution of the moon; and while they are passing from thence to the syzygies, it is continually increasing. Suppose the nodes in the octants at O and L, and the plane of AGBF, the orbit of the moon, so inclined to ADBC, the ecliptic, that if produced it will pass above the sun S. When the moon is nearer the sun than the earth is, it is attracted toward the sun more than the earth is; and when farther off, the earth is attracted more than the moon is, that is, the moon is, as it were, attracted the other way. Hence, whilst the moon is *ascending* from the ecliptic in passing from O to P, the disturbing force being toward S, and the orbit above S, the moon will not rise so high as P, and the inclination of its orbit will be diminished while it is passing over 90 degrees from the node O. In going from a point below P to the next quadrature B, which is 45 degrees, the disturbing force being still toward S, because the moon is as yet nearer the sun than the earth is, and the moon now *descending* toward the ecliptic, the attraction of the sun will hasten its descent, and therefore cause it to move in a plane which will make with the plane of the ecliptic a larger angle than before; that is, in passing from P to B the inclination of the orbit is increased. But when the moon has passed B, and is moving toward L, the disturbing force acting, in the plane of the ecliptic, *from*

the sun, and the moon still *descending* toward the ecliptic, the disturbing force, attracting the moon upward, will retard its descent to the ecliptic, and cause it to move in a plane which will make a less angle with the plane of the ecliptic than before; that is, while it is passing from B to the node L, the inclination of its orbit is diminished. Thus, while the moon passes from O to L, the inclination of its orbit is diminished during three fourths of the passage. In like manner, while the moon is *ascending* from L to I, because the disturbing force acts *from* the sun, the inclination of its orbit is diminished; and while it is *descending* from I to A, the disturbing force still acting from the sun, the inclination is increased. But while it is still descending from A to O, because the disturbing force acts *toward* the sun, the inclination is diminished. Add to this, that while the moon passes from O to P, and from L to I, the disturbing force is much greater than when it was passing from P to L, and from I to O, because the difference between the distances of the moon and of the earth from the sun is greater in the former case than in the latter. On the whole, therefore, while the nodes are between A and D, B and F; that is, while they are passing from syzygy to quadrature, the inclination of the lunar orbit is diminished; for, though the nodes have been supposed equally distant from the quadrature and syzygy, it is obvious that the like effects must happen, though different in degree, when they are nearer to the one than the other.

Next, let the nodes be in the octants I, P, between A and F, and B and G. While the moon is *ascending* from the node I toward the quadrature A, the disturbing force acting *from* the sun, it will be drawn upward, and the inclination of its orbit will be hereby increased. In *ascending* from A to O, the disturbing force acting *toward* the sun, its ascent will be diminished, or the inclination of its orbit lessened; but in *descending* from O to the node P, the disturbing force still acting toward the sun, it will be drawn downward, and, consequently, the inclination of its orbit will be increased. Thus, during one whole revolution of the moon in this position of its nodes, the inclination of its orbit will be increased through three fourths of its passage. And this will be true, as in the other case, when the nodes are not in the octants. Also, for the reason mentioned in the other case, the force which increases the inclination of the orbit is, while it acts, superior to that which diminishes it. While the nodes, therefore, are passing from the quadratures to the syzygies, the inclination of the moon's orbit is increasing. From all which it is manifest, that the inclination of the lunar orbit is the least when the line of the nodes is in quadrature, and the moon in syzygy, and greatest when the line of the nodes is in syzygy.

PROP. CLXVI. The nodes of the moon are at rest, when the line of the nodes is in syzygy; they move *in antecedentia*, or from east to west, when the line of the nodes is in quadrature; and also when it is between quadrature and syzygy; but their regress, in one revolution, is, in this case, less than when the line of the nodes is in quadrature.

When the line of the nodes is in syzygy, because the disturbing force acts in the plane of the moon's orbit, it cannot change the inclination of that plane to the ecliptic; whence the common intersection of the two planes, or the line of the nodes, is immovable. If, whilst the line of the nodes is in AB, the moon is passing from A through G to B, being constantly drawn toward the plane of the ecliptic by the disturbing force, it will come to the plane sooner than it would have done if no such force had acted upon it; that is, before it has described 180°, or is arrived at B. Plate 11. Fig. 12.

In like manner, while the moon is passing from B to A, through F, being drawn toward the plane of the ecliptic by the disturbing force, it will cross the ecliptic sooner than it would otherwise have done, that is, before it arrives at A. Consequently, the nodes will have changed their places, and moved in a contrary direction to the moon. In any other position of the line of the nodes, the disturbing force will, for the same reason, cause the line of the nodes to move *in antecedentia*, though in a less degree; because, whilst the moon is describing the greater part of its orbit, it is drawn by the disturbing force (as was shown in the last Prop.) toward the ecliptic, and consequently is made to cross the ecliptic sooner than it would otherwise have done, that is, the nodes are on the whole, in one revolution of the moon, made to move in a direction contrary to that of the moon; but this regress is less than when the line of the nodes is in quadrature, because, during part of the revolution in this oblique position of the line of the nodes, the nodes move *in consequentia*, or in the same direction with the moon, namely, whilst the disturbing force (as was shown in the last Prop.) draws the moon from the plane of the ecliptic; whereas, when the line of the nodes is in quadrature, they move *in antecedentia* during the whole revolution.

Schol. 1. The nodes perform one revolution, or pass through every part of the ecliptic, in about 19 years.

Schol. 2. All the irregularities of the moon are greater when the earth is in its perihelion, than

when it is in its aphelion, because the effect of the sun's action, whereby they are produced, is inversely as the cube of its distance from the earth. They are also greater when the moon is in conjunction with the sun, than in opposition, for the same reason; for the earth and moon, taken together, are nearer the sun in the former situation of the moon, than they are in the latter.

CHAPTER. III.

Of the Spheroidical Form of the Earth.

PROP. CLXVII. In the daily revolution of the earth round its axis, the centrifugal force diminishes the weight of bodies more at the equator than in any other place on the surface of the earth, in the duplicate ratio of the semidiameter to the cosine of the latitude of the place.

Plate 11. Fig. 13. Let PEP *e* be the earth, PP the axis, E *e* the equator. As the earth revolves upon its axis, every place on its surface, except the two poles, describes a circle, the plane of which is perpendicular to the axis, and the radius of which is the distance of that place from the axis. Thus, a body placed at A will in one revolution of the earth describe a circle, the semidiameter of which will be AB, which, with the plane in which it lies, will be perpendicular to the axis PP In like manner, CE is the semidiameter of a circle described by the revolution of a place in the equator. But CE is the semidiameter of the earth, and AB is the cosine of latitude of the place A; for AB is the sine of AP, the complement of AE, which is the latitude of the place. And a body at E, revolving in a circle whose radius is CE, performs its revolution in the same time with a body at A, revolving in a circle whose radius is AB. But where the periodical times are equal, the centrifugal forces are as the radii, (by Book II. Prop. LXXVII.) Whence the body at E has its centrifugal force as much greater than the body at A, as the radius CE is greater than the radius AB; and universally, the centrifugal force at the equator is to the centrifugal force at any other place on the surface of the earth, as the semidiameter of the earth to the cosine of the latitude of the place.

Moreover, if the centrifugal forces at E and A were equal, they would diminish the weights of bodies unequally, on account of the different directions in which these forces act The centrifugal force at A, acting obliquely upon the force of gravitation toward C, can only diminish this force by such a part of its action as is opposite to the direction of gravitation, that is, resolving A *b* which may express the centrifugal force at A into A *a*, *a b*, the part of the centrifugal force which will act to diminish the gravity of the body at A, will be to the whole centrifugal force at A, as A *a* to A *b*. Whereas at E, the whole centrifugal force, acting in direct opposition to the force of gravitation, will operate to diminish the weight of a body at E. Hence the force which acts to diminish the weight of a body, that is, the diminution at E is to the same at A, as the whole centrifugal force A *b* to the part A *a*. But A *b* is to A *a*, as CE to BA; for the triangles A *a b* and ABC being similar, A *b* is to A *a*, as AC or EC to BA. Therefore, from the different directions in which the centrifugal forces act at E and A, the weight at E is as much more diminished than at A, as EC, the semidiameter, is greater than AB, the cosine of the latitude of the place A.

The centrifugal force then diminishes the force of gravitation in the ratio of EC to AB, both because the centripetal force at E is greater than at A, and because it acts directly at E, but obliquely at A. Therefore the centrifugal force diminishes the weight of a body at E, more than at A, in the duplicate ratio, of CE to BA, that is, as much more at E than at A, as the square of CE is greater than the square of BA.

Schol. It is found by calculation from this Proposition, that gravity at the equator is diminished by the centrifugal force in the ratio of 288 to 289.

Cor. 1. If the diurnal motion of the earth round its axis were about 17 times faster than it is, the centrifugal force would, at the equator, be equal to the power of gravity, and all bodies there would entirely lose their weight. But if the earth revolved still quicker than this, they would all fly off.

Cor. 2. Since a place in the equator describes a circle of 24930 miles in 24 hours (Prop. III. Cor.) it is evident that the velocity with which that place moves, is at the rate of about 17.3 miles per minute. The velocity in any parallel of latitude decreases in the proportion of the cosine of latitude to radius. Thus, for the latitude of London, say, as Rad. : Cos 51° 30 : : velocity of the equator : velocity of London; by logarithms, as 10.00000 : 9.794150 : : 1.232046 : 1.026196 = 10.6 miles; that is, the city of London moves about the axis of the earth at the rate of more than $10\frac{1}{2}$ miles in a minute of time.

PROP. CLXVIII. The earth is an oblate spheroid, elevated at the equator, and depressed at the poles.

It has been found by observation, that a pendulum, shorter by 2.169 lines, is required to vibrate seconds at the equator, than at the poles; but (from Book II. Prop. XLIII. and XLIV.) the lengths of pendulums vibrating in the same time are as the gravities at the places where they vibrate; therefore the gravity at the poles is greater than at the equator. And it has been found by Sir I. Newton, that this difference of gravity is so much greater than would arise from the centrifugal force alone, that the ratio of the equatorial diameter of the earth to the polar diameter, must be as 230 to 229, which makes the equatorial diameter exceed the polar by about 34 miles.

Cor. 1. Hence bodies near the poles are heavier than the same bodies toward the equator; (1.) Because they are nearer the earth's centre, where the whole force of the earth's attraction is accumulated. (2.) Because their centrifugal force is less on account of their diurnal motion being slower. For both these reasons, bodies carried from the poles toward the equator, gradually lose their weight.

Cor. 2. The degrees of latitude upon the earth's surface are longer at the poles than at the equator. For an arc of a meridian near the poles is less curved than near the equator; that is, it is an arc of a larger circle; whence a degree measured upon that arc must be greater than upon an arc of the same meridian at the equator.

Cor. 3. The tendency of a heavy body, on any part of the surface of the earth between the poles and the equator, is not directly toward the centre, but toward some point between the centre and the equator.

Schol. The point toward which a body in any given place will tend may be determined.

For (by Prop. CLXVII.) as radius EC is to the cosine of latitude of the place AB, so is the centrifugal force at E to the centrifugal force at A in the direction A b. Produce, therefore, the line BA to b till A b has the same ratio to AC, as the quantity last found has to gravity upon the surface of the earth. Complete the parallelogram A b C c; the point sought will be c, and the tendency of the body will be along A c. Thus, suppose the latitude of the place 51° 46′; the centrifugal force at the equator is found to be to that of gravity, as 1 to 289; hence, as radius to the cosine of 51° 46′, so is 1 to 0.618, which is the centrifugal force at A. Consequently, the centrifugal force at A is to the force of gravity as 0.618 to 289; therefore, by the construction, A b or C c is to AC in that ratio. The ratio of AC to C c being thus found, as AC is to C c, or as 289 is to 0.618, so will the sine of the angle of latitude AC c, or 51° 46′, be to 5′ nearly, which is the angle required, measuring the deviation of the line of direction of falling bodies at the given latitude from a line drawn to the centre of the earth. Plate 11. Fig. 13.

CHAPTER IV.

Of the Precession of the Equinoxes.

Def. LXIV. A *Periodical Year* is the time in which the sun completes its revolution through the ecliptic.

Def. LXV. A *Tropical Year* is the time in which the sun completes its revolution, setting out from any solstitial or equinoctial point, and returning to the same.

PROP. CLXIX. The equinoctial points move *in antecedentia*, or go backward from east to west, contrary to the order of the signs.

It is found from observation, that the equator and ecliptic do not always intersect each other in the same points, but that the points of intersection change their place, moving from east to west, whilst the inclination of the planes remains the same. This motion is called the *precession of the equinoxes*, because it carries the equinoctial points *in precedentia signa.*

PROP. CLXX. The precession of the equinoxes makes the tropical year shorter than the periodical year.

If, while the sun moves in the order of the signs, the equinoctial point moves in the contrary direction, it is manifest, that the sun must arrive at the solstitial or equinoctial point from which it set out, before it arrives at the same place in the zodiac, or must complete the tropical year sooner than the periodical year.

The tropical year is observed to be 365 days, 5 hours, 49 minutes; the periodical year, 365 days, 6 hours, 9 minutes, 12 seconds.

PROP. CLXXI. The precession of the equinoxes causes the poles of the equator to describe a circle from east to west about the poles of the ecliptic.

In this precession, the plane of the equator revolves from east to west, cutting the ecliptic, which with its axis is at rest, in successive points. But while the plane of the equator is revolving, its axis must revolve with it the same way. And since the plane of the equator is always equally inclined to that of the ecliptic, the axis of the equator must always have the same inclination to the axis of the ecliptic; consequently, the poles of the equator will revolve round the poles of the ecliptic, always preserving the same distance from each other; that is, the poles of the equator will describe a circle about the poles of the ecliptic.

Exp. The precession of the equinoxes, and the revolution of the pole of the equator about that of the ecliptic, may be thus represented on the celestial globe. Let the broad wooden horizon represent the ecliptic; place the axis of the globe perpendicular to the wooden circle; the ecliptic on the globe will then make an angle of 23° 30′ with the wooden horizon; consequently, if the wooden horizon represent the ecliptic, the circle which commonly represents the ecliptic will now represent the equator; and the two points in which the circle cuts the wooden horizon will represent the equinoctial points. If the globe, in this position, be turned slowly round from east to west, these points of intersection will move round the same way, while the inclination of the circle which now represents the equator to that which represents the ecliptic remains the same; whence the precession of the equinoxes is properly represented. Again, the axis and poles of the globe now representing those of the ecliptic, the axis and poles of the ecliptic, marked on the globe, will represent those of the equator; and in turning the globe round from east to west, the points which represent the poles of the equator will revolve the same way round the poles of the globe which represent those of the ecliptic, and the axis of the supposed equator will always make the same angle with the plane of the supposed ecliptic.

PROP. CLXXII. The precession of the equinoxes is caused by the action of the sun and moon on that excess of matter about the equatorial parts of the earth, by which from a perfect sphere it becomes an oblate spheroid.

Plate 11. Fig. 12. Let ADCB be the plane of the ecliptic, S the sun, E the earth, and AFBG a ring encompassing the earth at any distance, as Saturn is encompassed by its ring. Let the half of this ring AGB toward the sun be above the plane of the ecliptic and the other half below it; then, a line passing through A and B will be the line of the nodes of this ring. If it be supposed that this ring moves round its centre E, the same way in which the moon moves round the earth, it is obvious that every point of this ring will be acted upon by the disturbing force of the sun in the same manner as the moon was shown to be acted upon in Prop. CLVIII. &c. Particularly, the motion of the nodes of this ring, and consequently of the whole ring which moves with these nodes, and its inclination to the plane in which its centre moves, will be affected in the same manner with the orbit of the moon; whence, its nodes when in syzygies will stand still, and its inclination will be greatest; but in all other situations, the nodes will go backward, and fastest of all when in the quadratures, at which time the inclination of the ring will be the least. This will be the case, whatever be the thickness of the ring, or its distance from the centre.

If this ring be supposed to adhere to the earth, it is obvious that it will still have the motions described above, and that, in this situation, the earth itself must participate of these motions. Now the earth being an oblate spheroid, having its equatorial diameter longer than that which passes through its poles, this redundancy of matter, by which the form of the earth departs from a perfect sphere, may be considered as a portion of the supposed ring, which receives from the action of the sun the motions abovementioned, and communicates them to the earth. Hence the equinoctial points, which are the nodes of the ring, when they are in syzygy, that is, at the equinox, will stand still, and the inclination of the equator to the plane of the ecliptic will be the greatest; in all other situations they will go backward, and fastest when in quadrature at the solstices; and the inclination of the plane of the equator to that of the ecliptic is then the least.

Cor. Hence the axis of the earth, being perpendicular to the plane of the equator, changes therewith its inclination to the plane of the ecliptic twice in every revolution of the earth about the sun. For instance, it increases whilst the earth is moving from the solstitial to the equinoctial, and diminishes as much in its passage from the equinoctial to the solstitial points; which phenomenon is called the Nutation of the Poles.

Schol. This precession of the equinoxes is found to be 50 seconds of a degree, every year, westward or contrary to the sun's annual motion; so that, with respect to the fixed stars, the equinoctial

points fall backward 30 degrees in 2160 years, whence the stars will appear to have gone 30 degrees forward with respect to the signs of the ecliptic, which are reckoned from the equinoctial point. Thus the stars which were formerly in Aries are now in Taurus, &c. This period is completed in 25920 years.

CHAPTER V.

Of the Tides.

PROP. CLXXIII. The tides are caused by the attraction of the moon and of the sun.

Let A *p* L *n* be the earth, and C its centre; let the dotted circle PN represent a mass of water covering the surface of the earth; let M, *m*, be the moon; S, *s*, the sun, in different situations. Because the power of gravity diminishes as the squares of the distances increase (by Prop. CXLVIII.), the waters on the side of the earth A are more attracted by the moon at M, than the central parts of the earth C, and the central parts are more attracted than the waters on the opposite side of the earth at L: consequently (as was shown concerning the moon) the waters on the side L will be as it were attracted *from* the centre of the earth, or will recede from thence. Therefore, while the moon is at M, the waters will rise toward *a* and *l* on the opposite sides of the earth A, L; while, by the oblique attraction of the moon, the waters at P and N will be depressed. Plate 11. Fig. 14.

Or thus; because (by Prop. CLV.) the moon and earth are continually revolving about their common centre of gravity, suppose *a*; the points A, C, L, describing circles about this common centre in the same periodical times, the forces required to retain them in these circles (as may be inferred from Book II. Prop. LXXV.) will be to each other as their distances from the centre *a* A, *a* C, *a* L. Consequently, the point L requires a greater force than C, and C than A, to retain it in its orbit. Now these points are retained in their respective circles by the moon at M; and consequently the point L, which is most remote, and therefore requires the greatest force, is attracted the least, whilst A, the nearest point, is attracted the most. Thus, the water about A being attracted too much, and that about L too little, both will have their gravity diminished by the action of the moon, and will endeavour to leave the centre C; while the water at P and N, having their gravity increased by the same cause, will subside. Hence the form of the water on the surface of the earth will become an oblong spheroid.

This oval of waters keeps pace with the moon in its monthly course round the earth; while the earth, by its daily rotation about its axis, presents each part of its surface to the direct action of the moon, twice each day, and thus produces two floods and two ebbs. But because the moon is in the mean time passing from west to east in its orbit, it comes to the meridian of any place later than it did the preceding day; whence the two floods and ebbs require nearly 25 hours to complete them. The tide is at the greatest height, not when the moon is in the meridian, but some time afterward, because the force, by which the moon raises the tide, continues to act for some time after it has passed the meridian.

As the moon thus raises the water in one place, and depresses it in another, the sun does the same; but in a much less degree, on account of the small ratio of the semidiameter of the earth to the distance of the sun; for, as it was shown of the moon that the force of the sun, by which it disturbs its motion, is as the distance of the moon from the earth to that of the sun from the same, so, in this case, the force of the sun to disturb the waters is as the semidiameter of the earth, to the distance of the sun, which ratio is very small.

Cor. The moon being nearest to the earth when in perigee, its attraction must then be strongest, and the effect or elevation of the waters, greatest. And the earth being in its perihelion at the winter solstice, the sun's power to produce tides is greatest at that time.

PROP. CLXXIV. The tides are greatest at the new and full moons, and least at the first and last quadratures; and the highest of the former tides are near the time of the equinoxes.

When the moon is in conjunction or opposition with the sun, as M, *m*, S, the tides which each endeavours to raise are in the same place; whence they are then greatest, and are called spring tides. But, when the moon is in the first or last quarter, the sun, being in the meridian when the moon is in the horizon, as M, *s*, depresses the water where the moon raises it; whence the tides are then Plate 11. Fig. 14.

least, and are called neap tides. On the full and new moons which happen about the equinoxes, when the luminaries are both in the equator or near it, the tides are the greatest of all; for, first, the two eminences of water are at the greatest distance from the poles, and hence the difference between ebb and flood is more sensible; for if those eminences were at the poles, it is obvious we should not perceive any tide at all; secondly, the equatorial diameter of the earth produced passes through the moon, which diameter is longer than any other, and consequently there is greater disproportion between the distances of the zenith, centre, and nadir, from the centre of gravity of the earth and moon, in this situation than in any other; and thirdly, the water rising higher in the open seas, rushes to the shores with greater force, where being stopped, it rises higher still; for it not only rises at the shores in proportion to the height it rises to in the open seas, but also according to the velocity with which it flows from thence against the shore. The spring tides, which happen a little before the vernal and after the autumnal equinox, are the greatest of all, because the sun is nearer the earth in the winter than in the summer.

PROP. CLXXV. When the moon is in the northern hemisphere, it produces a greater tide while it is in the meridian above the horizon, than when it is in the meridian below it; when in the southern hemisphere, the reverse.

Plate 11. Fig. 15. Let AFHD represent the earth, whose centre is T, and axis PO, the point P the north pole, and O the south pole, EQ the equator, FH a parallel to it on the south side, and KD another parallel to it on the north side. Let the fluid surrounding the earth form itself into an oblong spheroid, whose longer axis HK produced, passes through the moon at L. The right lines TK, or TH, drawn from the centre T, will represent the greatest height of the water in those places. Then, supposing NM perpendicular to KH, TN or TM will denote the least height, and will represent the height of the water in all parts of the globe through which the circle NM passes. The right lines TE, TF, TH, TQ, TD. will show the height of the water in those respective places E, F, H, Q, D.

Let us now consider some place in particular, which, by the diurnal motion of the earth, describes the parallel KD. When that place is at K, the height of the water TK is the greatest, that is, it will be high water, and the moon L will be in the meridian. But afterward, when that place comes to X, the height of the water will be the least, that is, it will be low water; and when the same place comes to D, it will be high water again. But because TK is greater than TD, therefore, in the present case, when the moon is on the north side of the equator, or in the northern signs, the height of the sea, or tide, will be greater when the moon is in the part of the meridian, which is above the horizon, than when it is in the meridian, and below it. Hence it is that the moon, when it is in the northern signs, makes the greatest tides on our side of the equator when it is above the earth.

Again, TH, on the south side of the equator, is longer than TF; and therefore, to a place that describes the parallel FH, the greatest height of the water, when the moon is in the northern signs, is when it is on that part of the meridian that is below the horizon of that place, and the least tides when it is above the horizon For the like reason, when the moon is in the southern signs, the greatest tides on the other side of the equator will be when it is below our horizon, and the least tides when it is above it.

Cor. Hence it is evident, that when the moon is in the equator, the two tides are equally high. For then the longer axis HK of the oblong spheroid coincides with the equator EQ, and the two points of high water on any parallel, being equally distant from the highest points of the tides, are consequently at the same height.

Schol. What has been said of the tides must be understood upon supposition, that the globe of of the earth is covered entirely with water to a considerable depth; but continents which stop the tide, straits between them, islands, and the shallowness of the sea in some places, which are all impediments to the course of the water, cause many exceptions to what hath been above laid down. These exceptions can only be explained from particular observations on the nature of tides at different places.

BOOK VII. PART III.

OF THE FIXED STARS.

Def. LXVII. Those bodies, which always appear in the heavens at the same distance from each other, are called *Fixed Stars*; because they do not appear to have any proper motion of their own.

PROP. CLXXVI. The fixed stars are luminous bodies.

Because they appear as points of small magnitude when viewed through a telescope, they must be at such immense distances, as to be invisible to the naked eye if they borrowed their light; as is the case with the satellites of Jupiter and Saturn, although they appear of very distinguishable magnitude through a telescope. Besides, from the weakness of reflected light, there can be no doubt but that the fixed stars shine with their own light. They are easily known from the planets, by their twinkling.

Schol. The number of stars, visible at any one time to the naked eye, is about 1000; but Dr. Herschel, by his magnificent improvements of the reflecting telescope, has discovered that the whole number is great beyond all conception.

PROP. CLXXVII. The fixed stars appear of different magnitudes.

The magnitudes of the fixed stars appear to be different from one another, which difference may arise either from a diversity in their real magnitudes, or distances; or from both these causes acting conjointly. The difference in the apparent magnitude of the stars is such as to admit of their being divided into six classes, the largest being called stars of the first magnitude, and the least, which are visible to the naked eye, stars of the sixth magnitude. Stars only visible by the help of glasses are called *telescopic* stars.

Schol. 1. It must not be inferred that all the stars of each class appear exactly of the same magnitude; there being great latitude given in this respect; even those of the first magnitude appear almost all different in lustre and size. There are also other stars of intermediate magnitudes, which as astronomers cannot refer to any one class, they therefore place them between two. *Procyon*, for instance, which Ptolemy makes of the first magnitude, and Tycho of the second, Flamstead lays down as between the first and second. So that instead of 6 magnitudes, we may say that there are almost as many orders of stars, as there are stars; such considerable varieties being observable in their magnitude, colour, and brightness.

Schol. 2. To the bare eye the stars appear of some sensible magnitude, owing to the glare of light arising from the numberless reflections of the rays in coming to the eye; this leads us to imagine that the stars are much larger than they would appear, if we saw them only by the few rays which come directly from them, so as to enter the eye without being intermixed with others.

Exp. Examine a fixed star of the first magnitude through a long and narrow tube; which, though it takes in as much of the sky as would hold a thousand such stars, scarcely renders that one visible.

Schol. 3. There seems but little reason to expect that the real magnitudes of the fixed stars will ever be discovered with certainty; we must, therefore, be contented with an approximation, deduced from their parallax (if this should ever be ascertained), and the quantity of light they afford us compared with the sun. To this purpose, Dr. Herschel informs us that with a magnifying power of 6450, and by means of his new micrometer, he found the apparent diameter of α Lyræ to be 0.335″.

Dr. Herschel's method of finding the annual parallax of the fixed stars is by observing how the angle between two stars very near to each other varies in opposite parts of the year. The following is the most simple case given by that great astronomer. Let G and E be two stars situated in a line with the earth at A, and supposed perpendicular to the diameter AB of the earth's orbit, and when the earth is at B, observe the angle GBE. Let P = the angle AGB, or the annual parallax of G; p the angle GBE found by observation; M, m, the angles under which the diameters of G and E appear, and draw GH perpendicular to BG. Then $p : P :: GH : AB :: GE : AE ::$ (because $M : m :: AE : AG$) $M - m : M$; hence $P = \frac{p \times M}{M - m}$ the parallax G. If G be a star of the first magnitude, and E one of the third, and $p = 1''$, then $P = 1\frac{1}{2}''$. See Phil. Trans. Vol. LXXII. Plate 11. Fig. 16.

This theory is only true upon the supposition that the stars are all of the same magnitude ; and that a star of the second magnitude is at double the distance of one of the first, and so on. These suppositions are certainly *not* founded on any analogy from the known and well established principles of that system of bodies to which we belong.

Schol. 4. The ingenious observation of Kepler upon the magnitudes and distances of the fixed stars deserves to be introduced here, and the more so, as he was followed in the conjecture by the great Dr. Halley. He observes, that there can be only 13 points upon the surface of a sphere as far distant from each other as from the centre; and supposing the nearest fixed stars to be as far from each other as from the sun, he concludes that there can be only 13 stars of the first magnitude. Hence at twice that distance from the sun there may be placed four times as many, or 52. At three times that distance, nine times as many, or 117; and so on. These numbers will give pretty nearly the number of stars of the first, second, third, &c. magnitudes. Dr. Halley farther remarks, that if the number of stars be finite, and occupy only a part of space, the outward stars would be continually attracted to those within, and in time would unite into one. But if the number be infinite, and they occupy an infinite space, all the parts would be nearly in equilibrio, and consequently each fixed star being drawn in opposite directions would keep its place, or move on till it had founed an equilibrium. Phil. Trans. No. 364. See also the introductory remarks of Dr. Herschel to a paper on the changes of the fixed stars. Phil. Trans. 1796.

PROP. CLXXVIII. The fixed stars are divided into constellations, or systems of stars.

The ancients, that they might the better distinguish the stars with regard to their situation in the heavens, divided them into several constellations, that is, systems of stars, each system consisting of such as are near each other. And to distinguish these systems from one another, they gave them the names of such men or things as they fancied the space they took up in the heavens represented. To these, several new constellations have been added by modern astronomers.

Schol. The following table contains the names of the constellations, and the number of stars observed in each by different astronomers.

The Ancient Constellations.		*Ptolemy.*	*Tycho.*	*Hevelius.*	*Flamstead.*
Ursa Minor	The Little Bear	8	7	12	24
Ursa Major	The Great Bear	35	29	73	87
Draco	The Dragon	31	32	40	80
Cepheus	Cepheus	13	4	51	35
Bootes, *Arctophilax*	Bootes	23	18	52	54
Corona Borealis	The Northern Crown	8	8	8	21
Hercules, *Engonasin*	Hercules kneeling	28	28	45	113
Lyra	The Harp	10	11	17	21
Cygnus, *Gallina*	The Swan	19	18	47	81
Cassiopea	The Lady in her Chair	13	26	37	55
Perseus	Perseus	29	29	46	59
Auriga	The Wagoner	14	9	40	66
Serpentarius, *Ophiuchus*	Serpentarius	29	15	40	74
Serpens	The Serpent	18	13	22	64
Sagitta	The Arrow	5	5	5	18
Aquila, *Vultur*	The Eagle	15	12	23	71
Antinous	Antinous		3	19	
Delphinus	The Dolphin	10	10	14	18
Equulus, *Equi sectio*	The Horse's Head	4	4	6	10
Pegasus, *Equus*	The Flying Horse	20	19	38	89
Andromeda	Andromeda	23	23	47	66
Triangulum	The Triangle	4	4	12	16
Aries	The Ram	18	21	27	66
Taurus	The Bull	44	43	51	141
Gemini	The Twins	25	25	38	85
Cancer	The Crab	23	15	29	83
Leo	The Lion	35	30	49	95
Coma Berenices	Berenice's Hair		14	21	43

The Ancient Constellations.		*Ptolemy.*	*Tycho.*	*Hevelius.*	*Flamstead.*
Virgo	The Virgin	32	33	50	110
Libra, *Chelæ*	The Scales	17	10	20	51
Scorpius	The Scorpion	24	10	20	44
Sagittarius	The Archer	31	14	22	69
Capricornus	The Goat	28	28	29	51
Aquarius	The Water-bearer	45	41	47	108
Pisces	The Fishes	38	36	39	113
Cetus	The Whale	22	21	45	97
Orion	Orion	38	42	62	78
Eridanus, *Fluvius*	Eridanus, *the River*	34	10	27	84
Lepus	The Hare	12	13	16	19
Canis major	The Great Dog	29	13	21	31
Canis minor	The Little Dog	2	2	13	14
Argo Navis	The Ship	45	3	4	64
Hydra	The Hydra	27	19	31	60
Crater	The Cup	7	3	10	31
Corvus	The Crow	7	4		9
Centaurus	The Centaur	37			35
Lupus	The Wolf	19			24
Ara	The Altar	7			9
Corona Australis	The Southern Crown	13			12
Piscis Australis	The Southern Fish	18			24

The New Southern Constellations.

Columba Noachi	Noah's Dove	10
Robur Carolinum	The Royal Oak	12
Grus	The Crane	13
Phœnix	The Phenix	13
Indus	The Indian	12
Pavo	The Peacock	14
Apus, *Avis Indica*	The Bird of Paradise	11
Apis, *Musca*	The Bee, or Fly	4
Chamæleon	The Camelion	10
Triangulum Australe	The South Triangle	5
Piscis volans, *Passer*	The Flying Fish	8
Dorado, *Xiphias*	The Sword Fish	6
Toucan	The American Goose	9
Hydrus	The Water Snake	10

Hevelius's Constellations made out of the unformed Stars.

		Hevelius.	*Flamstead.*
Lynx	The Lynx	19	44
Leo minor	The Little Lion		53
Asterion and Chara	The Grey Hounds	23	25
Cerberus	Cerberus	4	
Vulpecula & Anser	The Fox and Goose	27	35
Scutum Sobieski	Sobieski's Shield	7	
Lacerta	The Lizard	10	16
Camelopardalis	The Camelopard	32	58
Monoceros	The Unicorn	19	31
Sextans	The Sextant	11	41

Schol. 1. Stars not included in any constellation are called *unformed* stars. Besides the names o the constellations, the ancient Greeks gave particular names to some single stars, or small collection of stars; thus, the cluster of small stars in the neck of the bull, was called the *Pleiades*; five stars in the Bull's face, the *Hyades*; a bright star in the breast of *Leo*, the *Lion's Heart;* and a large star between the knees of *Bootes*, *Arcturus*.

Schol. 2. The constellations may be represented on two plane spheres projected on a great circle, or on the convex surface of a solid sphere, as on the celestial globe, or most perfectly on the concave surface of a hollow sphere. If the celestial globe be made use of, after rectifying it to the time of the night, the stars may be found, by conceiving a line drawn from the centre of the globe through any star in the heavens, and its representation upon the globe. Greek letters have been added by Bayer

to stars in the several constellations of his catalogue (α being affixed to the largest star), by means of which any star may be easily found.

Schol. 3. Twelve of these constellations lie upon the ecliptic, including a space about 15° broad, called the *Zodiac*, within which all the planets move. The constellation Aries, about 2000 years ago, lay in the first sign of the ecliptic; but on account of the precession of the equinoxes, it now lies in the second. Prop. CLXXII. Schol. In the foregoing table *Antinous* was made out of the unformed stars near *Aquila;* and *Coma Berenices* out of the unformed stars near the *Lion's* tail. They are both mentioned by Ptolemy, but as unformed stars. The constellations as far as the triangle, with *Coma Berenices*, are *northern;* those after *Pisces* are *southern*.

PROP. CLXXIX. The luminous part of the heavens, called the *Milky Way*, consists of fixed stars too small to be seen by the naked eye.

This is found from observations made with telescopes.

In a paper on the Constructions of the Heavens, Dr. Herschel says, "it is very probable, that the great stratum called the milky way is that in which the sun is placed, though perhaps not in the centre of its thickness, but not far from the place where some smaller stratum branches from it. Such a supposition will satisfactorily, and with great simplicity, account for all the phenomena of the milky way, which, according to this hypothesis, is no other than the appearance of the projection of the stars contained in this stratum, and its secondary branch."

In another paper on the same subject, he says, "that the milky way is a most extensive stratum of stars of various sizes, admits no longer of the least doubt; and that our sun is actually one of the heavenly bodies belonging to it is as evident."

"We will now retreat to our own retired station in one of the planets attending a star in the great combination with numberless others; and in order to investigate what will be the appearances from this contracted situation, let us begin with the naked eye. The stars of the first magnitude being in all probability the nearest, will furnish us with a step to begin our scale; setting off, therefore, with the distance of Sirius or Arcturus, for instance, as unity, we will at present suppose, that those of the second magnitude are at double, and those of the third at treble the distance, and so forth. Taking it, then, for granted, that a star of the seventh magnitude is about seven times as far from us as one of the first, it follows that an observer, who is enclosed in a globular cluster of stars, and not far from the centre, will never be able, with the naked eye, to see to the end of it. For since, according to the above estimations, he can only extend his view about seven times the distance of Sirius, it cannot be expected that his eyes should reach the borders of a cluster, which has, perhaps, not less than fifty stars in depth every where around him. The whole universe, therefore, to him, will be comprised in a set of constellations, richly ornamented with scattered stars of all sizes. Or, if the united brightness of a neighbouring cluster of stars should, in a remarkably clear night, reach his sight, it will put on the appearance of a small, faint, nebulous cloud, not to be perceived without the greatest attention. Allowing him the use of a common telescope, he begins to suspect that all the milkiness of the bright path which surrounds the sphere may be owing to stars. By increasing his power of vision, he becomes certain, that the milky way is indeed no other than a collection of very small stars, and the nebulæ nothing but clusters of stars."

Dr. Herschel then solves a general problem for computing the length of the visual ray; that of the telescope, which he uses, will reach to stars 497 times the distance of Sirius. Now (by Prop. B. Cor. 1. p. 323.) Sirius cannot be nearer than $100{,}000 \times 190{,}000{,}000$ miles; therefore Dr. Herschel's telescope will, at least, reach to $100{,}000 \times 190{,}000{,}000 \times 497$ miles. And Dr. Herschel says, that in the most crowded part of the milky way, he has had fields of view that contained no less than 588 stars, and these were continued for many minutes, so that, in a quarter of an hour, he has seen 116,000 stars pass through the field view of a telescope of only 15′ aperture; and at another time, in 41 minutes, he saw 258,000 stars pass through the field of his telescope. Every improvement in his telescopes has discovered stars not seen before, so that there appear no bounds to their number, nor to the extent of the universe. See Phil. Trans. Vol. lxxiv. and lxxvi.

Schol. 1. There are spots in the heavens, called *Nebulæ*, some of which consist of clusters of telescopic stars, others appear as luminous spots of different forms. The most considerable is one in the mid-way between the two stars on the blade of Orion's sword, marked θ by Bayer, discovered in the year 1656 by Huygens; it contains only seven stars, and the other part is a bright spot upon a dark ground, and appears like an opening into brighter regions beyond. Dr. Halley and others have discovered nebulæ in different parts of the heavens. In the *Connoissance des Temps* for 1783 and 1784, there is a catalogue of 103 nebulæ observed by Messier and Mechain. But to Dr. Herschel we are

indebted for catalogues of 2000 nebulæ and clusters of stars, which he himself has discovered. Some of them form a round compact system, others are more irregular, of various forms, and some are long and narrow. The globular systems ot stars appear thicker in the middle, than they would do if the stars were all at equal distances from each other; they are therefore condensed toward the centre. That stars should be thus accidentally disposed is too improbable a supposition to be admitted; he supposes, therefore, that they are brought together by their mutual attractions, and that the gradual condensation toward the centre is a proof of a central power of such a kind. He observes also, that there are some additional circumstances in the appearance of extended clusters and nebulæ, that very much favour the idea of a power lodged in the brightest part. For, although the form of them be not globular, it is plain that there is a tendency to sphericity. As the stars in the same nebulæ must be very nearly all at the same relative distances from us, and they appear nearly of the same size, their real magnitudes must be nearly equal. Granting, therefore, that these nebulæ and clusters of stars are formed by mutual attraction, Dr. Herschel concludes that we may judge of their relative age by the disposition of their component parts, those being the oldest which are most compressed. He supposes, and indeed offers powerful arguments to prove, that the milky way is the nebulæ, of which our sun is one of the component parts. See Phil. Trans. Vol. lxxvi. and lxxix.

Schol. 2. Dr. Herschel has also discovered other phenomena in the heavens which he calls *nebulous* stars; that is, stars surrounded with a faint luminous atmosphere of large extent. Those which have been thus styled by other astronomers, he says, ought not to have been so called, for on examination they have proved to be either mere clusters of stars plainly to be distinguished by his large telescopes, or such nebulous appearances as might be occasioned by a multitude of stars at a vast distance. The milky way consists entirely of stars; and he says, "I have been led on by degrees from the most evident congeries of stars to other groups in which the lucid points were smaller, but still very plainly to be seen; and from them to such wherein they could but barely be suspected, until I arrived at last to spots in which no trace of a star was to be discerned. But then the gradation to these latter was by such connected steps, as left no room for doubt but that all these phenomena were equally occasioned by stars variously dispersed in the immense expanse of the universe."

In the same paper is given an account of some nebulous stars, one of which is thus described. "Nov. 13, 1790. A most singular phenomenon! a star of the eighth magnitude, with a faint luminous atmosphere of a circular form, and about 3′ in diameter. The star is perfectly in the centre, and the atmosphere is so diluted, faint, and equal throughout, that there can be no surmise of its consisting of stars, nor can there be a doubt of the evident connexion between the atmosphere and the star. Another star, not much less in brightness, and in the same field of view with the above, was perfectly free from any such appearance." Hence Dr. Herschel draws the following consequences: Granting the connexion between the star and the surrounding nebulosity, if it consist of stars very remote which give the nebulous appearance, the central star which is visible, must be immensely greater than the rest; or if the central star be no bigger than common. how extremely small and compressed must be those other luminous points which occasion the nebulosity! As, by the former supposition, the luminous central point must far exceed the standard of what we call a star; so in the latter, the shining matter about the centre will be too small to come under the same denomination; we, therefore, either have a central body which is not a star, or a star which is involved in a shining fluid, of a nature totally unknown to us. This last opinion Dr. Herschel adopts. Light reflected from the star could not be seen at this distance. Besides, the outward parts are nearly as bright as those near the star. Moreover, a cluster of stars will not so completely account for the milkiness or soft tint of the light of these nebulæ, as a self-luminous fluid. "What a field of novelty," says Dr. Herschel, "is here opened to our conceptions! A shining fluid, of a brightness sufficient to reach us from the regions of a star of the 8th, 9th, 10th, 11th, 12th magnitude, and of an extent so considerable as to take up 3, 4, 5, or 6 minutes in diameter. He conjectures that this shining fluid may be composed of the light perpetually emitted from millions of stars. See Phil. Trans. Vol. lxxxi. p. 1, on Nebulous Stars properly so called.

Schol. 3. New stars sometimes appear while others disappear. Several stars, mentioned by ancient astronomers are not now to be found; several are now visible to the naked eye, which are not mentioned in the ancient catalogues; and some stars have suddenly appeared, and again, after a considerable interval, vanished; also a change of place has been observed in some stars.

The following are remarkable and well authenticated examples. The first new star we have an accurate account of, is that discovered by Cornelius Gemma, on Nov. 8, 1572, in the *Chair of Cassiopeia.* It exceeds *Sirius* in brightness, and *Jupiter* in apparent magnitude. Tycho Brahe observed it, and found that it had no sensible parallax It gradually decayed, and, after 16 months, disappeared.

On August 13, 1596, David Fabricius observed a new star in the *Neck of the Whale*, 25° 45′ of Aries, with 15° 54′ south latitude. It disappeared after October in the same year; was discovered again in 1637.

In the year 1600, William Jansenus discovered a changeable star in the neck of the *Swan*. It was seen by Kepler, who wrote a treatise upon it, and determined its place to be 16° 18′ ♒, and 55° 30′ or 32′ north latitude. Ricciolus saw it in 1616, 1621, and 1624, and is certain that it was invisible from 1640 to 1650. M. Cassini saw it again in 1655; it increased till 1660; then decreased; and at the end of 1661, it disappeared. In November, 1665, it appeared again, and disappeared in 1681. In 1715 it appeared, as it does at present, of the 6th magnitude.

In 1686 Kircher observed χ in the *Swan*, to be a changeable star; and from 20 years' observations, the period of the return of the same phases was found to be 405 days; the variations of its magnitude are subject to some irregularity.

In the year 1604, Kepler discovered a new star near the heel of *Serpentarius*, so very brilliant, that it exceeded every fixed star, and even Jupiter, in apparent magnitude.

Montanari discovered two stars in the *Ship*, marked β and γ by Bayer, to be wanting. He saw them in 1664, but lost them in 1668. He observed also that β, in *Medusa's Head*, varied in its magnitude.

Mr. Goodricke has determined the periodical variation of *Algol*, or β, in *Medusa's Head* (observed by Montanari to be variable), to be about 2 days 21 hours. Its greatest brightness is of the second magnitude, and least of the fourth. Phil. Trans. Vol. lxxiii.

Schol. 4. From an attentive examination of the stars with good telescopes, many, which appear only single to the naked eye, are found to consist of two, three, or more stars. Dr. Maskelyne had observed α *Herculis*, to be a double star. Dr. Hornsby found π *Bootis* to be double. Other astronomers had made similar discoveries. But Dr. Herschel, by his highly improved telescopes has found about 700, of which not more than 42 had been observed before.

The following are a few of the most remarkable;

α *Herculis*, Flam. 64, a beautiful double star; the two stars very unequal; the largest red, and the smallest blue, inclining to green.

α *Geminorum*, Flam. 66, double, a little unequal, both white; with a power of 146 their distance appears equal to the diameter of the smallest.

ϵ *Lyræ*, Flam. 4 and 5, a double-double star; at first sight it appears double at a considerable distance, and by a little attention each will appear double; one set are equal, and both white; the other unequal, the largest white, and the smallest inclined to red.

β *Lyræ*, Flam. 10, quadruple, unequal, white; but three of them a little inclined to red.

λ *Orionis*, Flam. 39, quadruple, or rather a double star, and has two more at a small distance; the double star considerably unequal, the largest white; smallest, pale rose-colour.

μ *Herculis*, Flam. 86, double, very unequal; the small star is not visible with a power of 278, but is seen very well with one of 460; the largest is inclined to a pale red; smallest, duskish.

α *Lyræ*, Flam. 3, double, very unequal, the largest a fine, brilliant white, the smallest dusky; it appears with a power of 227. Dr. Herschel measured the diameter of this fine star, and found it to be 0″.3553.

The examination of double stars with a telescope is a very excellent and ready method of proving its powers. Dr. Herschel recommends the following method. The telescope, and the observer, having been some time in the open air, adjust the focus of the telescope to some single star of nearly the same magnitude, altitude, and color of the star, to be examined; attend to all the phenomena of the adjusting star as it passes through the field of view; whether it be perfectly round, and well defined, or affected with little appendages playing about the edge, or any other circumstances of the like kind. Such deceptions may be detected by turning the object-glass a little in its cell, when these appendages will turn the same way. Thus you may detect the imperfections of the instrument, and therefore will not be deceived when you come to examine the double star. Phil. Trans. Vol. lxxii. and lxxv.

Schol. 5. The number of stars is unknown. The catalogue published by Bayer contains 1160; that by Flamstead, which includes many telescopic stars, contains 3000. But the most complete catalogue is that published by the Rev. Mr. Wollaston, in 1789.

PROP. CLXXX. The longitude of the fixed stars increases, while their latitude remains the same.

Because the vernal equinoctial point (by Prop. CLXXI.) moves westward, the distance between any given star and that point, that is, its longitude, will increase. But since this change is produced by the precession of the equinoxes, which is performed round the axis of the ecliptic, this motion will make no change in the distance of the fixed star from the ecliptic, that is, in its latitude.

Cor. Hence the constellations of the zodiac are to the east of those signs or arcs of the zodiac which

are called by the same names. The first part of the constellation Aries, by the precession of the equinoxes, has gone so far to the east, since the names were first given to the signs, that it is now 30° from the first degree of Aries in the line of the ecliptic.

Def. LXVIII. The *Annual Parallax* of a heavenly body is the change of its apparent place, as it is viewed from the earth in its annual motion.

If ADBC be the orbit of the earth, S the sun, and A, B, the earth in opposite parts of its orbit; the change in the apparent place of any body, as viewed from A and from B, is its annual parallax. Plate 11. Fig. 16.

PROP. CLXXXI. The annual parallax of any heavenly body is proportional to the angle which a diameter of the earth's orbit would subtend, if it were viewed from that body.

If when the earth is at A, the fixed star E appears at or near the pole, and when the earth has passed to the opposite point B, a different star F appears at or near the same pole, the star E will have changed its place in respect of the pole; for when the pole is at F, the star E, which was at or near it before, is at the distance EF from it; the apparent length of this distance EF (by Def. LXVIII.) is the star's annual parallax. Now, if AB, the diameter of the earth's orbit, were to be viewed from the star E, it would subtend the angle AEB; but because the axis of the earth is always parallel to itself, AE and BF, which coincide with the axis, are likewise parallel; whence (El. I. 29.) the angle EBF, subtended by EF, is equal to AEB, subtended by AB; and AEB is the parallactic angle.

Cor. The annual parallax of any heavenly body is inversely as its distance from the earth; for the angle AEB (by Book VI. Prop. LXIX.) is inversely as the distance of AB, the axis of the earth's orbit.

PROP. A. If the distance of an object be greater than 400,000 times the base, the angles at the stations will not sensibly differ from right angles; consequently the lines drawn from the object to the station are, physically speaking, parallel.

Suppose one of the angles to be 90°. Then, since the most accurate instruments for the mensuration of angles cannot be depended upon to less than 0″.5, the tangent of which is to radius as 1 to 403,132, if the base have a less ratio to the distance than this, that is, than about 1 to 400,000 (for the angle, the tangent of which is to radius in this ratio is 2″.06, or very little more than two seconds), the angle at the other station will not sensibly differ from 90°. See Hutton's Logarithms.

Schol. It has been seen that Dr. Herschel depends upon his instruments for the accuracy of measuring quantities much less than 0″.5.

PROP. B. The parallax of an object, the distance of which is above 400,000 times greater than that between the two stations of observation, is insensible.

If the object be at a greater distance from either station than 400,000 times the base, the angle at one of the stations being 90°, the angle at the other will be more than 89° 59′ 57.9″, the difference of which angle and 90°, being scarcely more than 0″.5, is too small to become sensible by observation.

Cor. 1. If the parallax of an object (observed with an instrument sufficiently exact to measure an angle of 0.″5) be insensible, the distance of it from either station cannot be less than 100,000 times the base from the extremities of which it is observed.

Schol. It is to be remarked, that, though the distance of the object cannot be less than 100,000 times the base, yet it may be greater in any assignable ratio.

Cor. 2. Lines drawn from any given points in a base, to an object, may be esteemed, in practice, parallel, without any sensible error, if the distance of the object be more than 100,000 times the base.

Cor. 3. Rays, therefore, diverging from any point in the sun's disk upon the surface of the earth, may be esteemed parallel, if their distance from each other do not exceed about 970 miles at the earth's surface; because 970 is to the distance of the earth from the sun in a proportion of 1 to 400,000.

PROP. CLXXXII. The fixed stars have no sensible annual parallax.

When the place of the star E is observed by the best instruments from opposite points of the earth's orbit, its apparent place in the heavens remains the same, which could not be the case if the angle of its parallax were so much as two seconds. Plate 11. Fig. 16.

Cor. 1. Hence it appears, that the fixed stars are so remote, that a diameter of the earth's orbit, bears no proportion to their distance, or (by Prop. CLXXXI.) that a diameter of the earth's orbit, if viewed from one of the fixed stars, would appear as a point.

Cor. 2. The distance of the stars must be greater than 400,000 times the base, from the extremities of which it is observed; that is, greater than 400,000 times the diameter of the orbit of the earth, or greater than 400,000 × 190,000,000.

Cor. 3. Two planes drawn parallel to each other, and passing through the extremities of a diameter of the orbit of the earth, if produced, will appear to coincide with the same great circle of the heavens; because the diameter of the earth's orbit, when seen from the fixed stars, subtends an angle less than $\frac{1}{2}''$. In the same manner, if a plane passing through the earth's centre, be parallel to a plane drawn to the surface, these planes, when produced, apparently coincide with the same great circle in the heavens.

Cor. 4. The parallax of a fixed star, being not more than $0''.5$, the sun, when viewed from that star, would appear under an angle less than $\frac{32'\ 6''}{200,000}$ or less than $\frac{1''}{100}$, and therefore could not be distinguished from a point.

Schol. Since bodies equal in magnitude and splendour to the sun, being placed at the distance of the fixed stars, would appear to us as the fixed stars now do, it may be supposed probable, that the fixed stars are bodies similar to the sun, which is the centre of the solar system. This being the case, the reason will appear, why a fixed star, when viewed through a telescope magnifying 200 times, appears no other than a point. For the apparent diameter of the star being less than $\frac{1}{100}$ part of a second when magnified 200 times, will subtend an angle less than $0''.5$, at the eye of a spectator, observing it in the telescope.

The parallax of the fixed star, when viewed from the opposite parts of the earth's orbit, is here assumed $0''.5$, but it is probable that the parallax of the nearest star is much less, and consequently the distance greater, in the same proportion, as the parallax is less.

PROP. CLXXXIII. The motion of the earth, and the progressive motion of light, will make a fixed star, which has no sensible parallax, deviate from its true place in the direction in which the earth moves.

Plate 11. Fig. 17. If a star passes through the zenith of any place when the earth is at A, it will (by last Prop.) pass through the zenith of the same place when the earth is at B, the opposite extremity of the earth's orbit. Consequently, such a star might be seen through a vertical telescope in the same perpendicular at any point of the earth's orbit, if the motion of light from the star were instantaneous. But the progressive motion of light will cause the star to deviate from the perpendicular; for, let the earth be moving from B to A, and let the velocity of light be to the velocity of the earth, as CA to BA, and let CB be the diagonal of the parallelogram formed from CA, BA. Then the direction of a telescope, in order to see the star S when the earth is arrived at A, must be AH, parallel to BC. For, suppose BC to be a very long, slender telescope, through which only one ray of light could pass at a time, or to be the axis of a larger telescope. The star S cannot be seen through this telescope, but through a telescope perpendicular to B, if the earth be stationary at B, and the progress of light instantaneous. But if the telescope in the position BC were to continue in this position, and to move along with the earth to A, so as to come into the situation AH, when the earth arrives at A, the star S might then be seen through it. For, since the straight course of the ray is the line CA, in which it must always be if it comes to the eye without interruption; and since the ray cannot come directly along CB, the axis of the telescope, and arrive at the eye in this axis, unless it is always in the axis; that is, since the ray, in order to come to the eye, must be always in the line CA, and also in the line CB, it must be always in the common intersection of these two lines. Now C is the common intersection when the earth is at B; *e* is the common intersection when the earth is at E; *f*, when it is at F; *g*, when it is at G; and A when at A; the telescope, at each station, being successively in the situations CB, EE, FF, GG, HA. Thus the common intersection descends down the line CA, while the earth moves from B to A; and since the velocity of light is to that of the earth, as CA to BA, a ray of light will likewise have descended down CA, while the earth is moving from B to A. Therefore, in the whole motion of the telescope, the ray will have been in the common intersection in the line CA, and the axis of the telescope, and consequently will have passed along the axis of the telescope, and will come without interruption to the eye at A.

Thus it appears, that by the progressive motion of light, a ray which, coming from S, enters, at C, a telescope in the situation CB, will arrive at the eye, when the telescope, carried along BA with a velocity which is to that of a ray of light, as BA to CA, is come into the situation HA; and consequently (Book VI. Prop. II.) the eye will see the star through the telescope in the direction AH, the axis of the telescope; that is, some point in the line AH produced will be the apparent place of the star. Thus the star's apparent place has deviated from its true place S in the direction BA, in which the earth was moving, so that, if the motion of the earth is from north to south, the star, which ap-

peared in the zenith of the place when the earth was at B, will appear to the southward of the zenith when the earth is arrived at A, and the reverse when the earth is moving from south to north.

According to Bradley's observations, made on the star γ in the constellation Dragon, this star deviated southward from the zenith from Dec. till March, when it had departed from the zenith 20″. From that time till June its southern deviation decreased, after which it deviated northward, and in September appeared about 20″ toward the north of its station in June, from which time till December it continued returning to its first situation. Thus the deviation of the star was always in the direction of the earth's motion, and contrary to that of any deviation which might be supposed to arise from the annual parallax of the star. But such a deviation could not happen unless the earth moved, and the motion of light was progressive; for if the earth did not move, since the star is fixed, no alteration could be made in the apparent place of the star by the progressive motion of its rays in a vertical direction; and if the earth moves, and the propagation, of light were instantaneous, the earth's velocity would be nothing in respect of the velocity of light, or BA with respect to CA would be nothing; whence the angle ACB, and its alternate angle CAH, would vanish, and AH wonld become coincident with AC, and consequently the star would have no deviation from its true place. Hence we may conclude from the deviation of the star above described, both that the earth moves, and that the motion of light is progressive.

Cor. From these observations it is found, that the velocity of star-light is such as carries it through a space equal to the sun's distance from the earth in 8′ 13″.

Schol. Sir Isaac Newton has shown that the sun, by its attractive power, retains the planets belonging to our system in their orbits; he has likewise pointed out the method whereby the quantity of matter contained in the sun may be accurately determined. Dr. Bradley has assigned the velocity of the solar light with a degree of precision exceeding our utmost expectation. Galileo and others have ascertained the rotation of the sun upon its axis, and determined the position of its equator. By means of the transit of Venus over the disk of the sun, our mathematicians have calculated its distance from the earth; its real diameter and magnitude; the density of the matter of which it is composed; and the laws of the fall of heavy bodies on its surface.

In the year 1779, there was a spot on the sun, which was large enough to be seen by the naked eye; it was divided into two parts, and must have been 50000 miles in diameter; this phenomenon may be accounted for, from some natural change of an atmosphere. For if some of the fluids which enter into its composition be of a shining brilliancy, whilst others are merely transparent, then any temporary cause which should remove the lucid fluid, will permit us to see the body of the sun through the transparent ones. If an observer were placed on the moon, he could see the solid body of the earth only in those places where the transparent fluids of our atmosphere would permit him. In others, the opaque vapours would reflect the light of the sun without permitting his view to penetrate the surface of our globe. He would probably find that our planet had occasionally some shining fluids in its atmosphere, such as the northern lights. And there is good reason to believe, that all the planets emit light in some degree; for the illumination which remains on the moon in a total eclipse cannot be entirely ascribed to the light which may reach it by the refraction of the earth's atmosphere. For, in some cases, as in the eclipse of 1790, the focus of the sun's rays refracted by the earth's atmosphere, must be many thousand miles beyond the moon.

There are appearances also which denote a phosphoric quality in the atmosphere of Venus.

Dr. Herschel supposes, that the spots in the sun are mountains on its surface, which, considering the great attraction exerted by the sun upon bodies placed at its surface, and the slow revolution it has about its axis, he thinks may be more than 300 miles high, and yet stand very firmly. He says, that in August, 1792, he examined the sun, with several powers from 90 to 500. And it evidently appeared, that the black spots are the opaque ground or body of the sun; and that the luminous part is an atmosphere, which being intercepted or broken, gives us a glimpse of the sun itself.

Hence he concludes, that the sun has a very extensive atmosphere, which consists of elastic fluids that are more or less lucid and transparent; and of which the lucid ones furnish us with light. This atmosphere, he thinks, is not less than 1843, nor more than 2765 miles in height; and he supposes that the density of the luminous solar clouds need not be exceedingly more than that of our aurora borealis, in order to produce the effects with which we are acquainted.

The sun, then, appears to be a very eminent, large, and originally luminous body, and the only one belonging to our system. Its similarity to the other globes of the solar system, with regard to its solidity;—its atmosphere;—its surface diversified with mountains and vallies;—its rotation on its axis;—and the fall of heavy bodies on its surface,—leads us to suppose that it is most probably inhabited, like the rest of the planets, by beings whose organs are adapted to the peculiar circumstances of that vast globe.

If it be objected, that from the effects produced at the distance of 97,000,000 miles, we may infer that every thing must be scorched up at its surface; we reply, that there are many facts in natural philosophy which show that heat is produced by the sun's rays only when they act on a calorific medium; they are the cause of the production of heat by uniting with the matter of fire which is contained in the substances that are heated; as the collision of the flint and steel will inflame a magazine of gunpowder, by putting all the latent fire which it contains into action.

On the tops of mountains of sufficient height, at the altitude where clouds can seldom reach to shelter them from the direct rays of the sun, we always find regions of ice and snow. Now if the solar rays themselves conveyed all the heat we find on this globe, it ought to be the hottest where their course is the least interrupted. Again; our aëronauts all confirm the coldness of the upper regions of the atmosphere; and since, therefore, even on our earth the heat of the situation depends upon the readiness of the medium to yield to the impression of the solar rays, we have only to admit, that on the sun itself, elastic fluids composing its atmosphere, and the matter on its surface, are of such a nature as not to be capable of any extensive affection of its own rays; and this seems to be proved by the copious emission of them; for if the elastic fluids of the atmosphere, or of the matter contained on the surface of the sun, were of such a nature as to admit of an easy chemical combination with its rays, their emission would be very much impeded. Another well known fact is, that the solar focus of the largest lens, thrown into the air, will occasion no sensible heat in the place where it has been kept for a considerable time, although its power of exciting combustion, when proper bodies are exposed, should be sufficient to fuse the most refractory substances.

It is by analogical reasoning that we consider the moon as inhabited. For it is a secondary planet of considerable size, its surface is diversified, like that of the earth, with hills and valleys. Its situation, with respect to the sun, is much like that of the earth; and by a rotation on its axis, it enjoys an agreeable variety of seasons, and of day and night. To the moon, our globe would appear a capital satellite, undergoing the same changes of illumination as the moon does to the earth. The sun, planets, and the starry constellations of the heavens, will rise and set there as they do here; and heavy bodies will fall on the moon as they do on the earth. There seems, then, only to be wanting, in order to complete the analogy, that it should be inhabited like the earth.

It may be objected, that, in the moon, there are no large seas; and its atmosphere (the existence of which is doubted by many) is extremely rare, and unfit for the purposes of animal life;—that its climates, its seasons, and the length of its days and nights, totally differ from ours;—that without dense clouds, which the moon has not, there can be no rain, perhaps no rivers and lakes.

In answer to this, it may be observed, that the very difference between the two planets strengthens the argument. We find even on our globe, that there is a most striking dissimilarity in the situation of the creatures that live upon it. While man walks on the ground, the birds fly in the air, and the fishes swim in the water. We cannot surely object to the conveniences afforded by the moon, if those that are to inhabit its regions are fitted to their conditions as well as we on this globe of ours. The analogy already mentioned establishes a high probability that the moon is inhabited.

Suppose, then, an inhabitant of the moon, who has not properly considered such analogical reasonings as might induce him to surmise that our earth is inhabited, were to give it as his opinion, that the use of that great body, which he sees in his neighbourhood, is to carry about his little globe, in order that it may be properly exposed to the light of the sun, so as to enjoy an agreeable and useful variety of illumination, as well as to give it light by reflection, when direct light cannot be had. Should we not condemn his ignorance and want of reflection? The earth, it is true, performs those offices which have been named, for the inhabitants of the moon, but we know that it also affords magnificent dwelling-places to numberless intelligent beings.

From experience, therefore, we affirm, that the performance of the most salutary offices to inferior planets, is not inconsistent with the dignity of superior purposes; and in consequence of such analogical reasonings, assisted by telescopic views, which plainly favour the same opinion, we do not hesitate to admit that the sun is richly stored with inhabitants.

This way of considering the sun, is of the utmost importance in its consequences. That stars are suns can hardly admit of a doubt. Their immense distance would effectually exclude them from our view, if their light were not of the solar kind. Besides, the analogy may be traced much farther; the sun turns on its axis; so does the star Algol; so do the stars called β Lyræ, δ Cephei, η Antinoi, o Ceti, and many more, most probably all. Now from what other cause can we, with so much probability, account for their periodical changes? Again; our sun's spots are changeable—so are the spots on the star o Ceti. But if stars are suns, and suns are inhabitable, we see at once what an extensive field for animation opens to our view.

It is true, that analogy may induce us to conclude, that since stars appear to be suns, and suns, according to the common opinion, are bodies that serve to enlighten, warm, and sustain a system of plan-

ets, we may have an idea of numberless globes that serve for the habitation of living creatures. But if these suns themselves are primary planets, we may see some thousands of them with the naked eyes, and millions with the help of telescopes; and, at the same time, the same analogical reasoning still remains in full force with regard to the planets which these suns may support.

Moreover, from the observations of Dr. Herschel, on the compressed clusters of stars, it appears, that in many instances there cannot be space for the revolutions of a system of planets and comets, and therefore it is highly probable that these suns are capital primary planets, which exist for themselves and are connected together in one great system for mutual support. See a very curious and valuable paper on the nature and construction of the sun and fixed stars, by Dr. Herschel; read to the Royal Society, Dec. 18, 1794. From this paper, the foregoing scholium has been taken. See also Dr. Herschel's paper on the periodical star, α Herculis; with remarks, tending to establish the rotatory motion of the stars on their axes. Phil. Trans. 1796.

APPENDIX TO THE ASTRONOMY.

CONTAINING SOLAR AND LUNAR TABLES, WITH THEIR EXPLANATION AND USE, AND THE PROJECTION OF ECLIPSES, SELECTED FROM "EWING'S PRACTICAL ASTRONOMY."

EXPLANATION OF THE TABLES.

Tables of the Mean Motions of Celestial Objects.

The idea that the sun, moon, and stars, performed all their motions in circles, was, perhaps, one of the first which men received concerning these very distant bodies. The regular returns of day and night, of the seasons of the year, and of almost all things in the visible world, would serve to confirm it; and although it is well known that the orbits of the planets are not perfect circles, it is equally known that they differ very little from circles; and therefore modern astronomers retain the idea, and form tables of the motions of the planets, as if their orbits were circles, and their motions always uniform, passing over equal spaces in equal times. Such are called *Tables of Mean Motion*; and the longitude of a planet computed from such Tables for any given time is called its *mean longitude.*

Table I. II. *Contain the Sun's Mean Motion, and the Precession of the Equinoctial Points in Julian Years.*

The astronomical year is that space of time wherein the earth moves round the sun, or wherein the sun seems to move round the whole ecliptic from any point, such as the vernal equinox, to the same again; which, according to the most accurate observations, consists of 365 natural days, 5 hours, 48 minutes, $43\frac{1}{2}$ seconds. But in civil reckoning there are two kinds of years, common and bissextile. A common year consists of 365 days, and the bissextile of 366. These are called *Julian Years*, from Julius Cæsar, who introduced this method of computation. Mr. Mayer makes the sun's mean motion in 365 days to be 11s. 29° 45′ 40″.7. Admitting his numbers, we have,

Year.	S.	°	′	″
1	11	29	45	40.7
2	11	29	31	20.4
3	11	29	17	2.1
4	0	0	1	51.1

For the 4th year,

	S.	°	′	″
To the 3d year	11	29	17	2.1
Add mo. for 1 year	11	29	45	40.7
And for 1 day			59	8.3
The 4th year =	0	0	1	51.1

That is, in four years the earth goes four times round the sun, and 1′ 51″ more. In the same manner, the sun's mean motion may be found for any number of Julian years, as in the Table.

The yearly motion of the earth's or sun's apogee is found to be 1′ 6″; which being subtracted from the sun's yearly motion in longitude, the remainder is the sun's mean anomaly for 1 year; and for any number of years the mean anomaly is found in the same manner as the mean longitude.

It has been found by observation, that the equinoctial points move backward, or contrary to the order of the signs, at the rate of 50″.3 yearly, called the *precession of the equinoxes*; and multiplying 50″.3 by 2, 3, 4, 5, &c. the numbers in Table II. are found.

Table III. *Contains the Sun's Mean Longitude and Anomaly, with the Obliquity of the Ecliptic, for years current of the Christian Era.*

The table is composed in this manner: The mean longitude and anomaly of the sun with the obliquity of the ecliptic, are found for the noon of the last day of Dec. 1760, which is accounted his mean longitude, &c. for the succeeding year 1761. To these numbers add the sun's mean motions for 20

years, taken from table I; the sums are the sun's mean longitude and anomaly for 1781. To these add the mean motions for 10 years, and the mean longitude and anomaly for 1791 are known. And for the following years add the sun's mean motions in one year to his mean longitude and anomaly for the preceding year continually; remembering to add the motion of 1 day more for every bissextile year until the number of years which the table is to contain be completed.

The annual differences of the obliquity of the ecliptic are very small, and not always regular. It appears by the Table that the difference is only about 25″ in 60 years; therefore the obliquity of the ecliptic may be stated at 23° 28′ during the next fifty years without sensible error.

When the sun's mean longitude and anomaly are wanted for any year which is not in the Table, take the mean motions for the intermediate years from Table I. and if the year is before 1761, subtract them; if after it, add them to the numbers for 1761, and the remainder or sum will be the mean longitude and anomaly required.

In these additions 12 signs are to be rejected as often as they occur; and in the subtractions 12 signs must be borrowed when necessary.

Table IV. Contains the sun's mean motions for the days of the year, distributed into 12 calendar months. At the noon of January 1st there is one astronomical day past, because it began at the noon of December 31st; and therefore the sun's mean motions in one day are put down. These being multiplied by 2, 3, 4, 5, &c. (allowance being made for the fractions in one day's motion) the products are the numbers of the Table; remembering to place them properly, *viz.* the motions for 31 days at the last day of January; for 32 days at the 1st day of February; and so on until the 31st of December.

Table V. Contains the sun's mean motions for hours, minutes, and seconds, which may be understood from what has been said of his motions for days.

Table VI. *Equations of the Sun's Centre.*

The sun's mean and true longitudes differ more or less in every point of the earth's orbit except two, *viz.* the aphelion and perihelion; and these differences are called *equations of the sun's centre.* One cause of the difference is, that the earth's orbit is an ellipse, in the periphery whereof the true longitude is reckoned from the vernal equinox, and the mean longitude is reckoned from the same point in the circumference of a circle, whose diameter is the greater axis of the ellipse. The circle and ellipse coincide only in the aphelion and perihelion; and therefore when the sun or earth is in either of these points, the mean and true longitudes are the same, and there is no equation.

Another cause of the difference between the mean and true longitudes arises from the unequal motion of the earth in its orbit; for while the earth proceeds from its aphelion to its perihelion, its motion is continually accelerated; and from its perihelion to its aphelion, its motion is continually retarded; and this is true of every planet.

When the sun appears in the earth's aphelion, his longitude is about 3s. 9°, and his anomaly is 0, because it is reckoned from that point; and when he appears in the earth's perihelion, his longitude is nearly 9s. 9°, and his anomaly is 6 signs. In the first six signs of anomaly, the equation found in the Table is to be subtracted from the mean longitude; in the other six signs it is to be added, and the remainder or sum is the true longitude.

Table VII. *Contains the Logarithms of the Earth's Distances from the Sun.*

Because the earth's orbit is an ellipse, and the sun in one of its foci, the earth's distance from the sun varies every moment; for it is greatest in aphelion, and decreases as the earth proceeds from thence toward the perihelion, where it is least. The several distances decrease in the first six signs of anomaly, and increase in the other six. When the anomaly is 3 or 9 signs, the earth is at its mean distance from the sun expressed by 1000000. In all other points the distance is either greater or less than the mean.

The earth's distance from the sun being calculated for every degree of anomaly, the logarithms of these distances are contained in the Table, and are of use in computing the longitudes of the other planets.

Table VIII. *The Sun's Declination to every Degree of his Longitude.*

At the time of either equinox the sun is in the equinoctial circle, but at all other times he appears at some distance from it, either north or south; and this distance is called his *declination.*

At the vernal equinox the sun has no declination; but from that point his declination increases northward, until he comes to the summer solstice, where it is greatest; and from thence it decreases until the sun is at the autumnal equinox, when again it is nothing; and then changes its name from north to south, and increases southward to the winter solstice, when again it is greatest; and from thence it decreases until the sun appears in the vernal equinox.

Those points of the ecliptic, which are equally distant from the equinoxes or solstices, being also equally distant from the equinoctial circle, have the same declination; and therefore the declination being calculated for every degree of the first quadrant of the ecliptic, answers for the whole; for the beginning of the sign Taurus, of Virgo, of Scorpio, and of Pisces, are all at the same distance from the equinoctial circle, and consequently these points have the same declination; only in the two first the declination is north, and in the two last it is south; and the same is true of all other points of the ecliptic which are equally distant from the equinoctial or solstitial points.

TABLE IX. *The Sun's Apparent Semidiameter and Hourly Motion.*

When the earth is in its aphelion, the sun's diameter appears least, and his apparent motion slowest; his diameter at that time is only 31′ 34″, and his hourly motion 2′ 23″. While the sun moves apparently from the aphelion to the perihelion, his diameter and hourly motion increase with his anomaly, and are greatest in the perihelion; his diameter being then 32′ 38″, and his hourly motion 2′ 33″. The difference between the aphelion and perihelion diameter is 64″, and between the hourly motions 10″.

The Table contains the sun's semidiameter and hourly motion to every 10° of mean anomaly; which by taking proportional parts for the intermediate degrees, will serve to find them for any given anomaly. They are frequently of use in Astronomy, especially in calculating eclipses.

TABLE X. Contains the equation of time for every degree of the sun's longitude; and by using proportion for the minutes and seconds, the equation may be found for any given longitude. For example, let it be required to find the equation of time when the sun's longitude is 6s. 10°. 50′ 30″. *Ans.* 11m. 12s.

s.	°		m.	s.		° s.	′ ″	s.
6	10	gives	10	57	As	1 : 18 ::	50 30 :	15
6	11		11	15	Add	-	10	57
	Diff.	-		18	True equation		11	12

TABLE XI. *The Sun's Longitude every day at Noon.*

The conveniency of having the sun's longitude nearly true for the noon of every day in the year inclined me to insert this Table. The reason why it could not be made accurate is, a common year of 365 days differs from the time wherein the sun seems to move round the whole ecliptic by almost 6 hours, or a day in four years; which causes a difference in the sun's longitude at noon on the same days of different years; and therefore, to have made the table more perfect, it must have been calculated for four years, which was not thought expedient.

TABLES *of the Moon's Mean Motions.*

These Tables are made in the same manner as those of the sun or earth, the moon's period being known; but her motions are more in number than those of the earth; for besides her mean longitude and anomaly, the longitude of her node must be known, in order to calculate her place for any given time.

The earth is in one of the foci of the moon's orbit, and is the centre of her motion; for the moon revolves round the earth in the same manner as the earth moves round the sun.

The moon's period, or the time wherein she moves once round the earth, is 27 days, 7 hours, 43 minutes, 5 seconds; and to find the moon's mean motion in a common year of 365 days the proportion is,

As the moon's period 27d. 7h. 43m. 5s.
Is to her whole orbit or 360°;
So is a common year of 365 days
To 13 revolutions, and 4s. 9°. 23′ 5″.

The 13 revolutions are rejected, and 4s. 9°. 23′ 5″ are taken for the moon's motion in 365 days.

For the Moon's Mean Motion in one day.

As the moon's period 27d. 7h. 43m. 5s.
Is to the whole circle or 360°;
So is one day
To the moon's mean motion in 1 day 13° 10′ 35″.

Having the moon's mean motion in 1 year and 1 day, the Table of her mean longitudes for Julian years is made by multiplying the motion for 1 year by 2, 3, 4, &c. and to the 4th adding the motion for 1 day, the products are the mean longitudes for these years; and by multiplying the motion for 1 year by

the several numbers of years, and adding the motion in 1 day to every 4th or bissextile year, the Table may be made to any extent.

To calculate the Moon's Mean Anomaly.

The moon's apogee moves once round her whole orbit in 8 years 309 days 8 hours 20 minutes; or (adding 2 days for leap years) in 3231 days 8 hours 20 minutes. Then,

As 3231d. 8h. 20m.
Is to the whole circle of 360°;
So is a common year of 365 days
To the mo. of the ☽'s apogee in 1 year 40° 39′ 50″.

	s.	°	′	″
From the ☽'s m. mot. in lon. during 1 year	4	9	23	5
Subtract the motion of her apogee in ditto	—1	10	39	50
There remains the ☽'s mean anomaly in 1 year	2	28	43	15

To find the Moon's Mean Anomaly for one Day.

Divide the motion of her apogee in a year, viz. 40° 39′ 50″ by 365, and the quotient 6′ 41″ is the motion of the moon's apogee for 1 day.

	°	′	″
From the moon's motion in longitude for 1 day	13	10	35
Subtract the motion of her apogee for ditto	—	6	41
There remains the moon's mean anom. for 1 day	13	3	54

Having found the moon's mean anomaly for 1 year and 1 day, the several mean anomalies in the Table are found in the same manner as the longitudes, viz. by multiplying the mean anomaly for 1 year by 2, 3, 4, &c. and adding the motion for 1 day to every 4th or bissextile year.

To find the Mean Motion of the Moon's Node.

The moon's node moves backward round her whole orbit in 18 years 224 days 5 hours; therefore, for its motion in one year,

As 18 years 224 days 5 hours
Is to the whole circle or 360°,
So is a year of 365 days
To the motion of the moon's node in 1 year 19° 19′ 43″.

For the motion of the moon's node in 1 day, divide 19° 19′ 43″ by 365, and the quotient, which is nearly 3′ 11″, is its motion for one day.

The motion of the moon's node for 1 year and 1 day being known, its motion for any number of years is found in the same manner as those of mean longitude and anomaly, viz. by multiplying one year's motion by 2, 3, 4, &c. and adding the motion of a day to every fourth or bissextile year.

Table II. *Containing the Moon's Mean Longitude and Anomaly, with the Longitude of her Node, for current Years.*

To make the Table, find, either by calculation or a good observation, the moon's mean longitude and anomaly, with the longitude of her node, on the noon of the last day of December preceding the year, where the Table is to begin, which here is the year 1760, and these are the mean places for 1761, the first in our Table. For 1781, the next year in the Table, take the mean motions for 20 years from Table I. and add the longitude and anomaly to those for 1761, but subtract the motion of the node; the results are the mean places for 1781; then take the mean motions for 10 years, and apply them in the same manner to the numbers for 1781, and the mean places for 1791 will be known.

For the following years the Table is carried on by the continual addition of the mean longitude and anomaly for one year, as also the motion for 1 day more every 4th or bissextile year, and subtracting the motion of the node.

Table III. *Contains the Moon's Mean Motions for Days.*

It begins with the mean motions for one day already found. These being multiplied by every number of days from 1 to 365, place the products at the proper days of the several months, and the table is made.

TABLE IV. *Contains the Moon's Mean Motions for Hours, Minutes, and Seconds.*

The numbers are found by dividing the mean motions for 1 day by 24, the quotients are the mean motions for an hour; and these again divided by 60, give the mean motions for 1 minute, &c. The motions for the different numbers of hours and minutes are found by multiplication.

The moon's motions are affected with many inequalities, and therefore many equations are necessary to reduce her mean longitude to the true. The method of finding the arguments, of taking out the equations, and applying them, is described in Prob. VII.

The Tables for finding the moon's latitude, parallax, diameter, hourly motion in longitude and latitude, at any given time, are sufficiently plain from their titles; and the method of applying them is described in Prob. VIII—XIII.

TABLES *of the Mean Motion of the Moon from the Sun.*

The moon's mean motion in a common year of 365 days is 4s. 9° 23′ 5″, over and above 13 revolutions; and the sun's apparent mean motion in the same time is 11s 29° 45′ 40″.

	S.	°	′	″
From	4	9	23	5
Subtract	11	29	45	40
The remainder	4	9	37	25

is the moon's mean motion from the sun in a common year.

In the same manner, the moon's mean motion from the sun may be found for a day, an hour, or for any number or parts of these.

Having the moon's mean motion from the sun in a year and a day, it may be found for any number of years by multiplication; remembering to add the motion for one day to every four years' motion for leap year.

To make a Table of the moon's mean motion from the sun for current years of the Christian era, subtract the sun's mean longitude from the moon's for every year which the Table is to contain; and the remainders will be the numbers expressing the moon's mean motion from the sun; as in the Table.

The table for months is made by subtracting the sun's mean longitude from the moon's on the last day of the preceding month; and the remainder is the moon's mean motion from the sun on the first day of the following month. The Tables for days, hours, minutes, and seconds, are made in the same manner; and the method of using these Tables is described in Prob. XIV.

TABLE *of Mean New Moons in March, with the Mean Anomalies of the Sun and Moon; and the Sun's Mean Distance from the Moon's Node.*

To make Table I. Calculate the time of mean new moon in March, for the year with which the Table is to begin, by the rule, Prob. XIV. or by any other method. Calculate also the sun and moon's mean anomalies for that time, and the sun's mean distance from the moon's node, from the Tables of mean motion, and write them down in order.

For the following years. If the new moon falls after the 11th day of March, add 12 lunations to the time for the former year; and if the next is a common year, subtract 365 days from the sum; the remainder is the time of mean new moon in March the next year; but if the next be a leap year, subtract 366 days. If the mean new moon falls before the 11th of March, add 13 mean lunations, and subtract 365 or 366 days from the sum, according as the next is a common or bissextile year.

Calculate also the sun and moon's mean anomalies, with the sun's mean distance from the moon's node for 12 or 13 lunations, and add them to those of the former year; and the numbers for the second year in the Table will be known. Proceed in the same manner for every succeeding year until the Table is completed.

TABLE II. *Contains 13½ Mean Lunations, with the Anomalies, and the sun's Mean Distance from the Moon's Node.*

The numbers are computed from the Tables of mean motion thus: Take the mean anomalies of the sun and moon out of the Tables for 29 days 12 hours 44 m. 3 s.; take also the mean motion of the sun, and of the moon's node, out of their proper Tables for the same time; their sum is the sun's mean distance from the moon's node. Having found the numbers for one lunation, multiply them by 2, 3, 4, &c. until there are 13 products, and divide the numbers for one lunation by 2 for the half lunation.

By these Tables the times of mean syzygies for any month of a given year, within the limits of Table I. may be found.

The Tables of equations for reducing the mean to the true times of new and full moons, in the following pages, depend on the mean anomalies of the sun and moon, with the sun's mean distance from

the moon's node, and are expressed in time for the conveniency of calculation. The method of applying them is given in Problem XVI.

Construction and Use of Logistical Logarithms.

Logistical Logarithms are artificial numbers deduced from common logarithms, and made for minutes and seconds either of degrees or time, used in working proportions, wherein some or all of the given terms are sexagesimals.

The numbers which compose the Table of logistical logarithms are regulated by its extent. For if the Table contains no more than the minutes and seconds in 1°, or 3600″, the logistical logarithm of 1″ is the logarithm of 3600″, viz. 3.5563; and all the following numbers are the differences between 3.5563 and the logarithms of the several numbers of seconds from 1 to 3600″. Therefore to calculate the logistical logarithm of any quantity less than 1°, subtract the logarithm of the given number of seconds from 3.5563: the remainder is the logistical logarithm required.

Exam. Let it be required to find the logistical logarithm of 10′, or 600″.

From the common log. of	3600 = 3.5563
Subtract the common log. of	600 = 2.7781
There remains the logistical log. of 10′ =	0.7782

But if the Table is to contain more than 1° such as in Dr. Maskelyne's Proportional Logarithms, which extends to 3°, or 10800″; then, in such a Table, the proportional or logistical logarithm of 1″ is the logarithm of 10800, viz. 4.0334; and all the rest are the differences between 4.0334 and the logarithms of the several numbers of seconds from 1 to 10800″.

Exam. Let it be required to find the logistical or proportional log. of 10′ or 600″.

From the log. of -	10800 = 4.0334
Subtract the log. of -	600 = 2.7781
Remains the prop. log. of 10′	1.2553

Hence there may be as many different systems of logistical or proportional logarithms as any one chooses to assume different extents of the Table. Our Table extends only from 1″ to 1°, which is sufficient for common use. Logistical logarithms consist of four figures beside the index.

The numbers on the head of the Table are either degrees or minutes; and those in the left hand column, on the side, are minutes or seconds; but if the numbers on the head be hours or minutes, those in the left hand column, on the side, will be minutes or seconds of time. Therefore these numbers change their denomination as occasion requires. The second line on the head of the Table are the numbers of seconds in the minutes which stand over them.

To take the logistical logarithms of any number of minutes and seconds out of the Table, the given number must be within the limits of the Table.

Find the minutes on the head of the Table, and the seconds in the left hand column on the side, and under the minutes and opposite to the seconds stands the logistical logarithm required.

Examples.

	′	″		
Given numbers	0	40	Logist. log.	1.9542
	1	10		1.7112
	7	40		8935
	9	15		8120
	50	0		792
	50	37		739

A logistical logarithm being given, to find the number of minutes and seconds answering.

If the given logarithm be found in the Table, the minutes are on the head of the column and the seconds on the side; but if it be not found exactly in the Table, take the next greater than it, and the minutes and seconds answering are found on the head and side of the table, as before.

Examples

L. Log.	Values. ′	″
1.7112	1	10
9823	6	15
6746	12	41
5175	18	13

When the logistical logarithm of any number less than 3600 is required, find it, or the nearest less, in the second line on the head of the Table; and below the number on the head, and opposite to its excess on the side, stands the logistical logarithm. Thus the logistical logarithm of 1276 is 4505.

Logistical logarithms are used in finding a fourth proportional, when some or all of the given terms, are minutes and seconds either of motion or time; and the method of operation is the same as in working with common logarithms, that is, by adding the logarithms of the second and third terms, and subtracting the logarithm of the first term from the sum, the remainder is the logarithm of the fourth term, or answer.

Exam. 1. Suppose the sun's hourly motion was 2′ 27″, and it were required to find his motion in 47′ 10″. The answer is 1′ 55″.

As 60 m.	0
To 2′ 27″	13890
So 47 m. 10 s.	1045
To 1′ 55″	14935

2. Suppose the hourly motion of the moon from the sun is 32′ 8″, and the moon is 58′ 12″ behind the sun. In what time will the moon overtake the sun? *Ans.* 1 h. 48 m. 40 s.

N. B. In such questions, when the fourth proportional would exceed 60 m. which is beyond the limit of the Table, take $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{4}$, of the third term, and multiply the fourth term, when found, by 2, 3, or 4.

As 32′ 8″	2712
To 60 m.	0
So 29′ 6″	3143
To 54 m. 20 s.	431
2	

Doubled is 1 h. 48 m. 40 s the answer.

In like manner, when two of the given terms exceed the limits of the Table, divide all the terms by some number, such as 2, 3, or 4, and multiply the answer by the same number which divided the given terms.

		Half of each term.	
Suppose we have,	As 76′ 34″	38′ 17″	1951
	Is to 24 h.	12 h.	6990
	So is 64′ 20″	32′ 10″	2707
	To 20 h. 10 m. nearly	10 h. 5 m.	9697
		Doubled 20 h. 10. m.	7746

PROBLEMS,

Showing the Use and Application of the Tables and Projection of Eclipses.

PROB. I. *The Longitudes of two Places, and the Time at one of them being given, to find the corresponding Time at the other.*

Reduce the difference of the longitudes to time, by allowing at the rate of 15° to an hour, and add it to the given time for a place toward the east; but subtract it for a place toward the west.

Exam. 1. Suppose the time at Greenwich to be 6 h. 7′ 8″ P. M. required the corresponding time at Cambridge, in longitude 71° 7′ 25″ west.

	h. ′ ″
Time at Greenwich	6 7 8 P. M.
Diff. long. 71° 7′ 25″ =	— 4 44 $29\frac{2}{3}$
Time at Cambridge	1 22 $38\frac{1}{3}$ P. M.

Exam. 2. The time at Cambridge being 7 h. 43′ 57″ A. M. required the corresponding time at Greenwich.

	h. ′ ″
Time at Cambridge	7 43 57 A. M.
Diff. long. 71° 7′ 25″ =	+ 4 44 29⅔
Time at Greenwich	28 26⅔ P. M.

EXAM. 3. The time at Hamburgh, in longitude 10° 38′ E. being 6 h. 30′ 40″ A. M. required the corresponding time at Cambridge, in longitude 71° 7′ 25″ W.

	h. ′ ″
Time at Hamburgh	6 30 40 A. M.
Diff. long. 81° 45′ 25″	5 27⅔
Time at Cambridge	1 3 38⅓ A. M.

PROB. II. *To calculate the true Longitude of the Sun for any given Time and Place by the Tables.*

When the given time is apparent, reduce it to mean time, and if the given place be not in the meridian of the Tables, that is, Greenwich, reduce it to that meridian by Prob. I. and then write down the years, months, days, hours, &c. under one another in a column on the left hand.

Take out of the Tables the sun's mean longitude and anomaly, answering to each part of the time, and write them down on the right hand of the former numbers; then add them as they stand in their several columns, rejecting 12 signs, or any multiple thereof, and the sum will be the mean longitude of the sun for the given time.

To reduce the mean longitude to the true. Enter the Table with the sun's mean anomaly, and take out the equation of the centre, making proportion for the odd minutes, &c. and if the sign of the sun's mean anomaly be at the head of the Table, subtract the equation from the mean longitude, but if it be at the foot of the Table, add it to the mean longitude, and the true longitude of the sun will be had.

EXAM. 1. Required the sun's longitude 1796, Feb. 19, 11 h. 21′ 30″ in the forenoon, apparent time, in the meridian of Edinburgh.

	D.	h.	′	″
Given time February - -	18	23	21	30
Edinburgh meridian W. -	+		12	25
Equation of time - - -	+		14	14
Mean time at Greenwich -	18	23	48	9

	M. Long. S. ° ′ ″	M. Anom. S. ° ′ ″
1796 - -	9 10 51 52	6 1 23 42
Feb. 18. bissex. -	1 17 18 40	1 17 18 31
23 h. - -	56 40	56 40
48′	1 58	1 58
9″ - -	10 29 9 10	7 19 40 51
Equation of the centre	+ 1 28 30	
Sun's true longitude	11 0 37 40	

2. Required the sun's longitude 1795, June 22, at noon, apparent time, at Greenwich.

	h. ′ ″
Mean time - - -	0 0 0
Equation of time - -	+ 1 28
	0 1 28

	M. Long. S ° ′ ″	M. Anom. S. ° ′ ″
1795 - -	9 10 7 2	6 0 39 58
June 22, - -	5 20 31 1	5 20 30 30
0h. 1′ - -	2	2
28″ - -	1	1
	3 0 38 6	11 21 10 31
Equation of the centre	+ 17 23	
Sun's true longitude	3 0 55 29	

PROB. III. *The Mean Anomaly of the Sun being given, to find the Logarithm of his Distance from the Earth.*

With the mean anomaly of the sun take out the logarithm of his distance from the earth from the Table.

Exam. Let the sun's mean anomaly be 7s. 19° 40′ 51″, what is the logarithm of his distance from the earth?

For 7s. 19° the log. is	-	4.995224
For 40′ 51″ the part is	-	+ 69
Answer	-	4.995293

PROB. IV. *To find the Apparent Semidiameter, and also the Hourly Motion of the Sun for any given Time.*

Calculate the sun's mean anomaly for the given time, and with the mean anomaly of the sun take out his semidiameter and hourly motion from the Table.

Exam. Required the sun's apparent semidiameter and hourly motion 1796, February 19th, 11h. 21′ in the forenoon.

The sun's mean anomaly at that time is 7s. 19° 42′, which gives his semidiameter 16′ 13″ and his hourly motion 2′ 31″.

PROB. V. *The Longitude of the Sun, or any Point of the Ecliptic, being given, to find the corresponding Declination.*

With the given longitude take the declination out of the Table.

Exam. Let the sun's longitude be 11s. 0° 37′ 40″, what is his declination? *Ans.* 11° 15′ 47″ south.

To longitude 11s. 0° the declination is 11° 29′ 5″ S. decreasing; the difference between this and the next is 21′ 12″. Then,

	° ′ ″	L.L.		° ′ ″
As	1 0 0	0		
To	0 21 12	4518	From	11 29 5
So is	0 37 40	2022	Subtract	— 13 18
To	0 13 18	6540	Declinat.	11 15 47 S.

PROB. VI. *The time of the Year and the Sun's Declination being given, to find his Longitude; or the Declination of a Star, or of any Point of the Ecliptic, being given, to find its Longitude.*

With the given declination take the longitude out of the same Table, as in the preceding problem. Thus, take the declination in the Table nearest less than that given, and write down the corresponding longitude; marking whether the declination be increasing or decreasing; then subtract the declination found in the Table from the next greater, the remainder is the difference of declination for 1° of longitude. Subtract also the declination found in the Table from that given; and then, as the difference of declination for 1° of longitude is to 1°, so is the difference between the given declination and that found in the Table to a part, to be applied to the longitude already found; the result is the true longitude.

Exam. February 19, 1796, the sun's declination is 11° 15′ 47″ S. what is his longitude? *Ans.* 11s. 0° 37′ 39″.

1st. To declin. 11° 7′ 53″ S. the long. is 11s. 1°, the declination decreasing.

		° ′ ″
2d.	From the given declin. - - -	11 15 47
	Subtract the next less in the Table -	11 7 53
	Difference - - - - - -	0 7 54
3d.	Also from the next greater - -	11 29 5
	Subtract the next less in the Table -	11 7 53
	Diff. declin. for 1° long. - -	0 21 12

		° ′ ″	L. L.
4th.	As - - - -	0 21 12	4518
	To - - - - -	1 0 0	0
	So is - - -	0 7 54	8805
	To - - - - -	0 22 21	4287

To be subtracted.

	s.	°	′	″
From the longitude found - -	11	1	0	0
Subtract the prop. part - -	—	—	22	21
There remains the true longitude - -	11	0	37	39

PROB. VII. *To calculate the true Longitude of the Moon for any given Time by the Tables.*

Calculate the true longitude of the sun and his mean anomaly for the given time by Prob. II.

From the Tables of the moon's mean motion, take out the mean longitude, anomaly, and ascending node, for the given year; under which write down the mean motions for the month, day, hours, &c.

Add the numbers in the columns of the mean longitude and anomaly, rejecting 12 signs or any multiples thereof when they occur; but from the longitude of the node for the given year subtract the sum of all the numbers below it, borrowing 12 signs when necessary; and thus the moon's mean longitude, mean anomaly, and the mean longitude of her node, will be obtained.

For the Arguments of the Equations.

The sun's mean anomaly is the first argument. Subtract the true longitude of the sun from the mean longitude of the moon, the remainder is the mean distance of the moon from the sun, of which take the double.

To and from the double distance of the moon from the sun add and subtract the first argument, the sum and remainder are the second and third arguments.

Add and subtract the moons mean anomaly to and from the double distance of the moon from the sun for the fourth and fifth arguments.

To and from the fifth argument add and subtract the first argument, and the sixth and seventh arguments are found.

Subtract the first argument from the moon's mean anomaly, the remainder is the eighth argument.

Subtract the moon's mean anomaly from the moon's mean distance from the sun, the remainder is the ninth argument.

Subtract the true longitude of the sun from the mean longitude of the moon's node, and the remainder is the tenth argument.

With these arguments take the several equations out of the Tables, and find their sum.

With the mean anomaly of the sun take out of the Tables the equations of the node and of the moon's mean anomaly. Apply the equation to the longitude of the node, and its equated longitude will be known. Correct the mean anomaly of the moon by its own equation, and also by the sum of the ten former equations, and the correct anomaly of the moon will be had; which is the eleventh argument, with which take out of the Table the equation of the centre. Correct the mean longitude of the moon both by the sum of the ten former equations and by the equation of the centre, and the equated longitude of the moon will be known.

Subtract the sun's true longitude from the equated longitude of the moon, the remainder is the twelfth argument, with which take the variation out of the Table, and apply it to the equated longitude of the moon.

Subtract the correct longitude of the node from the longitude of the moon twelve times corrected, and from double of the remainder subtract the moon's correct anomaly, the remainder is the 13th argument; with which take the thirteenth equation from the Table and apply it to the moon's longitude last found, which will give the moon's longitude in her orbit.

From the moon's longitude in her orbit subtract the correct longitude of the node, the remainder is the fourteenth argument; with which take out of the Table the reduction of the moon's place in her orbit to the ecliptic. And, lastly, with the mean longitude of the moon's node take the equation of the equinoxes from the Table; and apply both the reduction and the equation of the equinoxes to the moon's longitude in her orbit, according to their signs, and the true ecliptic longitude of the moon will be found.

Exam. It is required to find the moon's longitude 1795, June 22, at noon, apparent time, in the meridian of Greenwich.

The equation of time is $+ 1' 28''$, therefore the mean time is 0h. 1′ 28″.

Sun's longitude 3s. 0° 55′ 29″. Mean anomaly, 11s. 21° 10′ 31″.

	☽'s m. Long.	m. Anom.	☊		Equations.	
	S ° ′ ″	S ° ′ ″	S ° ′ ″		+ ′ ″	″
1795 -	1 5 37 57	6 13 36 29	4 9 57 43			
June 22 -	3 29 31 0	3 10 14 35		1		1 42
0h. 1m. -	33	33	— 9 9 40	2		0 48
28s. -	15	15		3		0 52
			4 0 48 3	4	0 50	
	5 5 9 45	9 23 51 52	—1 24	5	20 36	
Sum of 10 eqs.	+ 19 0	— 3 18		6		0 14
			4 0 46 39	7		0 19
	5 5 28 45	9 23 48 34		8		0 36
Equat. centre	+ 5 35 20	+ 19 0		9	1 14	
				10	0 51	
☽'s equat. long.	5 11 4 5	9 24 7 34=Arg. 11th.				
Variation	+ 20 48				+ 23 31	— 4 31
					— 4 31	
	5 11 24 53					
13th -	+ 46				+ 19 0	
☽'s orbit long.	5 11 25 39					
Reduction	—6 43					
	5 11 18 56					
Equinoxes	— 15					
☽'s eclip. long.	5 11 18 41					

Arguments found.	
	S ° ′ ″
☽ - -	5 5 9 45
☉ - -	3 0 55 29
☽ from ☉ -	2 4 14 16
Doubled -	4 8 28 32
☉ Anom. ± -	11 21 10 31
2d = - -	3 29 39 3
3d = -	4 17 18 1
	4 8 28 32
☽ Anom. ± -	9 23 51 52
4th = - -	2 2 20 24
5th = - -	6 14 36 40
	± 11 21 10 31

Arguments found.	
	S ° ′ ″
6th = - -	6 5 47 11
7th = - -	6 23 26 9
	9 23 51 52
	—11 21 10 31
8th = - -	10 2 41 21
	2 4 14 16
	—9 23 51 52
9th = - -	4 10 22 24
☊ - -	4 0 48 3
☉ - -	—3 0 55 29
10th = - -	0 29 52 34
☽'s eq. long. -	5 11 4 5

Arguments found.	
	S ° ′ ″
	—3 0 55 29
12th = - -	2 10 8 36
☽'s long. equat.	5 11 4 5
☊	—4 0 46 39
	1 10 17 26
Doubled - -	2 20 34 52
☽'s Anom. -	—9 24 7 34
13th = - -	4 26 27 18
☽ in orbit -	5 11 25 39
☊ - - -	—4 0 46 39
14th = - -	1 10 39 0

PROB. VIII. *The Moon's Longitude, Anomaly, and Node, being given, to find her Latitude by the Tables.*

Subtract the corrected longitude of the moon's node from the moon's longitude in her orbit, the remainder is the first argument of latitude.

From the moon's longitude in her orbit subtract the sun's longitude; and from double of the remainder subtract the first argument of latitude, the remainder is the second argument.

Subtract the moon's mean anomaly from the first argument of latitude, and the remainder is the third argument.

From the third argument subtract the moon's mean anomaly for the fourth argument; also subtract the moon's mean anomaly from the fourth argument, and the remainder is the fifth argument.

With these arguments take out the numbers with their proper signs from the Tables, and take the sum of those which are affirmative, and also the sum of the negative; the difference of these sums, with the sign of the greater, will be the latitude of the moon; and will be north if marked with the sign +, but south if marked with the sign —.

Exam. It is required to find the moon's latitude when her longitude in her orbit is 5s. 11° 25′ 39″, mean anomaly 9s. 23° 51′ 52″, and node 4s. 0° 46′ 39″. The sun's longitude is 3s. 0° 55′ 29″.

Arguments found.		Arguments found.			Equations.	
	S ° ′ ″		S ° ′ ″		+ ° ′ ″	— ″
☽'s orbit lon.	5 11 25 39	2d =	3 10 21 20			
☊	—4 0 46 39		——	1	3 21 23	
	——	1st	1 10 39 0	2	0 8 40	
Arg. 1st =	1 10 39 0	☽'s Anom.	—9 23 51 52	3		16
	——		——	4		3
☽	5 11 25 39	3d =	3 16 47 8	5		14
☉	—3 0 55 29		—9 23 51 52		——	—
	——		——		3 30 3	33
	2 10 30 10	4th =	5 22 55 16		— 33	
	——		—9 23 51 52		——	
Doubled	4 21 0 20		——		+3 29 30	
1st	—1 10 39 0	5th	7 29 3 24		*Ans.* The moon's lat. is 3°29′30″N.	

PROB. IX. *The Moon's Longitude with the Arguments for finding it, being known, to find the Moon's Equatorial Parallax or the Horizontal Parallax of the Moon for any Place on the Equator.*

With the 5th, 11th, and 12th arguments of longitude, take as many numbers from the Tables, with their proper signs; and from the sum of the affirmative numbers subtract the sum of the negative, the remainder is the moon's equatorial parallax, or the horizontal parallax of the moon to a place on the equator.

Exam. Let the moon's longitude be 5s. 11° 18′ 41″, and the arguments as below; required her equatorial parallax. *Ans.* 56′ 5″.

	S ° ′ ″	+ ′ ″	— ″
5th Arg.	6 14 36 40	0 36	
11th	9 24 7 34	55 49	
12th	2 10 8 36		20
		——	—
		+ 56 25	
		— 20	
		——	
Moon's equatorial parallax		56 5	

PROB. X. *The Equatorial Parallax of the Moon being given and the Latitude of a place, to find the Horizontal Parallax of the Moon at that place; and to reduce the Apparent Latitude of the Place to the Centre of the Earth, supposing the Earth is an Oblate Spheroid.*

Enter the Table, with the given apparent latitude of the place on the side, and the equatorial parallax at the top, and making proportion, if necessary, find the reduction of parallax; which subtracted from the equatorial parallax, leaves the horizontal parallax required.

With the apparent latitude of the place take the reduction of latitude out of the same Table; which being subtracted from the apparent latitude, leaves the latitude reduced.

N. B. The horizontal parallax and latitude thus found are what should be employed in computing the moon's parallaxes in longitude, latitude, and declination; which will now be performed by the common rules, founded on the supposition of the earth's being a sphere.

Exam. Let the equatorial parallax be 56′ 5″, and the apparent latitude of the place 51° 28′ 40″; the moon's horizontal parallax and the reduced latitude of the place are required.

	′ ″		° ′ ″
Equatorial parallax	56 5	Latitude	51 28 40
Reduction	— 9	Reduction	— 14 42
	——		——
Horizontal parallax	55 56	Reduced lat.	51 13 58

PROB. XI. *The Equatorial Parallax being given, to find the Moon's Horizontal Diameter.*

Enter the Table with the moon's equatorial parallax, and take out her horizontal diameter.

Exam. The moon's equatorial parallax being 56′ 5″, the diameter is 30′ 34″.

PROB. XII. *To find the Moon's Hourly Motion in Longitude; her Longitude in her Orbit, with the Arguments for finding it, being known.*

With the 5th, 11th, and 12th arguments of longitude, take out the numbers from the Tables; the sum of these, regard being had to their signs, is the moon's hourly motion in longitude.

Exam. Let the moon's longitude be 5s. 11° 25′ 39″, and the arguments as below; required her hourly motion in longitude.

	S	°	′	″	+ ′	″	— ″
5th Arg.	6	14	36	40	0	4	
11th	9	24	7	34	31	20	
12th	2	10	8	36			31
					+ 32	1	
					—	31	
Answer.					31	30	

PROB. XIII. *To find the Moon's Hourly Motion in Latitude, the Arguments for finding her Latitude being known.*

With the first and second arguments of latitude, take two numbers out of the Tables; their sum, if they have the same sign, or their difference, if they have contrary signs, is the moon's hourly motion in latitude; which tends to the north if it has the sign +, but to the south if it has the sign —.

Exam. It is required to find the moon's hourly motion in latitude, the arguments of latitude being,

	S.	°	′	″	′	″
1st Arg.	1	10	39	0	+ 2	15
2d	3	10	21	21	—	1
Answer					+2	14 tending north.

PROB. XIV. *To find the Time of the Mean Syzygies in any given Year and Month.*

From the Tables of the mean motion of the moon from the sun take out the motions for the given year and month; add them, and subtract the sum from 12 signs; then, if the time of mean conjunction or new moon be sought, the remainder, or the nearest less than it, being found in the Table of days, will give the day of mean new moon; and after subtracting the number found in the Table, the remainder or the next less is to be found in the Table of hours, which will give the hour of the day when it is mean new moon; and after another subtraction, the remainder is to be fouud in the Table of minutes for the minute of mean new moon, and the next remainder will give the seconds. But if the time of mean opposition or full moon be sought, add six signs to the first remainder, and the sum being found in the Table of days, will give the day of mean full moon; the hours and minutes are to be found as before.

Exam. 1. Required the time of mean new moon in January, 1797. *Ans.* Jan. 27d. 7h. 40m. 21s.

	S	°	′	″	‴
1797 - -	—0	26	57	10	
	12	0	0	0	
Remains -	11	3	2	50	
27 days - -	—10	29	9	0	
		3	53	50	
7 hours - -		—3	33	20	
			20	30	
40 minutes - -			—20	19	5
21 seconds - -				10	55

In this operation the Tables give the moon's mean distance from the sun on the 1st day of January 0s. 26° 57′ 10″, which is subtracted from 12 signs to find how much the moon wants of a conjunction with the sun; the remainder shows that she wants 11s. 3° 2′ 50″. In the Table of days it is found that she moves over 10s. 29° 9′ in 27 days; therefore the mean new moon happens on the 27th day of January.

Subtract 10s. 29° 9′ from 11s. 3° 2′ 50″, there remains 3° 53′ 50″, which in the Table of hours gives 7 hours; and subtracting 3° 33′ 20″ there remains 20′ 30″ which gives 40′; and by subtracting again there remains 10″ 55‴, and this gives 21 seconds.

2. It is required to find the time of mean full moon in January 1797. *Ans.* Jan. 12d. 13h. 18m. 18s.

	S	°	′	″	
1797	—0	26	57	10	
	12	0	0	0	
	11	3	2	50	
Add	6	0	0	0	
	5	3	2	50	
12 days	—4	26	17	20	
		6	45	30	
13 hours		—6	36	12	
			9	18	
18 minutes			—9	8	35‴
18 seconds				9	25

By this problem the times of mean new and full moon may be calculated for every month in the year, or any longer time, with very little trouble; for having found the time of mean conjunction and opposition in the month of January, add to these times a mean lunation, viz. 29d. 12h. 44m, 3s. continually, rejecting the days in the month wherein the mean new or full moon is required, and the times will be known.

Exam. Let it be required to find the mean new and full moons in every month of the year 1797.

Mean full Moon.					Mean new moon.				
	D.	h.	m.	s.		D.	h.	m.	s.
January	12	13	18	1	January	27	7	40	21
One lunation +	29	12	44	3	One lunation +	29	12	44	3
February	11	2	2	21	February	25	20	24	24
March	12	14	46	24	March	27	9	8	27
April	11	3	30	27	April	25	21	52	30
May	10	16	14	30	May	25	10	36	33
June	9	4	58	33	June	23	23	20	36
July	8	17	42	36	July	23	12	4	39
August	7	6	26	39	August	22	0	48	42
September	5	19	10	42	September	20	13	32	45
October	5	7	54	45	October	20	2	16	48
November	3	20	38	48	November	18	15	0	51
December	3	9	22	51	December	18	3	44	54

Note. The mean and true syzygies never happen at the same time, except when the moon is in or very near her apogee or perigee; for in these points many of her inequalities either vanish or are very small; when the moon is in any other point of her orbit there is some interval of time between the mean and true conjunctions or oppositions. The greatest is about 14 hours.

PROB. XV. *The Time of Mean Conjunction or New Moon being given, to find the True Time.*

Calculate the longitudes of both sun and moon for the time of mean conjunction (Prob. II.—VII.); and if they are equal to one another, the mean and true conjunctions happen at the same time; but if they differ, subtract the least from the greatest. Find the hourly motions of both sun and moon at the time (Prob. IV.—XII.), and subtract the sun's hourly motion from the moon's, the remainder is the hourly motion of the moon from the sun; and then,

As the moon's hourly motion from the sun
Is to one hour or 60 minutes;
So is the difference between the sun's and moon's long.
To the time between the mean and true conjunction.

If the moon's longitude be less than the sun's, the interval is to be added to the time of mean conjunction; but if it be greater, subtract the interval; the sum or remainder is the time of true conjunction.

When it is required to find the time of true conjunction very accurately, the sun's and moon's longitude must be calculated again for the time found; and their difference, if there be any, turned into time, and applied to the time last found, will give the true time of conjunction.

If the difference be great, it will be necessary to renew the operation again.

Exam. It is required to find the time of true conjunction in January 1797; the time of mean conjunction or new moon being January 27th, 7h. 40m. 21s. in the meridian of Greenwich. *Ans.* Jan. 27d. 13h. 40m.

The sun's longitude then is 10s. 8° 28′ 38″, and his mean anomaly 6s. 28° 3′ 51″.

For the Moon's Longitude.

	m. Long. S ° ′ ″	Anomaly. S ° ′ ″	☊ S ° ′ ″
1797	10 7 34 43	0 24 6 53	3 1 15 6
Jan. 27th	11 25 45 46	11 22 45 17	
7h.	3 50 35	3 48 38	— 1 25 47
40 m.	21 58	21 47	56
21s.	11	11	5
	10 7 33 13	0 21 2 46	2 29 48 18
10 equations	+ 25 41	— 10 25	4 25
			2 29 43 53
	10 7 58 54	0 20 52 21	
Equat. of centre	—2 9 10	+ 25 41	
	10 5 49 44	0 21 18 2 = 11th arg.	
Variation	—3 14		
	10 5 46 30		
13th	+ 1 5		
	10 5 47 35		
☉'s long.	10 8 28 38		
Short of conj.	2 41 3		

	Arguments. S ° ′ ″	Equations. + ′ ″	—
1	6 28 3 51		5′ 21″
2	6 26 13 1	0 25	
3	5 0 5 19		0 38
4	0 19 11 56	0 19	
5	11 7 6 24	30 53	
6	6 5 10 15		0 11
7	4 8 2 33	0 36	
8	5 22 58 55	0 5	
9	11 8 1 49	0 31	
10	4 21 19 40		0 58
12	11 27 21 6	+ 32 49	7 8
13	1 20 47 12	— 7 8	
		+ 25 41	

	′ ″
Moon's hourly motion - - - -	29 47
Sun's ditto - - - - -	2 33
Hourly motion ☽ from ☉ - -	27 14

As 27′ 14″
To 1h.
So 2° 41′ 3″
To 5h. 54m. 49s.

	D.	h.	m.	s.
Mean conjunction January	27	7	40	21
Interval add - - -	+	5	54	49
Ans. New Moon January	27	13	35	10

The time of true opposition or full moon is found in the same manner as the time of conjunction. The operation is to be continued until the difference of the sun and moon's longitude be 6 signs.

The rules here given for calculating the times of new and full moon are always good; but the tediousness of the operation has given occasion to the invention of other Tables, whereby the problem may be solved with less trouble. The best of these were published among Dr. Halley's Astronomical Tables in the year 1749, but not as his invention. The ingenious Mr. James Ferguson having adapted them to the Gregorian style, and made some alterations in the arrangement, printed them in his Astronomy about the year 1760. Such of them as were judged proper for this work, are to be found in the Tables of mean new moons, &c. extended to the year 1821.

These Tables contain the times of mean conjunction or new moon in March for every year, which is made the first month with the mean anomalies of the sun and moon, and also the sun's mean distance from the moon's node at the time, with equations for reducing the mean to the true time of conjunction or opposition, whereby these times may be found for any month of a given year within the limits of the Table.

PROB. XVI. *To calculate the Time of New or Full Moon for any Month of a given Year by the Tables.*

1st. For the Month of March.

Write down the time of mean new moon in March for the given year, with the mean anomalies of the sun and moon, and also the sun's mean distance from the moon's ascending node, out of the Table.

When the time of Full Moon in March is required.

If the new moon happens before the 15th of the month, add half a lunation, with the anomalies, &c. to the former numbers for new moon, the sum is the time of full moon; but if it happens after the 15th, subtract half a lunation with the anomalies from the numbers for new moon, and the time of mean full moon in March will be known.

2d. For any Month after March.

When the time of mean new or full moon is required in any month after March, take out the numbers for March as before, and under them write down as many lunations with their anomalies as the given month is after March; and by the sum of these, the time of mean new or full moon may be known, together with the mean anomalies and the sun's distance from the moon's node; which are the arguments for finding the several equations, to reduce the time of the mean syzygies to the true.

With the sum of the days enter the Table of days, under the given month; and opposite to that number in the left hand column is the day of the mean syzygies; but if the sum be less than any of those under the given month, add a lunation with the anomalies to the former numbers, and then enter the Table with the sum under the given month, and in the left hand column is the day of the month required.

The time of mean syzygy being known, to find the true by the Tables.

1. With the sun's mean anomaly enter the Table, and take out the first equation (making proportion for the odd minutes, &c.), and apply it to the time of the mean syzygy according to its sign.

2. With the sun's mean anomaly take the equation of the moon's mean anomaly out of the Table, and apply it according to its sign, and the moon's equated anomaly will be known.

3. With the moon's equated anomaly take the second equation out of the Table, which being applied to the former time, according to its sign, the result will be the time of the syzygy very nearly.

4. Subtract the moon's equated anomaly from the sun's mean anomaly, and with the remainder take the third equation out of the Table, and apply it to the former equated time according to its sign.

Lastly, with the sun's mean distance from the moon's node take the fourth equation out of the Table, and apply it to the last found equated time, according to its sign, and the result is the time of the true syzygy.

EXAM. 1. Required the time of full moon in January, 1797.

		☉'s anom.	☽'s anom.	☉'s dist. from ☊.
	D. h. m. s.	S. ° ′ ″	S. ° ′ ″	S. ° ′ ″
1796, March	8 11 39 44	8 8 3 47	3 7 2 14	8 0 26 47
½ lunation	+ 14 18 22 2	0 14 33 10	6 12 54 30	0 15 20 7
Full in March	23 6 1 46	8 22 36 57	9 19 56 44	8 15 46 54
Add 10 lunations	295 7 20 30	9 21 3 14	8 18 10 4	10 6 42 20
1797, January	12 13 22 16	6 13 40 11	6 8 6 48	6 22 29 14
1st equation	+ 1 0 30	—6 8 29 38	+ 22 50	arg. 4th equat.
	12 14 22 46	0 5 10 33	6 8 29 38	
2d equation	— 1 20 45	arg. 3d. equat.	arg. 2d. equat.	
	12 13 2 1			
3d equation	— 26			
4th equation	+ 1 6			
Ans. January	12 13 2 41			

2. It is required to find the time of conjunction, or new moon, in the month of August, 1820.

		☉'s Anomaly.	☽'s Anomaly.	☉'s dist. from ☽'s ☊.
	D. h. m. s.	S. ° ′ ″	S. ° ′ ″	S. ° ′ ″
1820, March	14 1 42 45	8 12 25 31	6 24 37 23	11 19 35 34
5th luna.	147 15 40 15	4 25 31 37	4 9 5 2	5 3 21 10
August	8 17 23 0	1 7 57 8	11 3 42 25	4 22 56 44
1st equat.	— 2 31 47	—11 2 44 51	— 57 34	arg. 4th. equat.
	8 14 51 13	2 5 12 17	11 2 44 51	
2d equat.	— 4 47 7	arg. 3d. equat.	arg. 2d. equat.	
	8 10 4 6			
3d equat.	— 4 27			
4th	— 1 29			
Ans. Aug.	8 9 58 10			

Note. 1. These examples are wrought for the meridian of Greenwich; but the answers may be reduced to the time under any other meridian by adding or subtracting the difference of longitude in time to or from them, according as the place is situated to the east or west of Greenwich.

2. These Tables give the times of new and full moon with little trouble, and sufficiently true for common use; being rarely above 1 or 2 minutes wide of the truth: but when it is required to find the moment of conjunction or opposition accurately, calculate the longitudes of the sun and moon for the time found by the Tables; and if they are the same or differ 6 signs, according as the new or full moon is required, the time is truly found; but if not, take their difference. Find the hourly motion of the moon from the sun; and then,

As the hourly motion of the moon from the sun
Is to 60 minutes;
So is the difference between the sun and moon's long.
To a number of minutes, &c.

which, applied to the time formerly found, will give the true time.

3. The precise moment of conjunction or opposition is seldom necessary except in calculating eclipses.

4. In the preceding examples the time of orbit conjunction or opposition is taken for the answer; that is, the time when the moon's longitude in her orbit is the same with the sun's longitude in the ecliptic, is taken for the time of conjunction; and the time when the moon's orbit longitude differs 6 signs from the sun's longitude, is taken for the time of opposition.

The answers are in mean time; and if the equation of time be applied to them with a contrary sign to that in the Table, the apparent time will be known.

To calculate more accurately the time of new moon in August, 1820. The answer by our Tables is August 8d. 9h. 58m. 10s. The sun's longitude at this time is 4s. 16° 9′ 44″, and his mean anomaly 1s. 7° 23′ 13″.

The calculation of the moon's longitude for the same time follows.

	☽'s m. Long. S. ° ′ ″	☽'s anom. S. ° ′ ″	☽'s ☊. S. ° ′ ″
1820	3 19 18 47	11 0 1 6	0 6 25 42
August 8th	0 18 48 26	11 24 17 51	
9h.	4 56 28	4 53 58	— 11 39 0
58m.	31 50	31 34	— 1 12
10s.	5	5	— 7
	4 13 35 36	10 29 44 34	11 24 45 23
10 equations	— 25 26	+ 12 54	+ 5 28
	4 13 10 10	10 29 57 28	11 24 50 51
Equat. centre	+ 3 1 4	— 25 26	
	4 16 11 14	10 29 32 2	
Variation	+ 1	arg. 11th.	
	4 16 11 15		
13th equation	— 1 1		
☽'s orb. long.	4 16 10 14		
Sun's long.	4 16 9 44		
Difference	30		

	Arguments. S. ° ′ ″	Equations. + ′ ″	Equations. — ′ ″
1	1 7 23 13	6 37	
2	1 2 14 57		0 29
3	10 17 28 31	0 51	
4	10 24 36 18		0 36
5	0 25 7 10		33 43
6	2 2 30 23	1 49	
7	11 17 43 57		0 10
8	9 22 21 21		0 39
9	0 2 40 29		0 3
10	7 8 35 39	0 57	
—			— 35 40
12	0 0 1 30	+ 10 14	+ 10 14
13	10 13 8 46		
			— 25 26

	′ ″
Moon's hourly motion	28 54
Sun's ditto.	2 24
Hourly motion ☽ from ⊙	26 30

	L. L.
As 26′ 30″	3549
To 60m.	0
So is 30″	20792
To 1m. 7s.	17243

	D. h. m. s.
Former time August	8 9 58 10
Interval subtract	— 1 7
True time of new moon	8 9 57 3, which is nearly the same with the former.

PROB. XVII. *To find out the number of Eclipses there are in any given Year, and in what Months and Days they happen.*

From the Table of the moon's mean motion take out the mean longitude of the moon's nodes for the given year; and then by the Table of the sun's longitude for every day of the year, find when the sun's longitude will be nearly the same with that of the nodes, for at these times the eclipses must happen. The months when the sun is in or near the places of the nodes may be called the *node-months.* In this inquiry it may be proper to remember that the moon's nodes move backward about 1° 38′ every month.

To find the Days when Eclipses happen.

Calculate the times of the mean syzygies in the node-months out of the Tables, and also the sun's distance from the moon's node; and if the sun's distance from the node at the time of new moon be less than 18 degrees, an eclipse of the sun may be expected; and if the sun's distance from the node at the time of full moon be less than 12 degrees, there may be an eclipse of the moon. We say there may be, because we speak only of mean motions here.

EXAM. It is required to find the number of eclipses in the year 1796; and on what days they happen.

The mean longitude of the moon's ascending node, on the 1st of January 1796, is 3s. 20° 35′, and of the descending node 9s. 20° 35′. The sun enters the sign Cancer about the 20th of June, and the sign Capricorn about the 22d of December; therefore the node-months are January, July, and December.

1. Calculate the time of mean new moon in January.

New moon.					☉'s dist. from ☽'s ☊				
	D.	h.	m.	s.	S.	°	′	″	
1795, March	20	2	51	8	7	22	24	0	
10 lunations	+ 295	7	20	30	10	6	42	20	
New ☽ 1796 Jan.	9	10	11	38	5	29	6	20	sun eclipsed.

The sun's mean distance from the moon's ascending node being 5s. 29° 6′ 20″, his distance from the descending node is only 53′ 40″; and therefore the sun will be eclipsed to some place of the earth.

2. For the time of mean new moon in July.

New moon.					☉'s dist. from ☽'s ☊				
	D.	h.	m.	s	S.	°	′	″	
1796, March	8	11	39	44	8	0	26	47	
4 lunations	+ 118	2	56	12	4	2	40	56	
New ☽ July	4	14	35	56	0	3	7	43	sun eclipsed.

3. For the time of mean full moon in December.

New Moon					Sun's dist. from ☊				
	D.	h	m.	s.	S.	°	′	″	
1796, March	8	11	39	44	8	0	26	47	
½ lunation	+ 14	18	22	2	+	15	20	7	
Full ☽ in March	23	6	1	46	8	15	46	54	
9 lunations	+ 265	18	36	27	+9	6	2	6	
Full ☽ Dec.	14	0	38	13	5	21	49	0	moon eclipsed.

The sun's distance from the moon's ascending node being 5s. 21° 49′, the moon's distance from her descending node is only 8° 11′, and therefore she is eclipsed.

4. For the time of mean new moon in December.

Add ½ lunation to the numbers for full moon.

New moon.					Sun's dist. from ☊.				
	D.	h.	m.	s.	S.	°	′	″	
Full ☽ Dec.	14	0	38	13	5	21	49	0	
½ lunation	+ 14	18	22	2	+	15	20	7	
New ☽ Dec.	28	19	0	15	6	7	9	7	sun eclipsed.

We have found that there are four eclipses in the year 1796, viz. three of the sun and one of the moon. The times of the mean conjunctions and oppositions are:

		D.	h.	m.	s.
Sun is eclipsed - -	Jan.	8	10	11	38
	July	4	14	35	56
	Dec.	28	19	0	15
And the moon is eclipsed - - —		14	0	38	13

PROB. XVIII. *To calculate an Eclipse of the Moon.*

1. Calculate the true time of opposition by Prob. XVI. and compute the longitude of the sun, and the orbit longitude of the moon, for that time; if these differ 6 signs, the time of opposition is truly found; but if their difference be less or greater than six signs, mark the defect or excess, and find the moon's hourly motion from the sun; then, as the hourly motion of the moon from the sun is to one hour, so is the defect or excess to the interval; which being applied to the time formerly found, gives the true

time of opposition. This operation must be repeated until the longitude of the sun and the orbit longitude of the moon differ precisely 6 signs.

2. Calculate the moon's latitude, horizontal parallax, and hourly motion, by the Tables; take also the sun and moon's semidiameters out of the Tables.

3. Add the sun's horizontal parallax (which is 9 seconds) to the moon's for the time, and from the sum subtract the sun's semidiameter; the remainder is the semidiameter of the earth's shadow; to which add the moon's semidiameter, and the sum is the semidiameter of the moon and earth's shadow; from which subtract the moon's latitude, and the remainder is called *the parts deficient*. If these be less than the moon's diameter, the eclipse will be partial; if they be equal to the moon's diameter, the eclipse will be total without continuance; and if they be greater than the moon's diameter, the eclipse will be total with continuance.

Exam. 1. It is required to calculate the eclipse of the moon, which will happen in December, 1797.

1. Calculate the time of opposition.

		Sun's Anom.	Moon's Anom.	☉'s dist. from ☊.
	D. h. m. s.	S. ° ′ ″	S. ° ′ ″	S. ° ′ ″
1797, new ☽ March	27 9 12 24	8 26 25 59	2 12 39 19	9 9 9 48
½ lunation	— 14 18 22 2	— 14 33 10	— 6 12 54 30	— 15 20 7
Full moon March	12 14 50 22	8 11 52 49	7 29 44 49	8 23 49 41
9 lunations	+265 18 36 27	8 21 56 54	7 22 21 4	9 6 2 6
Full moon December	3 9 26 49	5 3 49 43	3 22 5 53	5 29 51 47
1 equation	— 1 52 47	— 3 21 23 13	— 42 40	arg. 4th equat.
	3 7 34 2	1 12 26 30	3 21 23 13	Moon eclipsed.
2 equation	+ 8 53 8	arg 3d. equat.	arg. 2d equat.	
	3 16 27 10			
3 equation	— 3 11			
4 ditto	— 1			
Full moon December	3 16 23 58			

2. Calculate the sun and moon's longitude for the time of full moon.

For the Sun's Longitude.

	Sun's Long.	Sun's Anom.
	S. ° ′ ″	S. ° ′ ″
1797	9 10 37 33	6 1 8 17
December 3d	11 2 9 47	11 2 8 46
16h.	39 25	39 25
23m.	57	57
58s.	2	2
	8 13 27 44	5 3 57 27
Equation	— 51 45	
Sun's longitude	8 12 35 59	

For the Moon's Longitude.

	Moon's m. Long.	Moon's Anom.	Moon's ☊	Arguments		Equations. +	Equations. —
	S ° ′ ″	S. ° ′ ″	S. ° ′ ″		S. ° ′ ″	° ′ ″	′ ″
1797	10 7 34 43	0 24 6 53	3 1 15 6				
Dec. 3d.	4 0 26 45	2 22 54 4		1	5 3 57 27	0 5 0	
16h.	8 47 3	8 42 36	— 17 50 45	2	5 12 48 49		0 17
23m.	12 37	12 31	— 2 7	3	7 4 53 55	0 0 43	
58s.	32	32	— 3	4	4 4 47 58	0 0 47	
				5	8 12 54 46	1 17 15	
	2 17 1 40	3 25 56 36	2 13 22 11	6	1 16 52 13	0 1 30	
10 equations	+ 1 25 1	+ 9 44	+ 4 3	7	3 8 57 19	0 0 46	
				8	10 21 59 9		0 26
	2 18 26 41	3 26 6 20	2 13 26 14	9	2 8 29 5		0 19
Equat. centre.	— 5 45 57	+ 1 25 1		10	6 0 46 12	0 0 2	
	2 12 40 44	3 27 31 21		12	6 0 4 45	+ 1 26 3	— 1 2
Variation	+ 6	arg. 11th.		13	8 0 57 51	— 0 1 2	
				14	11 29 9 29		
	2 12 40 50					+ 1 25 1	
13th equation	— 1 13						
Moon's orb. lon.	2 12 39 37						
Sun's lon.	8 12 35 59						
Past opposition	3 38						

	′ ″
Moon's hourly motion - - -	35 18
Sun's ditto - - -	2 32
Hourly motion ☽ from ☉ - -	32 46
	L. L.
As hourly motion ☽ from ☉ 32′ 46″	2627
To one hour or 60m. - -	0
So is 3′ 38″ - - - - -	12178
To 6m. 39s. - - - - -	9551
	D. h. ′ ″
Former time December -	3 16 23 58
Interval subtract - - -	— 6 39
Time of full moon - -	3 16 17 19
	S ° ′ ″
Sun's longitude	8 12 35 43
Moon's ditto - -	2 12 35 43
Reduction - - -	+ 12

For the Moon's Latitude.

Arguments.	S. ° ′ ″	′ ″
1	11 29 13 23	— 4 11
2	0 0 53 23	+ 8
3	6 25 15 56	+ 6
4	2 29 19 20	— 25
5	11 3 22 44	— 7
Moon's lat. S.		— 4 29

For the Moon's Horizontal Parallax.

Arguments	S ° ′ ″	′ ″
1	8 12 54 46	+ 0 11
2	3 27 31 21	+ 58 32
3	6 0 4 45	+ 0 27
Equatorial Parallax		59 10
Reduction -		— 9
Horizontal Parallax		59 1

For the time of Reduction.

		L. L.
As moon's hourly motion from ⊙	32′ 46″	2627
To 1h. or 60′.		0
So is reduction 12″		24771
To time of reduction + 22″		22144

	′	″
Sun's semidiameter	16	17
Moon's semidiameter	16	8
Angle of the moon's path with the ecliptic	5°40′.	

Note. In eclipses, the disk of the sun or moon is supposed to be divided into twelve equal parts called *digits*, and each of these again divided into 60 equal parts or minutes. The magnitude of an eclipse is estimated by the number of digits.

Calculation of the Eclipse.

	′	″
Moon's horizontal parallax - -	59	1
Sun's ditto - - - - - - -		+ 9
Sum - - - - - - -	59	10
Sun's semidiameter subtract - -	—16	17
Semidiameter of the earth's shadow -	42	53
Moon's semidiameter add - -	+ 16	8
Semid. of the moon and earth's shadow	59	1
Moon's latitude subtract - - -	—4	29
Remains the parts deficient - -	54	32

The eclipse will be total.

For the Digits eclipsed.

	L. L.
As moon's semidiameter 16′ 8″ - -	5704
To 6 digits - - - - - -	10000
So parts deficient 54′ 32″ - - - - -	415
To digits eclipsed 20° 16′ - - -	4711

	D.	h.	′	″
Mean time of orbit opposition Dec.	3	16	17	19
Time of reduction add and subtract				± 22
Middle of the eclipse - - -	3	16	17	41
Ecliptic opposition - - - - -	3	16	16	57
To each of these add the equation of time			+ 9	22
Appar. time of the middle of the eclipse	3	16	27	3
——— of ecliptic opposition	3	16	26	19

To find the Scruples of Incidence, or that Part of the Moon's Path which she passes over between the beginning and middle of the Eclipse.

Reduce the sum of the semidiameters of the moon and earth's shadow to seconds, and also the moon's latitude; find their sum and difference; and to the logarithm of their sum add the logarithm of their difference, and divide the sum by 2, the quotient is the logarithm of the scruples of incidence in seconds.

In this example the semidiameter of the moon and earth's shadow is

	′ ″ ″	Logarithms.
	59 1 = 3541	
Moon's latitude	4 29 = 269	
Sum	3810	3.5809450
Difference	3272	3.5148133
		2)7.0957583
Scruples of incidence	3530	3.5478791
or 58′ 50″		

For the time of half duration.

	′ ″		L. L
As hourly mo. ☽ from ⊙ =	32 46		2627
To 1 hour or 60 minutes			0
So scruples of incidence	58 50	(29′ 25″)	3096
To time of half duration 1h. 47′ 42″.		h. ′ ″ (0 53 51)	469
		2	
		1 47 42	

Because the terms of this proportion are such, that the fourth term comes out too great for the Table of logistical logarithms, take half of the third term; and then double the fourth term for the answer.

	D.	h.	m.	s.
Apparent time of the middle of the eclipse	3	16	27	3
Time of half duration, subtract and add	±	1	47	42
Beginning - - - - - - -	3	14	39	21
End - - - - - - - -	3	18	14	45

In total eclipses of the moon, she continues some time in total darkness; to calculate this time, it is necessary to find the half length of that part of the moon's path which lies wholly within the earth's shadow, which some call the *scruples of total darkness.* The operation is performed thus:

Subtract the moon's semidiameter from that of the earth's shadow; reduce the remainder, and also the moon's latitude, to seconds; then find their sum and difference, and take the half sum of the logarithms of the sum and difference, and the number answering is the scruples of total darkness.

In this example,

	′ ″
The semidiameter of the earth's shadow is	42 53
The moon's diameter - -	— 16 8
	26 45

	′ ″ ″	Logarithms.
The difference is	26 45 = 1605	
Moon's latitude	4 29 = 269	
Sum - - - - - -	1874	3.2727696
Difference - - - - - -	1336	3.1258065
		2)6.3985761
Scruples - - - - -	1584	3.1992880
Or - - - - - 26′ 24″.		

For the time of half duration of total darkness.

	′ ″	L. L.
As hourly motion of moon from sun =	32 46	2627
To one hour or 60 - - -		0
So are scruples - -	26 24	3565
To time of half duration 48′ 21″ - - -		938

	D.	h.	m.	s.
Apparent time of the middle of the eclipse	3	16	27	3
Half duration of total darkness subtract and add		∓	48	21
Beginning of total darkness - -	3	15	38	42
End of ditto - - - -	3	17	15	23

From the calculation it appears, that the moon will be eclipsed at London, 1797, December 3d.

	Appar. time. h.	′
Beginning of the eclipse - - -	14	39
Beginning of total darkness - - -	15	38
Ecliptic opposition - - - -	16	26
Middle of the eclipse - - - -	16	27
End of total darkness - - -	17	15
End of the eclipse - - - -	18	14

Digits eclipsed 20° 16′ from S. side of earth's shadow.
Duration of the eclipse 3h. 35′.
Duration of total darkness 1h. 36′.

Exam. 2. Calculation of an eclipse of the moon which will happen 1802, Sept. 11th.

Mean time of orbit opposition, Sept. 11d. 10h. 40′. At that time sun's longitude 5s. 18° 25′ 16″; moon's longitude 11s. 18° 25′ 26″, latitude north 37′ 6″ ascending. Reduction = 1′ 40″, time of reduction 2′ 48″.

	′	″
Moon's hourly motion - - -	38	4
Sun's ditto - - -	— 2	26
Moon's hourly motion from sun -	35	38

	°	′	″
Moon's semidiameter - - -	0	16	43
Sun's ditto - - - -	9	15	57
Moon's horizontal parallax - -	0	66	17
Angle of the moon's path with the ecliptic is	5	38	0

Calculation of the Eclipse.

	′	″
Moon's horizontal parallax - - -	61	17
Sun's ditto - - - -	+	9
Sum - - - - - - -	61	26
Sun's semidiameter subtract - -	— 15	57
Semidiameter of the earth's shadow -	45	29
Moon's semidiameter, add - -	+ 16	43
Semidiameter of moon and earth's shadow -	62	12
Moon's latitude, subtract - -	—37	6
Parts deficient - - - -	25	6

For the digits eclipsed.

	°	′	″	L. L.
As moon's semidiameter - -	0	16	43	5550
To 6 digits - - - - -			-	10000
So parts deficient - - -	0	25	0	3785
To digits - - -	9	0	0	8235

	D. h. m. s.
Mean time of orbit opposition - -	11 10 40 0
Time of reduction, subtract and add -	∓ 2 48
Middle of the eclipse - - -	11 10 37 12
Ecliptic opposition - - -	11 10 42 48
Equation of time add to each - -	+ 3 31
Apparent middle of the eclipse - -	11 10 40 43
——— ecliptic opposition - -	11 10 46 19

For the scruples of incidence.

	′ ″	″	Logarithms.
Semid. of the ☽ and ⊕'s shadow	62 12 =	3732	
Moon's latitude - -	37 6 =	2226	
Sum - - - - -		5958	3.7751005
Difference - - - -		1506	3.1778250
			2)6.9529255
Scruples - - - - -		2905	3.4764627
Or - - -	49′ 45″		

For the time of half duration.

	′ ″	L. L.
As moon's hourly motion from sun	35 38	2263
To 1 hour or 60m. - - -		0
So $\frac{1}{5}$ scruples of incid. - -	9 59	7789
To $\frac{1}{5}$ of half duration - -	16 48	5526
	5	
Half duration - -	1h. 24m. 0s.	

	D. h. m. s.
Apparent middle of the eclipse - -	11 10 40 43
Time of half duration, subt. and add -	∓ 1 24 0
Beginning - - - -	11 9 16 43
End - - - -	11 12 4 43

1802, September 11th, the moon is eclipsed.

	Appar. time. h. m.
Beginning - - - -	9 16
Middle - - - - -	10 40
Ecliptic opposition - - -	10 46
End - - - - -	12 4

Duration 2 hours 48 minutes. Digits eclipsed 9°.

PROB. XIX. *To describe a Figure representing a lunar Eclipse.* (Plate XIV. Fig. 4.)

Exam. 1. It is required to project the eclipse, calculated in the 1st example of the last problem.

1. Draw the straight line AB for part of the ecliptic, in which take any point C, and draw CD at right angles to AB, for the axis of the ecliptic; which is drawn downward, because the moon's latitude at the beginning and middle of the ecliptic is south.

2. Take the semidiameter of the moon and earth's shadow, 59′ 1″, from any scale, and with it, as a radius, describe the circle ADB; the moon's centre is in one point of the circumference of this circle at the beginning; and in the opposite point of it, at the end of a total eclipse.

3. Take the semidiameter of the earth's shadow, 42′ 53″, from the same scale, and with it, from the centre C, describe the circle KLM, to represent the earth's shadow.

4. Make CD the radius of a line of chords on the sector, and from it take the angle of the moon's path with the ecliptic 5° 40′, and set it from D to E toward the left hand, (because the moon's latitude is south ascending northward) and draw the line CE for the axis of the moon's orbit.

5. Take the moon's latitude 4′ 29″ from the scale, which set from C to F on the line CE; and through F draw the straight line PN at right angles to CE; then the points N, F, and P are the places of the moon's centre at the beginning, middle, and end of the eclipse. Take the moon's semidiameter from the scale, and with it describe circles from the centres N, F, and P. These shall represent the moon at these points.

To mark the Hours on the Moon's Path.

The middle of the eclipse is December 3d. 16h. 27m. 3s. which, in civil reckoning, is 4h. 27m. in the morning of December 4th, marked on the moon's path by the point F. Now, for the place of four hours, say,

As one hour
Is to the moon's hourly motion from the sun 32′ 46″;
So is 27m. 3s.
To 14′ 46″, the distance of 4h. from F.

Take 14′ 46″ from the scale, and set it from F, toward the right hand for the place of 4 hours; then take the moon's hourly motion from the sun 32′ 46″ and set it on the moon's path from 4 to 3, and to 5, and also from 5 to 6, and the hours are marked. Divide each of the hour distances into six equal parts, and the place of the moon's centre will be known at every 10 minutes.

Exam. 2. Required the projection of the eclipse calculated in the 2d example of the last problem. (Plate XIV. Fig. 5.)

This is done in the same manner as the former; only, because the moon's latitude is north, the semicircles ADB and KLM must be above the straight line AB; and because the moon's latitude is ascending northward, the angle of her path with the ecliptic 5° 38′ must be set from D to E toward the right hand. Draw CE for the axis of the moon's orbit, and thereon set the moon's latitude 37′ 6″ from C to F; and through F draw the straight line NP at right angles to CE; and then the points N, F, and P, are the places of the moon's centre at the beginning, middle, and end of the eclipse, as in the former example.

Take the moon's semidiameter 16′ 43″ from the scale, and with it, for a radius, describe circles on the centres N, F, and and P, which will represent the moon's disk when she is at these points; and then place the hours on the moon's path, as directed in the former example.

Previous to the Calculation and Projection of Solar Eclipses, it will be proper to premise some things for explaining the terms used in the graphical delineation of them.

Although the earth is nearly a globe, yet when it is viewed from a very distant point, such as the centre of the sun, it will appear as a flat, round surface, like the moon, to us; and this appearance is called *the earth's disk.*

By the continual rotation of the earth round its axis, every point on its surface describes a circle parallel to the equator; and the circle described by any point or place is called *the path of the vertex of that place.*

The representations of the paths of places on the earth's disk are not always of the same form, but differ with the sun's longitude; for when the sun is in either of the equinoctial points, the paths of all places are represented by straight lines; and when the sun is in any other point of the ecliptic, they are represented by ellipses, more or less eccentric according as the sun is nearer to, or farther from, the equinoctial circle.

In describing these paths on the earth's disk, respect must be had to the position of the axes of the earth and ecliptic at the time.

When the sun is in any of the ascending signs ♑, ♒, ♓, ♈, ♉, ♊, the northern half of the earth's axis lies to the right hand of the axis of the ecliptic, as seen from the sun; but when he is in any of the descending signs ♋, ♌, ♍, ♎, ♏, ♐, it lies to the left hand. When the sun is in either of the solstitial points, the two axes coincide; and when he is in either of the equinoctial points, they form the greatest angle.

When the sun is in any of the northern signs, Aries, Taurus, &c. the north pole of the earth is in the upper or enlightened part of the earth's disk; and when he is in any of the southern signs, Libra, Scorpio, &c. the north pole of the earth is in the under or obscure part of the disk.

The transverse diameter of any path is always at right angles to the earth's axis, and the conjugate diameter coincides with or is a part of it.

PROB. XX. *The Latitude of a Place, the Time of the Year, and the Sun's Longitude, being given, to describe the Earth's Disk, and the Path of that Place thereon.*

Exam. It is required to describe the earth's disk, and the path of a place in latitude 51° 32′ N. 1803, August 17th, at 8h. 19m. in the morning, apparent time. (Plate XV. Fig. 2.)

Sun's longitude 4s. 23° 24′ 45″, distance from the solstice 53° 24′ 45″; declination N. 13° 43′ 43 .

Draw the straight line AB to represent a part of the ecliptic, and let C be the point therein, which is opposite to the sun at the time: from C draw CH at right angles to AB, and CH is the axis of the ecliptic, and H its pole.

From the centre C describe the semicircle AHB to represent the northern half of the earth's disk. Make CA or CB the radius of the line of chords on the sector, and take the chord of 23 $\frac{1}{2}$ degrees, which set from H both ways to *f* and *g*, and draw the straight line *f*V*g*; the north pole of the earth is always in this line.

Make *f*V the radius of the line of sines on the sector, and take the sine of 53° 24′ 45″, the sun's distance from the solstice, which set off from V to P toward the left hand, because the sun is in the sign Virgo, and draw the straight line CP*h* for the earth's axis; then P is the north pole of the earth. Or the angle contained between the earth's axis and that of the ecliptic may be found more accurately by calculation, thus;

		°	′	
As radius - - - - -	S.	90	0	10.
Is to the sine of the sun's dist. from the solstice -		53	25	9.9047106
So is the tan. of the distance of the poles		23	28	9.6376106
To the tan. of the angle contained by the axes		19	13	9.5423212

Now set off the chord of 19° 13′ from H to *h*, and join CH, which will cut *f g* in P, the place of the north pole.

Make CA the radius of the line of chords on the sector; take the chord of 38° 28′, the complement of the latitude, and set it off from *h*, both ways, to *e* and *n*, where make marks. Again, take the chord of the sun's declination 13° 43′ 43″ from the sector, and set it off from the points *e* and *n*, both ways, to D and F, and to M and G; then draw the lines DM, FG, cutting CP in 12 and 12; and the line 12 K 12 is the conjugate axis of the ellipse, which bisect in K, and through K draw the line 6 K 6 at right angles to CP.

Make CA the radius of the line of sines on the sector, and take the sine of the co-latitude 38° 28′, and set it off from K, both ways, to 6 and 6. These hours fall on the circumference of the disk, when the sun is in either of the equinoctial points; but at all other points they fall within it. The line 6 K 6 is the transverse axis of the ellipse.

Make K 6 the radius of the line of sines on the sector, and from it take the sines of 15°, 30°, 45°, 60°, and 75°, and set them off from K, both ways, toward 6 and 6; and through these points draw lines parallel to CP. Again, make K 12 the radius of the line of sines on the sector, and take from it the sines of 75°, 60°, 45°, 30°, and 15°, and set them off on the parallels, beginning with that next to 12 K 12, and through these points draw a curve line which will be the path of the given place. Mark the hours on the path as per Fig. and divide them into halves, quarters, and smaller parts, at pleasure.

The 12 next to P marks midnight, when the sun's declination is north; and that part of the path is below the disk; but the other 12 between K and C is noon or mid-day. The path touches the circumference of the disk a little before 5, which is the time of sun-rising; and again a little after 7, the time of sun-setting.

In north latitude, when the sun's declination is south, the 12 nearest to P is mid-day, and the other is midnight.

In the same manner, the parallel of latitude, or path of any place, may be described on the earth's disk for any given time, and is used in representing the appearance of an eclipse of the sun.

PROB. XXI. *To calculate an Eclipse of the Sun.*

1. Calculate the time of mean conjunction, with the sun's distance from the moon's node, and thereby find if an eclipse will happen; and if it do, calculate the true time of conjunction, and the sun and moon's longitudes at that time.
2. Calculate the moon's latitude, horizontal parallax, horary motion, semidiameter, and angle of her path with the ecliptic.
3. Calculate the sun's declination, hourly motion, and his distance from the solstice.

Having found these necessary requisites, write them down in order to be ready for use.

Exam. The sun will be eclipsed in August, 1803.
The time of mean conjunction is Aug. 17d. 7h. 12′ 24″.
Sun's mean distance from the moon's node 6s. 2° 8′ 8″.
And the true conjunction is Aug. 16d. 20h. 30′ 34″ mean time.

	S.	°	′	″
The sun's longitude then	4	23	25	3
The moon's orbit longitude	4	23	28	49
Past conjunction	0	0	3	46

Sun's mean anomaly 1s. 15° 8′ 15″.

	′	″
Moon's hourly motion	30	57
Sun's ditto subtract	2	24
Moon's hourly motion from sun	28	33

	L. L.
As moon's hourly motion from ☉ 28′ 33′	3225
To 1 hour or 60 minutes	0
So is difference of longitudes 3′ 46″	12022
To interval —7′ 54″	8796

	D.	h.	′	″
From the time formerly found	16	20	30	34
Subtract the interval			— 7	54
Correct time of conjunction	16	20	22	40
Equation of time			— 3	48
Apparent time of conjunction	16	20	18	54

or 8h. 18′ 54″ in the morning of August 17th.

For the Sun and Moon's places at the corrected time of Conjunction.

m. ′ ″ m. s. ″

As 60 : 2 24 : : 7 54 : 18 to be subtracted from ☉'s long.

60 : 30 57 : : 7 54 : 4′ 4″ to be subtracted from the ☽'s long.

	S.	°	′	″
And then the sun's longitude is	4	23	24	45
The moon's ditto	4	23	24	45

		°	′	″
2.	The moon's horizontal parallax, or semidiameter of the earth's disk	0	55	9
3.	Sun's distance from the nearest solstice	53	24	45
4.	Sun's declination north - -	13	43	43
5.	Angle of the moon's path with the ecliptic	5	43	0
6.	The moon's latitude south descending	0	0	46
7.	The moon's hourly motion from the sun	0	28	33
8.	The sun's semidiameter 15′ 52″			
9.	The moon's semidiameter 15 14			
10.	The semidiameter of the penumbra, or sum of the semidiameters of sun and moon	0	30	56

In this example the sun's distance from the nearest solstice is found by subtracting 3 signs from his longitude. The angle of the moon's path with the ecliptic is taken out of the Table.

The method of finding the other numbers has been taught formerly.

PROB. XXII. *To describe a Figure representing a Solar Eclipse, and to find its Magnitude, Beginning, Middle, and End at London.* (Plate XV. Fig. 2.)

Exam. Project the solar eclipse, calculated in the last problem.

Draw the straight line AB to represent a part of the ecliptic; take the semidiameter of the earth's disk from a scale, and with it as radius describe the semicircle AHB for the northern half of the earth's disk, and thereon describe the path of London, as has been taught in the preceding problem.

Make the semidiameter of the disk CA the radius of the line of chords on the sector, and take the chord of 5° 43′ the angle of the moon's path, which set off on the circumference of the disk, from H toward the right hand, to M, because the moon's latitude is south descending, and draw CM for the axis of the moon's path.

Set off the moon's latitude 46″ south on MC, produced below AB to X; and through X, at right angles to MC, draw the line of the moon's path, which cuts AB on the left hand of X. The point X is the time of conjunction by the Tables.

To find the place of 8 hours on the moon's path.

		L. L.
As 1 hour or 60 minutes - - - - -		0
To the ☽'s hourly motion from the ☉	28′ 33″	3225
So is the time of conjunction after 8h.	18′ 52″	5025
To the distance of 8 hours from X	8′ 58″	8250

Take 8′ 58″ from the scale, and set it off from X toward the left hand for the place of 8 hours on the moon's path.

Take also from the scale 28′ 33″, the moon's hourly motion from the sun, and set it off both ways from 8 to 9 and 7, and also from 7 to 6, &c. and the places of the hours on the moon's path will be known. Divide each hour distance into 12 equal parts by dots, and the place of the moon's centre, or rather the centre of the penumbra, will be known at every 5 minutes during the eclipse.

To find the middle of the Eclipse.

Apply one side of a square to the path of the moon or penumbra's centre, and move it backward and forward until the other side of the square cuts the same hour and minute on the path of London and of the penumbra's centre, which in this example is at y and u in 6 hours 32 minutes, the middle of the eclipse.

Take the sun's semidiameter 15′ 52″ from the scale, and with it describe a circle about the point y for the sun's disk; then take the moon's semidiameter 15′ 4″ from the scale, and with it describe a circle about the point u for the moon's disk at the middle of the eclipse. The part of the sun's disk cut off by the moon's is the magnitude of the eclipse as it will appear at London. In this example it is about $3\frac{1}{2}$ digits.

Lastly, take the semidiameter of the penumbra 30′ 56″ from the scale; and setting one foot of the compasses on the moon's path, and the other on the path of London, toward the left hand, carry that extent backward and forward until both points fall on the same hour and minute in each path, and that is the beginning of the eclipse at London. With the same extent of the compasses, and one foot on each path, carry them backward and forward toward the right hand; and where both points fall on the same time, that will be the end of the eclipse at London. These trials give,

	h.	m.
The beginning of the eclipse at - - -	5	50
The middle - - - - - - -	6	32
The end - - - - - - -	7	30

Duration 1h. 40m. Digits 3° 30′.

Note. 1. The projection of a solar eclipse will exhibit the appearance of it more naturally, if some alterations be made in the preceding process, adapted to the supposition, that the observer is on the earth. These alterations consist in drawing the axes of the earth and moon's path on the side of the axis of the ecliptic contrary to that which is required by the rule, and in numbering the hours in the opposite direction. For the relative position of the axes, as seen from the sun, is inverted with respect to an observer of the sun on the earth. If a projection, made according to the preceding rule, be turned over from right to left, it will appear, if visible through the paper, to correspond with one of this construction. Plate XV. Fig. 3, is an example, being the same eclipse projected in this manner, and requires no farther explanation.

Note 2. The situation of the point on the sun's limb, with respect to a vertical and a horizontal diameter, where an eclipse begins, may be easily determined by Projection. Thus, with the points in the respective paths, where the centres of the sun and moon are at the beginning of the eclipse, as centres, and their semidiameters as radii, describe circles, touching each other. Draw a line from the centre C (Fig. 2 or 3, Plate XV.) through the centre of the sun, and it will give the vertical diameter; and a diameter perpendicular to this will be horizontal. Then the line of chords on a sector being adapted to the semidiameter of the sun, the arc of the circumference, contained between the point of contact and an extremity of one of these diameters, measured on the chord line, will give the required situation of the point, where the eclipse begins.

SOLAR AND LUNAR TABLES.

TABLES OF THE SUN'S MEAN MOTIONS.

I. *The Sun's Mean Longitude and Anomaly in Julian Years.*									II. *The Precession of the Equinoctial Points.*		
Years.	M. Longitude.				M. Anomaly.						
	S.	°	′	″	S.	°	′	″	°	′	″
1	11	29	45	40	11	29	44	35	0	0	50.3
2	11	29	31	21	11	29	29	0	0	1	40.7
3	11	29	17	2	11	29	13	44	0	2	31
B 4	0	0	1	51	11	29	57	27	0	3	21.4
5	11	29	47	32	11	29	42	2	0	4	11.7
6	11	29	33	13	11	29	26	37	0	5	2.1
7	11	29	18	54	11	29	11	12	0	5	52.4
B 8	0	0	3	43	11	29	54	55	0	6	42.8
9	11	29	49	23	11	29	39	29	0	7	33.1
10	11	29	35	4	11	29	24	4	0	8	23.5
11	11	29	20	45	11	29	8	39	0	9	13.8
B 12	0	0	5	34	11	29	52	22	0	10	4.2
13	11	29	51	15	11	29	36	57	0	10	54.5
14	11	29	36	55	11	29	21	31	0	11	44.9
15	11	29	22	36	11	29	6	6	0	12	35.2
B 16	0	0	7	35	11	29	49	49	0	13	25.6
17	11	29	53	6	11	29	34	24	0	15	15.9
18	11	29	38	47	11	29	18	59	0	15	6.3
19	11	29	24	27	11	29	3	33	0	15	56.6
B 20	0	0	9	17	11	29	47	17	0	16	47
B 40	0	0	18	33	11	29	34	33	0	33	34
B 60	0	0	27	50	11	29	21	50	0	50	21
B 80	0	0	37	6	11	29	9	6	1	7	8
B 100	0	0	46	23	11	28	56	23	1	23	55
B 200	0	1	32	46	11	27	52	46	2	47	50
B 300	0	2	19	9	11	26	49	9	4	11	45
B 400	0	3	5	32	11	25	45	32	5	35	40
B 500	0	3	51	55	11	24	41	55	6	59	35

III. *Sun's Mean Longitude and Anomaly in Years current.*									*Obliquity of the Ecliptic 1st January.*		
Years.	M. Longitude.				M. Anomaly.						
A. D.	S.	°	′	″	S.	°	′	″	°	′	″
1761	9	10	20	51	6	1	31	11	23	28	16
1781	9	10	30	7	6	1	18	27	23	28	9
1791	9	10	5	12	6	0	42	32	23	27	50
B 1792	9	10	50	1	6	1	26	15	23	27	48
1793	9	10	35	41	6	1	10	49	23	27	47
1794	9	10	21	22	6	0	55	24	23	27	48
1795	9	10	7	3	6	0	39	59	23	27	50
B 1796	9	10	51	52	6	1	23	42	23	27	52
1797	9	10	37	33	6	1	8	17	23	27	54
1798	9	10	23	13	6	0	52	51	23	27	55
1799	9	10	8	54	6	0	37	26	23	27	58
1800	9	9	54	35	6	0	22	1	23	28	0
1801	9	9	40	16	6	0	6	36	23	28	1
1802	9	9	25	56	5	29	51	10	23	28	0
1803	9	9	11	37	5	29	35	45	23	28	0
B 1804	9	9	56	26	6	0	19	28	23	27	58
1805	9	9	42	6	6	0	4	2	23	27	55
1806	9	9	27	48	5	29	48	38	23	27	51
1807	9	9	13	29	5	29	33	13	23	27	48
B 1808	9	9	58	17	6	0	16	48	23	27	44
1809	9	9	43	57	6	0	1	31	23	27	42
1810	9	9	29	37	5	29	45	57	23	27	40
1811	9	9	15	17	5	29	30	32	23	27	39
B 1812	9	10	0	5	6	0	14	15	23	27	39
1813	9	9	45	45	5	29	58	49	23	27	40
1814	9	9	31	25	5	29	43	26	23	27	41
1815	9	9	17	5	5	29	27	58	23	27	43
B 1816	9	10	1	53	6	0	11	41	23	27	46
1817	9	9	47	33	5	29	56	15	23	27	48
1818	9	9	33	13	5	29	40	50	23	27	50
1819	9	9	18	53	5	29	25	24	23	27	52
B 1820	9	10	3	41	6	0	9	7	23	27	52
1821	9	9	49	22	5	29	55	42	23	27	51
1841	9	9	58	39	5	29	42	59	23	27	59

TABLE IV. *The Sun's Mean Longitude and Anomaly for Months and Days.*

Days.	January.								Days.	February.							
	Longitude.				Anomaly.					Longitude.				Anomaly.			
	S.	°	'	"	S.	°	'	"		S.	°	'	"	S.	°	'	'
1	0	0	59	8	0	0	59	8	1	1	1	32	27	1	1	32	21
2	0	1	58	17	0	1	58	17	2	1	2	31	35	1	2	31	29
3	0	2	57	25	0	2	57	24	3	1	3	30	43	1	3	30	37
4	0	3	56	33	0	3	56	32	4	1	4	29	52	1	4	29	46
5	0	4	55	42	0	4	55	41	5	1	5	29	0	1	5	28	53
6	0	5	54	50	0	5	54	49	6	1	6	28	8	1	6	28	1
7	0	6	53	58	0	6	53	57	7	1	7	27	17	1	7	27	10
8	0	7	53	7	0	7	53	6	8	1	8	26	25	1	8	26	17
9	0	8	52	15	0	8	52	14	9	1	9	25	33	1	9	25	26
10	0	9	51	23	0	9	51	21	10	1	10	24	42	1	10	24	35
11	0	10	50	32	0	10	50	30	11	1	11	23	50	1	11	23	42
12	0	11	49	40	0	11	49	38	12	1	12	22	58	1	12	22	50
13	0	12	48	48	0	12	48	46	13	1	13	22	7	1	13	21	59
14	0	13	47	57	0	13	47	54	14	1	14	21	15	1	14	21	7
15	0	14	47	5	0	14	47	2	15	1	15	20	23	1	15	20	15
16	0	15	46	13	0	15	46	10	16	1	16	19	31	1	16	19	23
17	0	16	45	22	0	16	45	19	17	1	17	18	40	1	17	18	31
18	0	17	44	30	0	17	44	27	18	1	18	17	48	1	18	17	39
19	0	18	43	38	0	18	43	35	19	1	19	16	56	1	19	16	47
20	0	19	42	47	0	19	42	43	20	1	20	16	5	1	20	15	56
21	0	20	41	55	0	20	41	51	21	1	21	15	13	1	21	15	4
22	0	21	41	3	0	21	40	59	22	1	22	14	21	1	22	14	11
23	0	22	40	12	0	22	40	8	23	1	23	13	30	1	23	13	20
24	0	23	39	20	0	23	39	16	24	1	24	12	38	1	24	12	28
25	0	24	38	28	0	24	38	23	25	1	25	11	46	1	25	11	36
26	0	25	37	37	0	25	37	32	26	1	26	10	55	1	26	10	45
27	0	26	36	45	0	26	36	40	27	1	27	10	3	1	27	9	52
28	0	27	35	53	0	27	35	48	28	1	28	9	11	1	28	9	0
29	0	28	35	2	0	28	34	57									
30	0	29	34	10	0	29	34	5									
31	1	0	33	18	1	0	33	12									

In the months January and February of bissextile years take away one day from the time.

Days.	March.								Days.	April.							
	Longitude.				Anomaly.					Longitude.				Anomaly.			
	S.	°	'	"	S.	°	'	"		S.	°	'	"	S.	°	'	"
1	1	29	8	20	1	29	8	9	1	2	29	41	38	2	29	41	22
2	2	0	7	28	2	0	7	17	2	3	0	40	46	3	0	40	30
3	2	1	6	36	2	1	6	25	3	3	1	39	55	3	1	39	38
4	2	2	5	45	2	2	5	34	4	3	2	39	3	3	2	38	46
5	2	3	4	53	2	3	4	42	5	3	3	38	11	3	3	37	54
6	2	4	4	1	2	4	3	49	6	3	4	37	20	3	4	37	3
7	2	5	3	10	2	5	2	58	7	3	5	36	28	3	5	36	11
8	2	6	2	18	2	6	2	6	8	3	6	35	36	3	6	35	18
9	2	7	1	26	2	7	1	14	9	3	7	34	45	3	7	34	27
10	2	8	0	35	2	8	0	22	10	3	8	33	53	3	8	33	35
11	2	8	59	43	2	8	59	30	11	3	9	33	1	3	9	32	43
12	2	9	58	51	2	9	58	38	12	3	10	32	10	3	10	31	52
13	2	10	58	0	2	10	57	47	13	3	11	31	18	3	11	30	59
14	2	11	57	8	2	11	56	55	14	3	12	30	26	3	12	30	7
15	2	12	56	16	2	12	56	3	15	3	13	29	35	3	13	29	16
16	2	13	55	25	2	13	55	11	16	3	14	28	43	3	14	28	24
17	2	14	54	33	2	14	54	19	17	3	15	27	51	3	15	27	32
18	2	15	53	41	2	15	53	27	18	3	16	27	0	3	16	26	40
19	2	16	52	50	2	16	52	36	19	3	17	26	8	3	17	25	48
20	2	17	51	58	2	17	51	44	20	3	18	25	16	3	18	24	56
21	2	18	51	6	2	18	50	51	21	3	19	24	25	3	19	24	5
22	2	19	50	15	2	19	50	0	22	3	20	23	38	3	20	23	13
23	2	20	49	23	2	20	49	8	23	3	21	22	41	3	21	22	21
24	2	21	48	31	2	21	48	16	24	3	22	21	50	3	22	21	29
25	2	22	47	40	2	22	47	25	25	3	23	20	37	3	23	20	37
26	2	23	46	48	2	23	46	33	26	3	24	20	6	3	24	19	45
27	2	24	45	56	2	24	45	40	27	3	25	19	15	3	25	18	54
28	2	25	45	5	2	25	44	53	28	3	26	18	23	3	26	18	2
29	2	26	44	13	2	26	43	57	29	3	27	17	31	3	27	17	9
30	2	27	43	21	2	27	43	5	30	3	28	16	40	3	28	16	18
31	2	28	42	30	2	28	42	14									

The Sun's mean Longitude and Anomaly for Months and Days.

Days.	May.		Days.	June.		Days.	July.		Days.	August.	
	Longitude.	Anomaly.		Longitude.	Anomaly.		Longitude.	Anomaly.		Longitude.	Anomaly.
	S. ° ′ ″	S. ° ′ ″		S. ° ′ ″	S. ° ′ ″		S. ° ′ ″	S. ° ′ ″		S. ° ′ ″	S. ° ′ ″
1	3 29 15 48	3 29 15 26	1	4 29 49 6	4 29 48 39	1	5 29 23 16	5 29 22 43	1	6 29 56 34	6 29 55 55
2	4 0 14 56	4 0 14 34	2	5 0 48 15	5 0 47 47	2	6 0 22 24	6 0 21 51	2	7 0 55 43	7 0 55 4
3	4 1 14 5	4 1 13 43	3	5 1 47 23	5 1 46 55	3	6 1 21 33	6 1 21 0	3	7 1 54 51	7 1 54 12
4	4 2 13 13	4 2 12 51	4	5 2 46 31	5 2 46 3	4	6 2 20 41	6 2 20 8	4	7 2 53 59	7 2 55 20
5	4 3 12 21	4 3 11 58	5	5 3 45 40	5 3 45 12	5	6 3 19 49	6 3 19 15	5	7 3 53 8	7 3 52 29
6	4 4 11 30	4 4 11 7	6	5 4 44 48	5 4 44 20	6	6 4 18 58	6 4 18 24	6	7 4 52 16	7 4 51 37
7	4 5 10 38	4 5 10 15	7	5 5 43 56	5 5 43 28	7	6 5 18 6	6 5 17 32	7	7 5 51 24	7 5 50 44
8	4 6 9 46	4 6 9 23	8	5 6 43 5	5 6 42 37	8	6 6 17 14	6 6 16 40	8	7 6 50 33	7 6 49 54
9	4 7 8 55	4 7 8 32	9	5 7 42 13	5 7 41 44	9	6 7 16 23	6 7 15 49	9	7 7 49 41	7 7 49 1
10	4 8 8 3	4 8 7 40	10	5 8 41 21	5 8 40 52	10	6 8 15 31	6 8 14 56	10	7 8 48 49	7 8 48 9
11	4 9 7 11	4 9 6 47	11	5 9 40 30	5 9 40 1	11	6 9 14 39	6 9 14 4	11	7 9 47 58	7 9 47 18
12	4 10 6 20	4 10 5 56	12	5 10 39 38	5 10 39 8	12	6 10 13 48	6 10 13 13	12	7 10 47 6	7 10 46 25
13	4 11 5 28	4 11 5 4	13	5 11 38 46	5 11 38 16	13	6 11 12 56	6 11 12 21	13	7 11 46 14	7 11 45 33
14	4 12 4 36	4 12 4 12	14	5 12 37 55	5 12 37 25	14	6 12 12 4	6 12 11 29	14	7 12 45 23	7 12 44 42
15	4 13 3 45	4 13 3 21	15	5 13 37 3	5 13 36 33	15	6 13 11 13	6 13 10 38	15	7 13 44 31	7 13 43 50
16	4 14 2 53	4 14 2 28	16	5 14 36 11	5 14 35 41	16	6 14 10 21	6 14 9 45	16	7 14 43 39	7 14 42 58
17	4 15 2 1	4 15 1 36	17	5 15 35 20	5 15 34 50	17	6 15 9 29	6 15 8 53	17	7 15 42 48	7 15 42 7
18	4 16 1 10	4 16 0 45	18	5 16 34 28	5 16 33 57	18	6 16 8 38	6 16 8 2	18	7 16 41 56	7 16 41 16
19	4 17 0 18	4 16 59 53	19	5 17 33 36	5 17 33 5	19	6 17 7 46	6 17 7 10	19	7 17 41 4	7 17 40 22
20	4 17 59 26	4 17 59 1	20	5 18 32 45	5 18 32 14	20	6 18 6 54	6 18 6 18	20	7 18 40 13	7 18 39 31
21	4 18 58 35	4 18 58 10	21	5 19 31 53	5 19 31 32	21	6 19 6 3	6 19 5 26	21	7 19 39 21	7 19 38 39
22	4 19 57 43	4 19 57 17	22	5 20 31 1	5 20 30 30	22	6 20 5 11	6 20 4 34	22	7 20 38 29	7 20 37 47
23	4 20 56 51	4 20 56 25	23	5 21 30 10	5 21 29 39	23	6 21 4 19	6 21 3 42	23	7 21 37 38	7 21 36 56
24	4 21 56 0	4 21 55 34	24	5 22 29 18	5 22 28 46	24	6 22 3 28	6 22 2 51	24	7 22 36 46	7 22 36 4
25	4 22 55 8	4 22 54 42	25	5 23 28 26	5 23 27 54	25	6 23 2 36	6 23 1 59	25	7 23 35 54	7 23 35 11
26	4 23 54 16	4 23 53 50	26	5 24 27 35	5 24 27 3	26	6 24 1 44	6 24 1 7	26	7 24 35 3	7 24 34 20
27	4 24 53 25	4 24 52 58	27	5 25 26 43	5 25 26 11	27	6 25 0 53	6 25 0 15	27	7 25 34 11	7 25 33 28
28	4 25 52 33	4 25 52 6	28	5 26 25 51	5 26 25 19	28	6 26 0 1	6 25 59 23	28	7 26 33 19	7 26 32 30
29	4 26 51 41	4 26 51 14	29	5 27 24 59	5 27 24 26	29	6 26 59 9	6 26 58 31	29	7 27 32 28	7 27 31 44
30	4 27 50 50	4 27 50 23	30	5 28 24 8	5 28 23 35	30	6 27 58 18	6 27 57 40	30	7 28 31 36	7 28 30 52
31	5 28 49 58	4 28 49 31				31	6 28 57 26	6 28 56 48	31	7 29 30 44	7 29 30 0

The Sun's Mean Longitude and Anomaly for Months and Days.

Days.	September. Longitude.				September. Anomaly.				Days.	October. Longitude.				October. Anomaly.			
	S.	°	′	″	S.	°	′	″		S.	°	′	″	S.	°	′	″
1	8	0	29	53	8	0	29	9	1	9	0	4	3	9	0	3	13
2	8	1	29	1	8	1	28	17	2	9	1	3	11	9	1	2	21
3	8	2	28	9	8	2	27	25	3	9	2	2	19	9	2	1	26
4	8	3	27	18	8	3	26	33	4	9	3	1	28	9	3	0	38
5	8	4	26	26	8	4	25	41	5	9	4	0	36	9	3	59	46
6	8	5	25	34	8	5	24	49	6	9	4	59	44	9	4	58	54
7	8	6	24	43	8	6	23	58	7	9	5	58	53	9	5	58	2
8	8	7	23	51	8	7	23	6	8	9	6	58	1	9	6	57	10
9	8	8	22	59	8	8	22	14	9	9	7	57	9	9	7	56	18
10	8	9	22	8	8	9	21	22	10	9	8	56	18	9	8	55	27
11	8	10	21	16	8	10	20	30	11	9	9	55	26	9	9	54	35
12	8	11	20	24	8	11	19	38	12	9	10	54	34	9	10	53	42
13	8	12	19	33	8	12	18	47	13	9	11	53	43	9	11	52	51
14	8	13	18	41	8	13	17	55	14	9	12	52	51	9	12	51	59
15	8	14	17	49	8	14	17	2	15	9	13	51	59	9	13	51	7
16	8	15	16	58	8	15	16	11	16	9	14	51	8	9	14	50	16
17	8	16	16	6	8	16	15	19	17	9	15	50	16	9	15	49	24
18	8	17	15	14	8	17	14	27	18	9	16	49	24	9	16	48	31
19	8	18	14	23	8	18	13	35	19	9	17	48	33	9	17	47	40
20	8	19	13	31	8	19	12	43	20	9	18	47	41	9	18	46	48
21	8	20	12	39	8	20	11	51	21	9	19	46	49	9	19	45	56
22	8	21	11	48	8	21	11	0	22	9	20	45	58	9	20	45	5
23	8	22	10	56	8	22	10	8	23	9	21	45	6	9	21	44	13
24	8	23	10	4	8	23	9	16	24	9	22	44	14	9	22	43	20
25	8	24	9	13	8	24	8	25	25	9	23	43	23	9	23	42	29
26	8	25	8	21	8	25	7	32	26	9	24	42	31	9	24	41	37
27	8	26	7	29	8	26	6	40	27	9	25	41	39	9	25	40	45
28	8	27	6	38	8	27	5	49	28	9	26	40	48	9	26	39	54
29	8	28	5	46	8	28	4	57	29	9	27	39	56	9	27	39	2
30	8	29	4	54	8	29	4	5	30	9	28	39	4	9	28	38	9
									31	9	29	38	13	9	29	37	18

Days.	November. Longitude.				November. Anomaly				Days.	December. Longitude.				December. Anomaly.			
	S.	°	′	″	S.	°	′	″		S.	°	′	″	S.	°	′	″
1	10	0	37	21	10	0	36	26	1	11	0	11	31	11	0	10	30
2	10	1	36	29	10	1	35	34	2	11	1	10	39	11	1	9	38
3	10	2	35	38	10	2	34	43	3	11	2	9	47	11	2	8	46
4	10	3	34	46	10	3	33	50	4	11	3	8	56	11	3	7	55
5	10	4	33	54	10	4	32	58		11	4	8	4	11	4	7	3
6	10	5	33	3	10	5	32	7	6	11	5	7	12	11	5	6	11
7	10	6	32	11	10	6	31	15	7	11	6	6	21	11	6	5	20
8	10	7	31	19	10	7	30	23	8	11	7	5	29	11	7	4	27
9	10	8	30	27	10	8	29	30	9	11	8	4	37	11	8	3	35
10	10	9	29	36	10	9	28	39	10	11	9	3	46	11	9	2	44
11	10	10	28	44	10	10	27	47	11	11	10	2	54	11	10	1	52
12	10	11	27	52	10	11	26	55	12	11	11	2	2	11	11	1	0
13	10	12	27	1	10	12	26	4	13	11	12	1	11	11	12	0	8
14	10	13	26	9	10	13	25	12	14	11	13	0	19	11	12	59	16
15	10	14	25	17	10	14	24	19	15	11	13	59	27	11	13	58	24
16	10	15	24	26	10	15	23	28	16	11	14	58	36	11	14	57	33
17	10	16	23	34	10	16	22	36	17	11	15	57	44	11	15	56	41
18	10	17	22	42	10	17	21	44	18	11	16	56	52	11	16	55	48
19	10	18	21	50	10	18	20	52	19	11	17	56	1	11	17	54	57
20	10	19	20	59	10	19	20	0	20	11	18	55	9	11	18	54	5
21	10	20	20	7	10	20	19	8	21	11	19	54	17	11	19	53	13
22	10	21	19	16	10	21	18	17	22	11	20	53	26	11	20	52	22
23	10	22	18	24	10	22	17	25	23	11	21	52	34	11	21	51	29
24	10	23	17	32	10	23	16	33	24	11	22	51	42	11	22	50	37
25	10	24	16	41	10	24	15	41	25	11	23	50	51	11	23	49	46
26	10	25	15	49	10	25	14	49	26	11	24	49	59	11	24	48	54
27	10	26	14	57	10	26	13	57	27	11	25	49	7	11	25	48	2
28	10	27	14	6	10	27	13	6	28	11	26	48	16	11	26	47	11
29	10	28	13	14	10	28	12	14	29	11	27	47	24	11	27	46	18
30	10	29	12	22	10	29	11	22	30	11	28	46	32	11	28	45	26
									31	11	25	45	41	11	29	44	35

TABLE V. *Of the Sun's Mean Longitude and Anomaly for Hours, Minutes, and Seconds.*

H ′ ″	′ ″ ‴	″ ‴ iv	‴ iv v	H ′ ″	° ′ ″	′ ″ ‴	″ ‴ iv
1	2	27	50	31	1	16	23
2	4	55	41	32	1	18	51
3	7	23	31	33	1	21	19
4	9	51	22	34	1	23	47
5	12	19	12	35	1	26	14
6	14	47	2	36	1	28	42
7	17	14	53	37	1	31	10
8	19	42	43	38	1	33	38
9	22	10	34	39	1	36	6
10	24	38	24	40	1	38	34
11	27	6	14	41	1	41	1
12	29	34	5	42	1	43	29
13	32	1	55	43	1	45	57
14	34	29	46	44	1	48	25
15	36	57	36	45	1	50	53
16	39	25	27	46	1	53	21
17	41	53	17	47	1	55	48
18	44	21	7	48	1	58	16
19	46	48	58	49	2	0	44
20	49	16	48	50	2	3	12
21	51	44	39	51	2	5	40
22	54	12	29	52	2	8	8
23	56	40	19	53	2	10	36
24	59	8	19	54	2	13	3
25	61	36	9	55	2	15	31
26	64	3	59	56	2	17	59
27	66	31	49	57	2	20	27
28	68	59	31	58	2	22	55
29	71	27	21	59	2	25	23
30	73	55	11	60	2	27	50

TABLE VI. *Equations of the Sun's Centre.*

Argument. **Sun's Mean Anomaly.**

Signs.	— 0			— 1			— 2			— 3			— 4			— 5			
°	°	′	″	°	′	″	°	′	″	°	′	″	°	′	″	°	′	″	°
0	0	0	0	0	56	47	1	39	6	1	55	37	1	41	12	0	58	53	30
1	0	1	59	0	58	30	1	40	7	1	55	39	1	40	12	0	57	7	29
2	0	3	57	1	0	12	1	41	6	1	55	38	1	39	10	0	55	19	28
3	0	5	56	1	1	53	1	42	3	1	55	36	1	38	6	0	53	30	27
4	0	7	54	1	3	33	1	42	59	1	55	31	1	37	0	0	51	40	26
5	0	9	52	1	5	12	1	43	52	1	55	24	1	35	52	0	49	49	25
6	0	11	50	1	6	50	1	44	44	1	55	15	1	34	43	0	47	57	24
7	0	13	48	1	8	27	1	45	34	1	55	3	1	33	32	0	46	5	23
8	0	15	46	1	10	2	1	46	22	1	54	50	1	32	19	0	44	11	22
9	0	17	43	1	11	36	1	47	8	1	54	35	1	31	4	0	42	16	21
10	0	19	40	1	13	9	1	47	53	1	54	17	1	29	47	0	40	21	20
11	0	21	37	1	14	41	1	48	35	1	53	57	1	28	29	0	38	25	19
12	0	23	33	1	16	11	1	49	15	1	53	36	1	27	9	0	36	28	18
13	0	25	29	1	17	40	1	49	54	1	53	12	1	25	48	0	34	30	17
14	0	27	25	1	19	8	1	50	30	1	52	46	1	24	25	0	32	32	16
15	0	29	20	1	20	34	1	51	5	1	52	18	1	23	0	0	30	33	15
16	0	31	15	1	21	59	1	51	37	1	51	48	1	21	34	0	28	33	14
17	0	33	9	1	23	22	1	52	8	1	51	15	1	20	6	0	26	33	13
18	0	35	2	1	24	44	1	52	36	1	50	41	1	18	36	0	24	33	12
19	0	36	55	1	26	5	1	53	3	1	50	5	1	17	5	0	22	32	11
20	0	38	47	1	27	24	1	53	27	1	49	26	1	15	33	0	20	30	10
21	0	40	39	1	28	41	1	53	50	1	48	46	1	13	59	0	18	28	9
22	0	42	30	1	29	57	1	54	10	1	48	3	1	12	24	0	16	26	8
23	0	44	20	1	31	11	1	54	28	1	47	19	1	10	47	9	14	24	7
24	0	46	9	1	32	24	1	54	44	1	46	32	1	9	9	0	12	21	6
25	0	47	57	1	33	35	1	54	58	1	45	44	1	7	29	0	10	18	5
26	0	49	45	1	34	45	1	55	10	1	44	53	1	5	49	0	8	14	4
27	0	51	32	1	35	53	1	55	20	1	44	1	1	4	7	0	6	11	3
28	0	53	18	1	36	59	1	55	28	1	43	7	1	2	24	0	4	7	2
29	0	55	3	1	38	3	1	55	34	1	42	10	1	0	39	0	2	4	1
30	0	56	47	1	39	6	1	55	37	1	41	12	0	58	53	0	0	0	0
Signs	+ 11			+ 10			+ 9			+ 8			+ 7			+ 6			

TABLE VII. *Logarithms of the Sun's Distance from the Earth.*

Argument. Sun's Mean Anomaly.

Signs.	0	1	2	3	4	5	
0°	5.007286	5.006347	5.003749	5.000124	4.996405	4.993620	30°
1	5.007285	5.006284	5.003640	4.999995	4.996292	4.993555	29
2	5.007282	5.006220	5.003531	4.999867	4.996180	4.993491	28
3	5.007277	5.006154	5.003420	4.999738	4.996069	4.993429	27
4	5.007269	5.006087	5.003307	4.999611	4.995959	4.993369	26
5	5.007260	5.006018	5.003194	4.999483	4.995850	4.993311	25
6	5.007249	5.005946	5.003080	4.999354	4.995742	4.993255	24
7	5.007235	5.005872	5.002965	4.999227	4.995636	4.993201	23
8	5.007218	5.005797	5.002849	4.999099	4.995531	4.993150	22
9	5.007200	5.005720	5.002732	4.998971	4.995427	4.993102	21
10	5.007180	5.005642	5.002614	4.998844	4.995325	4.993055	20
11	5.007158	5.005562	5.002495	4.998717	4.995224	4.993009	19
12	5.007134	5.005480	5.002375	4.998590	4.995126	4.992966	18
13	5.007107	5.005397	5.002254	4.998463	4.995028	4.992926	17
14	5.007079	5.005312	5.002134	4.998336	4.994932	4.992888	16
15	5.007048	5.005225	5.002012	4·998210	4.994836	4.992852	15
16	5.007015	5.005136	5.001890	4.998084	4.994743	4.992818	14
17	5.006980	5.005047	5.001767	4.997960	4.994652	4.992786	13
18	5.006943	5.004956	5.001643	4.997837	4.994562	4.992757	12
19	5.006905	5 004863	5.001518	4.997714	4.994474	4.992731	11
20	5.006864	5.004768	5.001393	4.997591	4.994387	4.992706	10
21	5.006821	5.004672	5.001268	4.997468	4.994302	4.992683	9
22	5.006776	5.004575	5.001142	4.997347	4.994219	4.992663	8
23	5.006730	5.004477	5.001016	4.997226	4.994138	4.992646	7
24	5.006681	5.004377	5.000889	4.997106	4.994058	4.992631	6
25	5.006630	5.004275	5.000762	4.996987	4.993980	4.992618	5
26	5.006577	5.004173	5.000635	4.996868	4.993904	4.992607	4
27	5.006522	5.004069	5.000508	4.996750	4.993831	4.992599	3
28	5.006466	5.003963	5.000380	4.996634	4.993759	4.992593	2
29	5.006408	5.003857	5.000252	4.996519	4.993688	4.992590	1
30	5.006347	5.003749	5.000124	4.996405	4.993620	4.992589	0
Signs.	11	10	9	8	7	6	

TABLE VIII. *The Sun's Declination to every Degree of his Longitude.*

Argument. Sun's Longitude.

Deg.	0 ♈ North. 6 ♎ South. ☉ Declin.	1 ♉ North. 7 ♏ South. ☉ Declin.	2 ♊ North. 8 ♐ South. ☉ Declin.	Deg.
	° ′ ″	° ′ ″	° ′ ″	
0	0 0 0	11 29 5	20 10 25	30
1	0 23 53	11 50 6	20 22 57	29
2	0 47 47	12 10 56	20 35 7	28
3	1 11 39	12 31 34	20 46 55	27
4	1 35 30	12 51 59	20 58 20	26
5	1 59 20	13 12 12	21 9 21	25
6	2 23 8	13 32 12	21 19 59	24
7	2 46 54	13 51 58	21 30 13	23
8	3 10 37	14 11 30	21 40 3	22
9	3 34 17	14 30 48	21 49 29	21
10	3 57 54	14 49 52	21 58 30	20
11	4 21 27	15 8 40	22 7 6	19
12	4 44 57	15 27 13	22 15 17	18
13	5 8 22	15 45 30	22 23 3	17
14	5 31 42	16 3 31	22 30 24	16
15	5 54 57	16 21 16	22 37 18	15
16	6 18 6	16 38 44	22 43 47	14
17	6 41 9	16 55 55	22 49 50	13
18	7 4 6	17 12 48	22 55 27	12
19	7 26 57	17 29 23	23 0 38	11
20	7 49 41	17 45 40	23 5 22	10
21	8 12 17	18 1 38	23 9 39	9
22	8 34 45	18 17 18	23 13 29	8
23	8 57 5	18 32 38	23 16 53	7
24	9 19 17	18 47 38	23 19 50	6
25	9 41 19	19 2 18	23 22 20	5
26	10 3 12	19 16 37	23 24 22	4
27	10 24 56	19 30 35	23 25 57	3
28	10 46 30	19 44 13	23 27 5	2
29	11 7 53	19 57 30	23 27 46	1
30	11 29 5	20 10 25	23 28 0	0
D.	☉ Declin. 11 ♓ South. 5 ♍ North.	☉ Declin. 10 ♒ South. 4 ♌ North.	☉ Declin. 9 ♑ South. 8 ♋ North.	D.

TABLE IX. *Of the Sun's Apparent Semidiameter and Hourly Motion.*

Argument. Sun's Mean Anomaly.

Signs.	0		1		2		3		4		5		
	Semidiam.	Hour. M.	Semidiam.	Hour. M.	Semidiam.	Hour. M.	Semidiam.	Hour. M.	Semidiam.	Hour.M.	Semidiam.	Hour. M.	
°	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	°
0	15 47	2 23	15 49	2 24	15 55	2 25	16 3	2 28	16 11	2 30	16 17	2 32	30
10	15 47	2 23	15 51	2 24	15 57	2 26	16 5	2 29	16 13	2 31	16 18	2 33	20
20	15 49	2 23	15 52	2 25	16 0	2 27	16 8	2 30	16 15	2 32	16 19	2 33	10
30	15 49	2 24	15 55	2 25	16 3	2 28	16 11	2 30	16 17	2 32	16 19	2 33	0
Signs.	11		10		9		8		7		6		

TABLE X. *Of the Equation of Time fitted to each Degree of the Ecliptic. Place of the Apogee* 3s. 9°.

Argument. Sun's Longitude.

Deg.	♈ 0 +—	♉ 1	♊ 2 —+	♋ 3 +	♌ 4 +	♍ 5 +	♎ 6 —	♏ 7 —	♐ 8 —	♑ 9 —+	♒ 10 +	♓ 21 +
	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.	m. s.
0	7 36	1 9	3 51	1 13	5 57	2 20	7 38	15 31	13 33	1 11	11 28	14 19
1	7 17	1 23	3 47	1 26	5 59	2 4	7 58	15 39	13 17	0 42	11 45	14 13
2	6 58	1 36	3 41	1 40	6 0	1 48	8 19	15 46	13 0	— 12	12 1	14 6
3	6 39	1 48	3 37	1 55	6 1	1 31	8 40	15 52	12 42	+ 17	12 17	13 59
4	6 20	2 0	3 32	2 7	6 1	1 14	9 1	15 57	12 23	0 46	12 32	13 51
5	6 1	2 11	3 26	2 20	6 0	0 56	9 21	16 2	12 4	1 16	12 46	13 43
6	5 42	2 22	3 19	2 33	5 59	0 38	9 41	16 6	11 44	1 45	12 59	13 34
7	5 24	2 32	3 12	2 45	5 57	0 20	10 1	16 9	11 23	2 14	13 12	13 24
8	5 5	2 42	3 4	2 58	5 54	+ 1	10 20	16 11	11 1	2 43	13 24	13 14
9	4 47	2 51	2 56	3 11	5 51	—18	10 39	16 13	10 39	3 11	13 35	13 3
10	4 28	3 0	2 47	3 23	5 47	0 37	10 57	16 13	10 16	3 39	13 45	12 51
11	4 9	3 8	2 38	3 35	5 42	0 57	11 15	16 13	9 53	4 7	13 54	12 39
12	3 50	3 16	2 29	3 46	5 37	1 17	11 33	16 12	9 29	4 35	14 2	12 27
13	3 32	3 23	2 19	3 58	5 31	1 38	11 51	16 10	9 5	5 2	14 9	12 14
14	3 13	3 30	2 8	4 9	5 24	1 58	12 8	16 7	8 40	5 29	14 16	12 0
15	2 55	3 36	1 57	4 19	5 17	2 19	12 25	16 4	8 14	5 56	14 22	11 46
16	2 37	3 41	1 46	4 29	5 9	2 40	12 41	16 0	7 48	6 22	14 27	11 31
17	2 19	3 46	1 35	4 39	5 1	3 1	12 57	15 55	7 22	6 48	14 31	11 16
18	2 1	3 50	1 23	4 48	4 52	3 22	13 12	15 49	6 55	7 13	14 35	11 1
19	1 43	3 53	1 11	4 57	4 43	3 44	13 27	15 42	6 28	7 37	14 38	10 46
20	1 26	3 56	0 59	5 5	4 33	4 5	13 42	15 35	6 0	8 1	14 40	10 30
21	1 9	3 58	0 46	5 13	4 22	4 26	13 56	15 26	5 32	8 24	14 41	10 14
22	0 52	4 0	0 34	5 20	4 11	4 47	14 9	15 17	5 4	8 47	14 42	9 58
23	0 36	4 1	0 21	5 27	3 50	5 9	14 21	15 7	4 36	9 9	14 41	9 41
24	0 20	4 1	— 8	5 33	3 46	5 30	14 33	14 56	4 8	9 31	14 40	9 24
25	+ 4	4 1	+ 5	5 39	3 33	5 32	14 44	14 44	3 39	9 53	14 39	9 6
26	—11	4 0	0 19	5 44	3 19	6 13	14 53	14 31	3 10	10 14	14 37	8 48
27	0 26	3 59	0 31	5 48	3 4	6 35	15 5	14 17	2 41	10 34	14 34	8 30
28	0 40	3 57	0 46	5 52	2 50	6 56	15 14	14 3	2 11	10 53	14 30	8 12
29	0 53	3 54	0 59	5 55	2 35	7 17	15 23	13 48	1 41	11 11	14 25	7 54
30	1 9	3 51	1 13	5 57	2 20	7 28	15 31	13 33	1 11	11 28	14 19	7 36

The equations with + are to be added to the apparent time to have the mean time; those with — are to be subtracted from the apparent for the mean time.

TABLE XI. *The Sun's Longitude for every Day in the Year, at Noon.*

Days.	January.	February.	March.	April.	May.	June.	July.	August.	Sept.	October.	Novem.	Decem.
	S. ° ′	S. ° ′	S. ° ′	S. ° ′	S. ° ′	S. ° ′	S. ° ′	S. ° ′	S. ° ′	S. ° ′	S. ° ′	S. ° ′
1	9 11 21	10 12 54	11 11 8	0 11 55	1 11 12	2 11 2	3 9 41	4 9 17	5 9 7	6 8 26	7 9 15	8 9 32
2	9 12 23	10 13 55	11 12 8	0 12 54	1 12 10	2 12 0	3 10 38	4 10 14	5 10 6	6 9 25	7 10 15	8 10 33
3	9 13 24	10 14 56	11 13 8	0 13 53	1 13 9	2 12 57	3 11 35	4 11 11	5 11 4	6 10 24	7 11 16	8 11 34
4	9 14 25	10 15 57	11 14 8	0 14 52	1 14 7	2 13 54	3 12 32	4 12 9	5 12 2	6 11 24	7 12 16	8 12 35
5	9 15 26	10 16 57	11 15 8	0 15 51	1 15 5	2 14 52	3 13 30	4 13 6	5 13 0	6 12 23	7 13 16	8 13 36
6	9 16 27	10 17 58	11 16 8	0 16 50	1 16 3	2 15 49	3 14 27	4 14 4	5 13 59	6 13 22	7 14 16	8 14 37
7	9 17 29	10 18 59	11 17 8	0 17 49	1 17 1	2 16 46	3 15 24	4 15 2	5 14 57	6 14 21	7 15 17	8 15 38
8	9 18 30	10 20 0	11 18 8	0 18 48	1 17 59	2 17 44	3 16 21	4 15 59	5 15 55	6 15 21	7 16 17	8 16 39
9	9 19 31	10 21 0	11 19 8	0 19 47	1 18 56	2 18 41	3 17 18	4 16 57	5 16 53	6 16 20	7 17 17	8 17 40
10	9 20 32	10 22 1	11 20 8	0 20 45	1 19 54	2 19 39	3 18 15	4 17 54	5 17 52	6 17 19	7 18 18	8 18 41
11	9 21 33	10 23 1	11 21 7	0 21 44	1 20 52	2 20 35	3 19 13	4 18 52	5 18 50	6 18 19	7 19 18	8 19 42
12	9 22 34	10 24 2	11 22 7	0 22 43	1 21 50	2 21 33	3 20 10	4 19 49	5 19 49	6 19 18	7 20 19	8 20 43
13	9 23 35	10 25 3	11 23 7	0 23 41	1 22 48	2 22 30	3 21 7	4 20 47	5 20 47	6 20 18	7 21 19	8 21 44
14	9 24 36	10 26 3	11 24 6	0 24 40	1 23 45	2 23 28	3 22 4	4 21 45	5 21 46	6 21 17	7 22 20	8 22 45
15	9 25 37	10 27 4	11 25 6	0 25 39	1 24 43	2 24 25	3 23 2	4 22 42	5 22 44	6 22 17	7 23 20	8 23 47
16	9 26 39	10 28 4	11 26 6	0 26 37	1 25 41	2 25 22	3 23 59	4 23 40	5 23 43	6 23 17	7 24 21	8 24 48
17	9 27 39	10 29 4	11 27 5	0 27 36	1 26 39	2 26 19	3 24 56	4 24 38	5 24 42	6 24 16	7 25 21	8 25 49
18	9 28 41	11 0 5	11 28 5	0 28 34	1 27 36	2 27 17	3 25 53	4 25 36	5 25 40	6 25 16	7 26 22	8 26 50
19	9 29 42	11 1 5	11 29 4	0 29 33	1 28 34	2 28 14	3 26 51	4 26 33	5 26 39	6 26 16	7 27 23	8 27 51
20	10 0 43	11 2 6	0 0 4	1 0 31	1 29 32	2 29 11	3 27 48	4 27 31	5 27 38	6 27 16	7 28 23	8 28 52
21	10 1 44	11 3 6	0 1 3	1 1 30	2 0 29	3 0 8	3 28 45	4 28 29	5 28 37	6 28 15	7 29 24	8 29 54
22	10 2 45	11 4 6	0 2 3	1 2 28	2 1 27	3 1 6	3 29 43	4 29 27	5 29 35	6 29 15	8 0 25	9 0 55
23	10 3 46	11 5 7	0 3 2	1 3 26	2 2 25	3 2 3	4 0 40	5 0 25	6 0 34	7 0 15	8 1 26	9 1 56
24	10 4 47	11 6 7	0 4 1	1 4 25	2 3 22	3 3 0	4 1 37	5 1 23	6 1 33	7 1 15	8 2 26	9 2 57
25	10 5 48	11 7 7	0 5 1	1 5 23	2 4 20	3 3 57	4 2 35	5 2 21	6 2 32	7 2 15	8 3 27	9 3 58
26	10 6 49	11 8 7	0 6 0	1 6 21	2 5 17	3 4 55	4 3 32	5 3 19	6 3 31	7 3 15	8 4 28	9 4 59
27	10 7 50	11 9 8	0 6 59	1 7 20	2 6 15	3 5 52	4 4 29	5 4 17	6 4 30	7 4 15	8 5 29	9 6 1
28	10 8 51	11 10 8	0 7 59	1 8 18	2 7 12	3 6 49	4 5 27	5 5 15	6 5 29	7 5 15	8 6 30	9 7 2
29	10 9 52		0 8 58	1 9 16	2 8 10	3 7 46	4 6 24	5 6 13	6 6 28	7 6 15	8 7 31	9 8 3
30	10 10 52		0 9 57	1 10 14	2 9 7	3 8 43	4 7 22	5 7 11	6 7 27	7 7 15	8 8 32	9 9 4
31	10 11 53		0 10 56		2 10 5		4 8 19	5 8 9		7 8 15		9 10 5

LUNAR TABLES.

TABLE I. *Of the Moon's Mean Motions in Julian Years.*

	Years.	☽'s M. Long.				☽'s M. Ano.				☽'s ☊ Retro.			
		S.	°	′	″	S.	°	′	″	S.	°	′	″
	1	4	9	23	5	2	28	43	15	0	19	19	43
	2	8	18	46	11	5	27	26	30	1	8	39	26
	3	0	28	9	16	8	26	9	45	1	27	59	9
B	4	5	20	42	57	0	7	56	54	2	17	22	3
	5	10	0	6	2	3	6	40	9	3	6	41	46
	6	2	9	29	7	6	5	23	23	3	26	1	29
	7	6	18	52	13	9	4	6	39	4	15	21	12
B	8	11	11	25	53	0	15	53	47	5	4	44	6
	9	3	20	48	59	3	14	37	3	5	24	3	49
	10	8	0	12	4	6	13	20	17	6	13	23	32
	11	0	9	35	9	9	12	3	32	7	2	43	15
B	12	5	2	8	50	0	23	50	17	7	22	6	9
	13	9	11	31	55	3	22	33	56	8	11	25	52
	14	1	20	55	1	6	21	17	11	9	0	45	35
	15	6	0	18	6	9	20	0	26	9	20	5	18
B	16	0	22	51	46	1	1	47	34	10	9	28	12
	17	3	2	14	52	4	0	30	50	10	28	47	55
	18	7	11	37	57	6	29	14	4	11	18	7	38
	19	11	21	1	2	9	27	57	19	0	7	27	21
B	20	4	13	34	43	1	9	44	30	0	26	50	15
B	40	8	27	9	26	2	19	28	56	1	23	40	30
B	60	1	10	44	9	3	29	13	24	2	20	30	45
B	80	5	24	18	52	5	8	57	52	3	17	21	0
B	100	10	7	53	35	6	18	42	20	4	14	11	15
B	200	8	15	47	10	1	7	24	40	8	28	22	13
B	300	6	23	40	45	7	26	7	0	1	12	33	45
B	400	5	1	34	20	2	14	49	20	5	26	45	0
B	500	3	9	27	55	9	3	31	40	10	10	56	15

TABLE II. *The Moon's Mean Longitude and Anomaly for Current Years.*

	A. D.	Mean Long.				Mean Anom.				Long. ☊.			
		S.	°	′	″	S.	°	′	″	S.	°	′	″
	1761	7	1	8	8	10	12	34	50	2	7	33	33
	1781	11	14	42	54	11	22	19	18	1	10	43	18
	1791	7	14	54	59	6	5	39	35	6	27	19	46
	1792	0	7	28	40	9	17	26	44	6	7	56	52
	1793	4	16	51	45	0	16	9	59	5	18	37	9
	1794	8	26	14	51	3	14	53	14	4	29	17	26
	1795	1	5	37	57	6	13	36	29	4	9	57	43
B	1796	5	28	11	37	9	25	23	38	3	20	34	49
	1797	10	7	34	43	0	24	6	53	3	1	15	6
	1798	2	16	57	48	3	22	50	8	2	11	55	23
	1799	6	26	20	54	6	21	33	23	1	22	35	40
	1800	11	5	44	0	9	20	16	38	1	3	15	57
	1801	3	15	7	5	0	18	59	52	0	13	56	14
	1802	7	24	30	11	3	17	43	7	11	24	36	31
	1803	0	3	53	16	6	16	26	22	11	5	16	48
B	1804	4	26	26	57	9	28	13	31	10	15	53	54
	1805	9	5	50	2	0	26	56	46	9	26	34	11
	1806	1	15	13	8	3	25	40	1	9	7	14	28
	1807	5	24	36	14	6	24	23	16	8	17	54	45
B	1808	10	17	9	54	10	6	10	25	7	28	31	51
	1809	2	26	33	0	1	4	53	40	7	9	12	8
	1810	7	5	56	5	4	3	36	55	6	19	52	25
	1811	11	15	19	11	7	2	20	9	6	0	32	42
B	1812	4	7	52	52	10	14	7	18	5	11	9	48
	1813	8	17	15	57	1	12	50	33	4	21	50	5
	1814	0	26	39	3	4	11	33	48	4	2	30	22
	1815	5	6	2	8	7	10	17	3	3	13	10	39
B	1816	9	28	35	49	10	22	4	12	2	23	47	45
	1817	2	7	58	55	1	20	47	27	2	4	28	2
	1818	6	17	22	0	4	19	30	42	1	15	8	19
	1819	10	26	45	6	7	18	13	57	0	25	48	36
B	1820	3	19	18	47	11	0	1	6	0	6	25	42
	1821	7	28	41	54	1	28	44	21	11	17	5	59
	1841	0	12	16	37	3	8	28	51	10	20	15	44

TABLE III. *The Moon's Mean Motions for Months and Days.*

January.

Days.	M. Motion ☽. S. ° ′ ″	Anomaly. S. ° ′ ″	Mot. ☊ S. ′ ″
1	0 13 10 35	0 13 3 54	0 3 11
2	0 26 21 10	0 26 7 48	0 6 21
3	1 9 31 45	1 9 11 42	0 9 32
4	1 22 42 20	1 22 15 36	0 12 43
5	2 5 52 55	2 5 19 30	0 15 53
6	2 19 3 30	2 18 23 24	0 19 4
7	3 2 14 5	3 1 27 18	0 22 14
8	3 15 24 40	3 14 31 12	0 25 25
9	3 28 35 15	3 27 35 6	0 28 36
10	4 11 45 50	4 10 39 0	0 31 46
11	4 24 56 25	4 23 42 53	0 34 57
12	5 8 7 0	5 6 46 47	0 38 8
13	5 21 17 35	5 19 50 41	0 41 18
14	6 4 28 10	6 2 54 35	0 14 29
15	6 17 38 45	6 15 58 29	0 47 40
16	7 0 49 20	6 29 2 23	0 50 50
17	7 13 59 55	7 12 6 17	0 54 1
18	7 27 10 30	7 25 10 11	0 57 11
19	8 10 21 5	8 8 14 5	1 0 22
20	8 23 31 41	8 21 17 59	1 3 33
21	9 6 42 16	9 4 41 53	1 6 43
22	9 19 52 51	9 17 25 47	1 9 54
23	10 3 3 26	10 0 29 41	1 13 5
24	10 16 14 1	10 13 33 35	1 16 15
25	10 29 24 36	10 26 37 29	1 19 26
26	11 12 35 11	11 9 41 23	1 22 37
27	11 25 45 46	11 22 45 17	1 25 47
28	0 8 56 21	0 5 49 11	1 28 58
29	0 22 6 56	0 18 53 5	1 32 8
30	1 5 17 31	1 1 56 59	1 35 19
31	1 18 28 6	1 15 0 53	1 38 30

February.

Days.	M. Motion ☽. S. ° ′ ″	Anomaly. S. ° ′ ″	Mot. ☊. ° ′ ″
1	2 1 38 41	1 28 4 47	1 41 40
2	2 14 49 16	2 11 8 41	1 44 51
3	2 27 59 51	2 24 12 35	1 48 2
4	3 11 10 26	3 7 16 29	1 51 12
5	3 24 21 1	3 20 20 22	1 54 23
6	4 7 31 36	4 3 24 16	1 57 34
7	4 20 42 11	4 16 28 10	2 0 44
8	5 3 52 46	4 29 32 4	2 3 55
9	5 17 3 21	5 12 35 58	2 7 5
10	6 0 13 56	5 25 39 52	2 10 16
11	6 13 24 31	6 8 43 46	2 13 27
12	6 26 35 6	6 21 47 40	2 16 37
13	7 9 45 41	7 4 51 34	2 19 48
14	7 22 56 16	7 17 55 28	2 22 59
15	8 6 6 51	8 0 59 22	2 26 9
16	8 19 17 26	8 14 3 16	2 29 20
17	9 2 28 1	8 27 7 10	2 32 31
18	9 15 38 36	9 10 11 4	2 35 41
19	9 28 49 11	9 23 14 58	2 38 52
20	10 11 59 46	10 6 18 52	2 42 3
21	10 25 10 21	10 19 22 46	2 45 13
22	11 8 20 56	11 2 26 40	2 48 24
23	11 21 31 31	11 15 30 34	2 51 34
24	0 4 42 7	11 28 34 28	2 54 45
25	0 17 52 42	0 11 38 22	2 57 56
26	1 1 3 17	0 24 42 16	3 1 6
27	1 14 13 52	1 7 46 10	3 4 17
28	1 27 24 27	1 20 50 4	3 7 28

March.

Days.	M. Motion ☽. S. ° ′ ″	Anomaly. S. ° ′ ″	Mot. ☊. ° ′ ″
1	2 10 35 2	2 3 53 57	3 10 38
2	2 23 45 37	2 16 57 51	3 13 59
3	3 6 56 12	3 0 1 45	3 17 0
4	3 20 6 47	3 13 5 39	3 20 10
5	4 3 17 22	3 26 9 33	3 23 21
6	4 16 27 57	4 9 13 27	3 26 31
7	4 29 38 32	4 22 17 21	3 29 42
8	5 12 49 7	5 5 21 15	3 32 53
9	5 25 59 42	5 18 25 9	3 36 3
10	6 9 10 17	6 1 29 3	3 39 14
11	6 22 20 52	6 14 32 57	3 42 25
12	7 5 31 27	6 27 36 51	3 45 35
13	7 18 42 2	7 10 40 45	3 48 46
14	8 1 52 37	7 23 44 39	3 51 57
15	8 15 3 12	8 6 48 33	3 55 7
16	8 28 13 47	8 19 52 27	3 58 18
17	9 11 24 22	9 2 56 21	4 1 28
18	9 24 34 57	9 16 0 15	4 4 39
19	10 7 45 32	9 29 4 9	4 7 50
20	10 20 56 7	10 12 8 3	4 11 0
21	11 4 6 42	10 25 11 57	4 14 11
22	11 17 17 17	11 8 15 51	4 17 22
23	0 0 27 52	11 21 19 45	4 20 32
24	0 13 38 27	0 4 23 39	4 23 43
25	0 26 49 2	0 17 27 32	4 26 34
26	1 9 59 37	1 0 31 26	4 30 4
27	1 23 10 12	1 13 35 20	4 33 15
28	2 6 20 47	1 26 39 14	4 36 25
29	2 19 31 22	2 9 43 8	4 39 36
30	3 2 41 57	2 22 47 2	4 42 47
31	3 15 52 33	3 5 50 56	4 45 57

In January and February of a bissextile year, subtract 1 from the number of days.

TABLE III. *The Moon's Mean Motions for Months and Days.*

April.

Days.	M. Motion ☽. S. ° ′ ″	Anomaly. S. ° ′ ″	Mot. ☊. ° ′ ″
1	3 29 3 8	3 18 54 50	4 49 8
2	4 12 13 43	4 1 58 44	4 52 19
3	4 25 24 18	4 15 2 38	4 55 29
4	5 8 34 53	4 28 6 32	4 58 40
5	5 21 45 28	5 11 10 26	5 1 51
6	6 4 56 3	5 24 14 20	5 5 1
7	6 18 6 38	6 7 18 14	5 8 12
8	7 1 17 13	6 20 22 8	5 11 23
9	7 14 27 48	7 3 26 2	5 14 33
10	7 27 38 23	7 16 29 56	5 17 44
11	8 10 48 58	7 29 33 50	5 20 54
12	8 23 59 33	8 12 37 44	5 24 5
13	9 7 10 8	8 25 41 38	5 27 16
14	9 20 20 43	9 8 45 32	5 30 26
15	10 3 31 18	9 21 49 26	5 33 37
16	10 16 41 53	10 4 53 20	5 36 48
17	10 29 52 28	10 17 57 14	5 39 58
18	11 13 3 3	11 1 1 7	5 43 9
19	11 26 13 38	11 14 5 1	5 46 20
20	0 9 24 13	11 27 8 55	5 49 30
21	0 22 34 48	0 10 12 49	5 52 41
22	1 5 45 23	0 23 16 43	5 55 51
23	1 18 55 58	1 6 20 37	5 59 2
24	2 2 6 33	1 19 24 31	6 2 13
25	2 15 17 8	2 2 28 25	6 5 23
26	2 28 27 43	2 15 32 19	6 8 34
27	3 11 38 18	2 28 36 13	6 11 45
28	3 24 48 53	3 11 40 7	6 14 55
29	4 7 59 28	3 24 44 1	6 18 6
30	4 21 10 3	4 7 47 55	6 21 17

May.

Days.	M. Motion ☽. S. ° ′ ″	Anomaly. S. ° ′ ″	Mot. ☊. ° ′ ″
1	5 4 20 38	4 20 51 49	6 24 27
2	5 17 31 13	5 3 55 43	6 27 38
3	6 0 41 48	5 16 59 37	6 30 48
4	6 13 52 23	6 0 3 31	6 33 59
5	6 27 2 59	6 13 7 25	6 37 10
6	7 10 13 34	6 26 11 19	6 40 20
7	7 23 24 9	7 9 15 13	6 43 31
8	8 6 34 44	7 22 19 7	6 46 42
9	8 19 45 19	8 5 23 1	6 49 52
10	9 2 55 54	8 18 26 55	6 53 3
11	9 16 6 29	9 1 30 49	6 56 14
12	9 29 17 4	9 14 34 42	6 59 24
13	10 12 27 32	9 27 38 36	7 2 35
14	10 25 38 14	10 10 42 30	7 5 46
15	11 8 48 49	10 23 46 24	7 8 56
16	11 21 59 24	11 6 50 18	7 12 7
17	0 5 9 59	11 19 54 12	7 15 17
18	0 18 20 34	0 2 58 6	7 18 28
19	1 1 31 9	0 16 2 0	7 21 39
20	1 14 41 44	0 29 5 54	7 24 49
21	1 27 52 19	1 12 9 48	7 28 0
22	2 11 2 54	1 25 13 42	7 31 11
23	2 24 13 29	2 8 17 36	7 34 21
24	3 7 24 4	2 21 21 30	7 37 32
25	3 20 34 39	3 4 25 24	7 40 43
26	4 3 45 14	3 17 29 18	7 43 53
27	4 16 55 49	4 0 33 12	7 47 4
28	5 0 6 24	4 13 37 6	7 50 14
29	5 13 16 59	4 26 41 0	7 53 25
30	5 26 27 34	5 9 44 54	7 56 36
31	6 9 38 9	5 22 48 48	7 59 46

June.

Days.	M. Motion ☽. S. ° ′ ″	Anomaly. S. ° ′ ″	Mot. ☊. ° ′ ″
1	6 22 48 44	6 5 52 42	8 2 57
2	7 5 59 19	6 18 56 36	8 5 8
3	7 19 9 54	7 2 0 30	8 9 18
4	8 2 20 29	7 15 4 24	8 12 29
5	8 15 31 4	7 28 8 17	8 15 40
6	8 28 41 39	8 11 12 11	8 18 50
7	9 11 52 14	8 24 16 5	8 22 1
8	9 25 2 49	9 7 19 59	8 25 11
9	10 8 13 25	9 20 23 53	8 28 22
10	10 21 24 0	10 3 27 47	8 31 33
11	11 4 34 35	10 16 31 41	8 34 43
12	11 17 45 10	10 29 35 35	8 37 54
13	0 0 55 45	11 12 39 29	8 41 5
14	0 14 6 20	11 25 43 23	8 44 5
15	0 27 16 55	0 8 47 17	8 47 26
16	1 10 27 30	0 21 51 11	8 50 37
17	1 23 38 5	1 4 55 5	8 53 47
18	2 6 48 40	1 17 58 59	8 56 58
19	2 19 59 15	2 1 2 53	9 0 8
20	3 3 9 50	2 14 6 47	9 3 19
21	3 16 20 25	2 27 10 41	9 6 30
22	3 29 31 0	3 10 14 35	9 9 40
23	4 12 41 35	3 23 18 29	9 12 51
24	4 25 52 10	4 6 22 23	9 16 2
25	5 9 2 45	4 19 26 17	9 19 12
26	5 22 13 20	5 2 30 11	9 22 23
27	6 5 23 55	5 15 34 5	9 25 34
28	6 18 34 30	5 28 37 59	9 28 44
29	7 1 45 5	6 11 41 52	9 31 55
30	7 14 55 40	6 24 45 46	9 35 6

TABLE III. *The Moon's Mean Motions for Months and Days.*

Days.	July. M. Motion ☽.	July. Anomaly.	July. Mot. ☊.
	S. ° ′ ″	S. ° ′ ″	° ′ ″
1	7 28 6 15	7 7 49 40	9 38 16
2	8 11 16 50	7 20 53 34	9 41 27
3	8 24 27 25	8 3 57 28	9 44 37
4	9 7 38 0	8 17 1 22	9 47 48
5	9 20 48 35	9 0 5 16	9 50 59
6	10 3 59 10	9 13 9 10	9 54 9
7	10 17 9 45	9 26 13 4	9 57 20
8	11 0 20 20	10 9 16 58	10 0 31
9	11 13 30 55	10 22 20 52	10 3 41
10	11 26 41 30	11 5 24 46	10 6 52
11	0 9 52 5	11 18 28 40	10 10 3
12	0 23 2 40	0 1 32 34	10 13 13
13	1 6 13 15	0 14 36 28	10 16 24
14	1 19 23 51	0 27 40 21	10 19 34
15	2 2 34 26	1 10 44 16	10 22 45
16	2 15 45 1	1 23 48 10	10 25 56
17	2 28 55 36	2 6 52 4	10 29 6
18	3 12 6 11	2 19 55 58	10 32 17
19	3 25 16 46	3 2 59 52	10 35 28
20	4 8 27 21	3 16 3 46	10 38 38
21	4 21 37 56	3 29 7 40	10 41 49
22	5 4 48 31	4 12 11 34	10 45 0
23	5 17 59 6	4 25 15 27	10 48 10
24	6 1 9 41	5 8 19 21	10 51 21
25	6 14 20 16	5 21 23 15	10 54 32
26	6 27 30 51	6 4 27 9	10 57 42
27	7 10 41 26	6 17 31 3	11 0 53
28	7 23 52 1	7 0 34 54	11 4 3
29	8 7 2 36	7 13 38 51	11 7 14
30	8 20 13 11	7 26 42 45	11 10 25
3	9 3 23 46	8 9 46 39	11 13 35

Days.	August. M. Motion ☽.	August. Anomaly.	August. Mot. ☊.
	S. ° ′ ″	S. ° ′ ″	° ′ ″
1	9 16 34 21	8 22 50 33	11 16 46
2	9 29 44 56	9 5 54 27	11 19 57
3	10 12 55 31	9 18 58 21	11 23 7
4	10 26 6 6	10 2 2 15	11 26 18
5	11 9 16 41	10 15 6 9	11 29 29
6	11 22 27 16	10 28 10 3	11 32 39
7	0 5 37 51	11 11 13 57	11 35 50
8	1 18 48 26	11 24 17 51	11 39 0
9	1 1 59 1	0 7 21 45	11 42 11
10	1 15 9 36	0 20 25 39	11 45 22
11	1 28 20 11	1 3 29 33	11 48 32
12	2 11 30 46	1 16 33 27	11 51 43
13	2 24 41 21	1 29 37 21	11 54 54
14	3 7 51 56	2 12 41 15	11 58 4
15	3 21 2 31	2 25 45 9	12 1 15
16	4 4 13 6	3 8 49 3	12 4 26
17	4 17 23 41	3 21 52 56	12 7 36
18	5 0 34 17	4 4 56 50	12 10 47
19	5 13 44 52	4 18 0 44	12 13 57
20	5 26 55 27	5 1 4 38	12 17 8
21	6 10 6 2	5 14 8 32	12 20 19
22	6 23 16 37	5 27 12 26	12 23 29
23	7 6 27 12	6 10 16 20	12 26 40
24	7 19 37 47	6 23 20 14	12 29 51
25	8 2 48 22	7 6 24 8	12 33 1
26	8 15 58 57	7 19 28 2	12 36 12
27	8 29 9 32	8 2 31 56	12 39 23
28	9 12 20 7	8 15 35 50	12 42 33
29	9 25 30 42	8 28 39 44	12 45 44
30	10 8 41 17	9 11 43 38	12 48 54
31	10 21 51 52	9 24 47 32	12 52 5

Days.	September. M. Motion ☽.	September. Anomaly.	September. Mot. ☊.
	S. ° ′ ″	S. ° ′ ″	° ′ ″
1	11 5 2 27	10 7 51 26	12 55 16
2	11 18 13 2	10 20 55 20	12 58 26
3	0 1 23 37	11 3 59 14	13 1 37
4	0 14 34 12	11 17 3 8	13 4 48
5	0 27 44 47	0 0 7 2	13 7 58
6	1 10 55 22	0 13 10 56	13 11 9
7	1 24 5 57	0 26 14 50	13 14 20
8	2 7 16 32	1 9 18 44	13 17 30
9	2 20 27 7	1 22 22 38	13 20 41
10	3 3 37 42	2 5 26 31	13 23 52
11	3 16 48 17	2 18 30 26	13 27 2
12	3 29 58 52	3 1 34 19	13 30 13
13	4 13 9 27	3 14 38 13	13 33 23
14	4 26 20 2	3 27 42 7	13 36 34
15	5 9 30 37	4 10 46 1	13 39 45
16	5 22 41 12	4 23 49 55	13 42 55
17	6 5 51 47	5 6 53 49	13 46 6
18	6 19 2 22	5 19 57 43	13 49 17
19	7 2 12 57	6 3 1 37	13 52 27
20	7 15 23 32	6 16 5 31	13 55 38
21	7 28 34 7	6 29 9 25	13 58 49
22	8 11 44 43	7 12 13 19	14 1 59
23	8 24 55 18	7 25 17 13	14 5 10
24	9 8 5 53	8 8 21 7	14 8 20
25	9 21 16 28	8 21 25 1	14 11 31
26	10 4 27 3	9 4 28 55	14 14 42
27	10 17 37 38	9 17 32 49	14 17 52
28	11 0 48 13	10 0 36 43	14 21 3
29	11 13 58 48	10 13 40 37	14 24 14
30	11 27 9 23	10 26 44 31	14 27 24

TABLE III. *The Moon's Mean Motions for Months and Days.*

October.

Days.	M. Motion ☽. S. ° ′ ″	Anomaly. S. ° ′ ″	Mot. ☊. ° ′ ″
1	0 10 19 58	11 9 48 25	14 30 35
2	0 23 30 33	11 22 52 19	14 33 46
3	1 6 41 8	0 5 56 13	14 36 56
4	1 19 51 43	0 19 0 6	14 40 7
5	2 3 2 18	1 2 4 0	14 43 17
6	2 16 12 53	1 15 7 54	14 46 28
7	2 29 23 28	1 28 11 48	14 49 39
8	3 12 34 3	2 11 15 42	14 52 49
9	3 25 44 38	2 24 19 36	14 56 0
10	4 8 55 13	3 7 23 30	14 59 11
11	4 22 5 48	3 20 27 24	15 2 21
12	5 5 16 23	4 3 31 18	15 5 32
13	5 18 26 58	4 16 35 12	15 8 43
14	6 1 37 33	4 29 39 6	15 11 53
15	6 14 48 8	5 12 43 0	15 15 4
16	6 27 58 43	5 25 46 54	15 18 15
17	7 11 9 18	6 8 50 48	15 21 25
18	7 24 19 53	6 21 54 42	15 24 36
19	8 7 30 28	7 4 58 36	15 27 46
20	8 20 41 3	7 18 2 30	15 30 57
21	9 3 51 38	8 1 6 24	15 34 8
22	9 17 2 13	8 14 10 18	15 37 18
23	10 0 12 48	8 27 14 12	15 40 29
24	10 13 23 23	9 10 18 6	15 43 40
25	10 26 33 58	9 23 22 0	15 46 50
26	11 9 44 33	10 6 25 54	15 50 1
27	11 22 55 8	10 19 29 48	15 53 12
28	0 6 5 44	11 2 33 41	15 56 22
29	0 19 16 19	11 15 37 35	15 59 33
30	1 2 26 54	11 28 41 29	16 2 43
31	1 15 37 29	0 11 45 23	16 5 54

November.

Days.	M. Motion ☽. S. ° ′ ″	Anomaly. S. ° ′ ″	Mot. ☊. ° ′ ″
1	1 28 48 4	0 24 49 17	16 9 5
2	2 11 58 39	1 7 53 11	16 12 15
3	2 25 9 14	1 20 57 5	16 15 26
4	3 8 19 49	2 4 0 59	16 18 37
5	3 21 30 24	2 17 4 53	16 21 47
6	4 4 40 59	3 0 8 47	16 24 58
7	4 17 51 34	3 13 12 41	16 28 9
8	5 1 2 9	3 26 16 35	16 31 19
9	5 14 12 44	4 9 20 29	16 34 30
10	5 27 23 19	4 22 24 23	16 37 40
11	6 10 33 54	5 5 28 17	16 40 51
12	6 23 44 29	5 18 32 11	16 44 2
13	7 6 55 4	6 1 36 5	16 47 12
14	7 20 5 39	6 14 39 59	16 50 23
15	8 3 16 14	6 27 43 53	16 53 34
16	8 16 26 49	7 10 47 47	16 56 44
17	8 29 37 24	7 23 51 41	16 59 55
18	9 12 47 59	8 6 55 35	17 3 6
19	9 25 58 34	8 19 59 29	17 6 16
20	10 9 9 9	9 3 3 23	17 9 27
21	10 22 19 44	9 16 7 16	17 12 38
22	11 5 30 19	9 29 11 10	17 15 48
23	11 18 40 54	10 12 15 4	17 18 59
24	0 1 51 29	10 25 18 58	17 22 9
25	0 15 2 4	11 8 22 52	17 25 20
26	0 28 12 39	11 21 26 46	17 28 31
27	1 11 23 14	0 4 30 40	17 31 41
28	1 24 33 49	0 17 34 34	17 34 52
29	2 7 44 24	1 0 38 28	17 38 3
30	2 20 54 59	1 13 42 22	17 41 13

December.

Days.	M. Motion ☽. S. ° ′ ″	Anomaly. S. ° ′ ″	Mot. ☊. ° ′ ″
1	3 4 5 34	1 26 46 16	17 44 24
2	3 17 16 10	2 9 50 10	17 47 35
3	4 0 26 45	2 22 54 4	17 50 45
4	4 13 37 20	3 5 57 58	17 53 56
5	4 26 47 55	3 19 1 52	17 57 6
6	5 9 58 30	4 2 5 46	18 0 17
7	5 23 9 5	4 15 9 40	18 3 28
8	6 6 19 40	4 28 13 34	18 6 38
9	6 19 30 15	5 11 17 28	18 9 49
10	7 2 40 50	5 24 21 22	18 13 0
11	7 15 51 25	6 7 25 16	18 16 10
12	7 29 2 0	6 20 29 10	18 19 21
13	8 12 12 35	7 3 33 4	18 22 32
14	8 25 23 10	7 16 36 58	18 25 42
15	9 8 33 45	7 29 40 51	18 28 53
16	9 21 44 20	8 12 44 45	18 32 3
17	10 4 54 55	8 25 48 39	18 35 14
18	10 18 5 30	9 8 52 33	18 38 25
19	11 1 16 5	9 21 56 27	18 41 35
20	11 14 26 40	10 5 0 21	18 44 46
21	11 27 37 15	10 18 4 15	18 47 56
22	0 10 47 50	11 1 8 9	18 51 7
23	0 23 58 25	11 14 12 3	18 54 18
24	1 7 9 0	11 27 15 57	18 57 29
25	1 20 19 35	0 10 19 51	19 0 39
26	2 3 30 10	0 23 23 45	19 3 50
27	2 16 40 45	1 6 27 39	19 7 0
28	2 29 51 20	1 19 31 33	19 10 11
29	3 13 1 55	2 2 35 27	19 13 22
30	3 26 12 30	2 15 39 21	19 16 32
31	4 9 23 5	2 28 43 15	19 19 43

TABLE IV. *The Moon's Mean Motions for Hours, Minutes, and Seconds.*

H	M. Mo. ☽.			Anomaly.			Mot. ☊.			M. Mo. ☽.			Anomaly.			Mot. ☊.	
	°	'	"	°	'	"	'	"									
'	'	"	'''	'	"	'''	"	'''	'	'	"	'''	'	"	'''	'	"
"	"	'''	''''	"	'''	''''	'''	''''	"	"	'''	''''	"	'''	''''	"	'''
1	0	32	56	0	32	39	0	8	31	17	1	10	16	52	32	4	6
2	1	5	53	1	5	20	0	16	32	17	34	7	17	25	13	4	14
3	1	38	49	1	37	59	0	24	33	18	7	3	17	58	0	4	22
4	2	11	46	2	10	39	0	32	34	18	40	0	18	30	31	4	30
5	2	44	42	2	43	18	0	40	35	19	12	55	19	3	12	4	37
6	3	17	39	3	15	59	0	48	36	19	45	53	19	35	50	4	47
7	3	50	35	3	48	38	0	56	37	20	18	48	20	8	30	4	54
8	4	23	32	4	21	18	1	4	38	20	51	47	20	41	10	4	54
9	4	56	29	4	53	58	1	12	39	21	24	41	21	13	49	5	1
10	5	29	25	5	26	38	1	19	40	21	57	37	21	46	30	5	12
11	6	2	21	5	59	17	1	27	41	22	30	36	22	19	12	5	25
12	6	35	18	6	31	57	1	35	42	23	3	30	22	51	48	5	31
13	7	8	14	7	4	37	1	43	43	23	36	30	23	24	30	5	42
14	7	41	10	7	37	16	1	51	44	24	9	24	23	57	6	5	50
15	8	14	7	8	9	56	1	59	45	24	42	19	24	29	48	6	0
16	8	47	3	8	42	36	2	7	46	25	15	18	25	2	30	6	6
17	9	20	0	9	15	16	2	15	47	25	48	12	25	35	6	6	13
18	9	52	56	9	47	55	2	23	48	26	21	12	26	7	48	6	23
19	10	25	53	10	20	35	2	31	49	26	54	6	26	40	30	6	30
20	10	58	49	10	53	15	2	39	50	27	27	0	27	13	6	6	36
21	11	31	46	11	25	55	2	47	51	28	0	0	27	45	48	6	48
22	12	4	42	11	58	34	2	55	52	28	32	55	28	18	30	6	54
23	12	37	39	12	31	14	3	3	53	29	5	54	28	51	6	7	0
24	13	10	35	13	3	54	3	11	54	29	38	48	29	23	48	7	6
25	13	43	32	13	36	34	3	19	55	30	11	48	29	56	24	7	18
26	14	16	28	14	9	13	3	26	56	30	44	42	30	29	6	7	24
27	14	49	24	14	41	53	3	34	57	31	17	36	31	1	48	7	30
28	15	22	21	15	14	33	3	42	58	31	50	36	31	34	24	7	48
29	15	55	17	15	47	13	3	50	59	32	23	30	32	7	6	7	48
30	16	28	14	16	19	52	3	58	60	32	56	39	32	39	48	7	56

TABLES OF THE MOON'S EQUATIONS.

1. *Annual Equation of the Moon's Node.*

Argument. Sun's Mean Anomaly.

S.	+ 0		+ 1		+ 2		+ 3		+ 4		+ 5		S.
°	'	"	'	"	'	"	'	"	'	"	'	"	°
0	0	0	4	30	7	52	9	12	8	4	4	42	30
5	0	47	5	10	8	15	9	11	7	38	3	58	25
10	1	34	5	48	8	34	9	6	7	9	3	13	20
15	2	19	6	23	8	49	8	56	6	37	2	26	15
20	3	4	6	56	9	1	8	43	6	1	1	38	10
25	3	48	7	26	9	8	8	25	5	23	0	49	5
30	4	30	7	52	9	12	8	4	4	42	0	0	0
S.	—11		—10		—9		—8		—7		—6		S.

Annual Equation of Moon's Mean Anomaly.

Argument. Sun's Mean Anomaly.

S.	+ 0		+ 1		+ 2		+ 3		+ 4		+ 5		S.
°	'	"	'	"	'	"	'	"	'	"	'	"	°
0	0	0	10	37	18	33	21	42	19	1	11	5	30
5	1	51	12	11	19	27	21	39	18	1	9	23	25
10	3	40	13	41	20	13	21	27	16	53	7	36	20
15	5	29	15	4	20	49	21	5	15	37	5	45	15
20	7	15	16	21	21	16	20	34	14	13	3	52	10
25	8	58	17	31	21	34	19	52	12	42	1	56	5
30	10	37	18	33	21	42	19	1	11	5	0	0	0
S.	—11		—10		—9		—8		—7		—6		S.

I. *For the Moon's Longitude.*

Argument. Sun's Mean Anomaly.

S.	+ 0		+ 1		+ 2		+ 3		+ 4		+ 5		S.
°	'	"	'	"	'	"	'	"	'	"	'	"	°
0	0	0	5	27	9	31	11	9	9	47	5	42	30
5	0	57	6	15	9	59	11	8	9	16	4	49	25
10	1	53	7	1	10	23	11	1	8	41	3	54	20
15	2	49	7	44	10	41	10	50	8	2	2	58	15
20	3	43	8	23	10	55	10	34	7	19	1	59	10
25	4	36	8	59	11	4	10	13	6	32	1	0	5
30	5	27	9	31	11	9	9	47	5	42	0	0	0
S.	—11		—10		—9		—8		—7		—6		S.

II. *For the Moon's Longitude.*

Arg. II. 2 ☽ from ⊙ + Arg. I.

S.	— 0	— 1	— 2	S.
S.	+ 6	+ 7	+ 8	S.
°	"	"	"	°
0	0	28	48	30
5	5	32	51	25
10	10	36	52	20
15	14	39	54	15
20	19	43	55	10
25	24	46	56	5
30	28	48	56	0
S.	+11	+10	+ 9	S.
S.	—5	—4	—3	S.

III. *For the Moon's Longitude.*

Arg. III. 2 ☽ from ⊙ — Arg. I.

S.	— 0		— 1		— 2		S.
S.	+ 6		+ 7		+ 8		S.
°	'	"	'	"	'	"	°
0	0	0	0	38	1	5	30
5	0	7	0	43	1	8	25
10	0	13	0	48	1	11	20
15	0	19	0	53	1	13	15
20	0	26	0	58	1	14	10
25	0	32	1	2	1	15	5
30	0	38	1	5	1	15	0
S.	+11		+10		+ 9		S.
S.	—5		—4		—3		S.

IV. *For the Moon's Longitude.*

Arg. IV. 2 ☽ from ⊙ + ☽'s Mean Anomaly.

S.	+ 0	+ 1	+ 2	S.
S.	—6	—7	—8	S.
°	"	"	"	°
0	0	29	50	30
5	5	33	52	25
10	10	37	54	20
15	15	41	56	15
20	20	44	57	10
25	24	47	57	5
30	29	50	58	0
S.	—11	—10	— 9	S.
S.	+ 5	+ 4	+ 3	S.

V. *For the Moon's Longitude. Evection.*

Argument V.
2 ☽ from ⊙ — ☽'s Mean Anomaly.

S.	— 0			— 1			— 2			— 3			— 4			— 5			S.
°	°	'	"	°	'	"	°	'	"	°	'	"	°	'	"	°	'	"	°
0	0	0	0	0	39	44	1	9	11	1	20	28	1	10	12	0	40	45	30
1	0	1	23	0	40	56	1	9	53	1	20	29	1	9	30	0	39	31	29
2	0	2	46	0	42	7	1	10	34	1	20	28	1	8	46	0	38	16	28
3	0	4	9	0	43	18	1	11	14	1	20	25	1	8	1	0	37	0	27
4	0	5	32	0	44	27	1	11	52	1	20	21	1	7	15	0	35	45	26
5	0	6	55	0	45	36	1	12	29	1	20	16	1	6	28	0	34	27	25
6	0	8	17	0	46	45	1	13	5	1	20	9	1	5	40	0	33	10	24
7	0	9	40	0	47	52	1	13	39	1	20	1	1	4	50	0	31	52	23
8	0	11	2	0	48	59	1	14	12	1	19	51	1	3	59	0	30	33	22
9	0	12	24	0	50	4	1	14	44	1	19	40	1	3	7	0	29	14	21
10	0	13	46	0	51	9	1	15	15	1	19	27	1	2	13	0	27	54	20
11	0	15	8	0	52	13	1	15	44	1	19	13	1	1	19	0	26	34	19
12	0	16	30	0	53	16	1	16	11	1	18	57	1	0	23	0	25	13	18
13	0	17	51	0	54	18	1	16	38	1	18	40	0	59	26	0	23	51	17
14	0	19	12	0	55	19	1	17	3	1	18	21	0	58	28	0	22	29	16
15	0	20	32	0	56	19	1	17	26	1	18	1	0	57	29	0	21	7	15
16	0	21	52	0	57	18	1	17	48	1	17	40	0	56	29	0	19	44	14
17	0	23	12	0	58	16	1	18	9	1	17	17	0	55	28	0	18	22	13
18	0	24	31	0	59	13	1	18	29	1	16	53	0	54	26	0	16	58	12
19	0	25	50	1	0	9	1	18	47	1	16	27	0	53	22	0	15	34	11
20	0	27	9	1	1	4	1	19	3	1	16	0	0	52	18	0	14	10	10
21	0	28	27	1	1	58	1	19	18	1	15	31	0	51	13	0	12	46	9
22	0	29	44	1	2	51	1	19	32	1	15	1	0	50	7	0	11	22	8
23	0	31	1	1	3	42	1	19	44	1	14	30	0	49	0	0	9	57	7
24	0	32	18	1	4	33	1	19	55	1	13	57	0	47	51	0	8	32	6
25	0	33	34	1	5	22	1	20	4	1	13	23	0	46	42	0	7	7	5
26	0	34	49	1	6	10	1	20	12	1	12	47	0	45	33	0	5	42	4
27	0	36	4	1	6	57	1	20	18	1	12	10	0	44	22	0	4	16	3
28	0	37	18	1	7	43	1	20	23	1	11	32	0	43	10	0	2	51	2
29	0	38	31	1	8	28	1	20	26	1	10	53	0	41	58	0	1	25	1
30	0	39	44	1	9	11	1	20	28	1	10	12	0	40	45	0	0	0	0
S.	+ 11			+ 10			+ 9			+ 8			+ 7			+ 6			S.

VI. *For the ☽'s Longitude.*

Argument.

Arg. V. + Arg. I.

S.	+ 0	+ 1	+ 2	S.
S.	— 6	— 7	— 8	S.
°	′ ″	′ ″	′ ″	°
1	0 0	1 2	1 47	30
5	0 11	1 11	1 52	25
10	0 21	1 19	1 56	20
15	0 32	1 27	1 59	15
20	0 42	1 35	2 2	10
25	0 52	1 41	2 3	5
30	1 2	1 47	2 3	0
S.	—11	—10	— 9	S.
S.	+ 5	+ 4	+ 3	S.

VII. *For the ☽'s Longitude.*

Argument.

Arg. V. — Arg. I.

S.	+ 0	+ 1	+ 2	S.
S.	— 6	— 7	— 8	S.
°	″	″	″	°
0	0	23	40	30
5	4	27	42	25
10	8	30	44	20
15	12	33	45	15
20	16	36	46	10
25	20	38	46	5
30	23	40	47	0
S.	—11	—10	— 9	S.
S.	+ 5	+ 4	+ 3	S.

VIII. *For the ☽'s Longitude.*

Argument.

☽'s Mean Anom. — Arg. 1.

S.	+ 0	+ 1	+ 2	S.
S.	— 6	— 7	— 8	S.
°	″	″	″	°
0	0	21	36	30
5	4	24	38	25
10	7	27	40	20
15	11	30	41	15
20	14	32	41	10
25	18	34	42	5
30	21	36	42	0
S.	—11	—10	— 9	S.
S.	+ 5	+ 4	+ 3	S.

IX. *For the Moon's Longitude.*

Argument IX.

Mean Dist. ☽ from ☉ — ☽'s Mean Anom.

S.	— 0	— 1	∓ 2	+ 3	+ 4	+ 5	S.
°	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	°
0	0 0	0 38	0 30	0 23	1 9	1 1	30
5	8	41	23	33	1 12	0 54	25
10	16	42	16	42	1 14	0 45	20
15	23	41	7	51	1 13	0 35	15
20	29	39	+ 3	58	1 11	23	10
25	34	35	13	1 5	1 7	12	5
30	38	30	23	1 9	1 1	0 0	0
S.	+11	+10	± 9	— 8	— 7	— 6	S.

X. *For the Moon's Longitude.*

Argument X.

Mean Long. ☊. — True Long. ☉.

S.	+ 0	+ 1	+ 2	S.
S.	+ 6	+ 7	+ 8	S.
°	″	′ ″	″	°
0	0	0 52	52	30
5	10	0 57	46	25
10	21	0 59	39	20
15	30	1 0	30	15
20	39	1 0	21	10
25	46	0 57	10	5
30	52	0 52	0	0
S.	—11	—10	— 9	S.
S.	— 5	— 4	— 3	S.

XI. *For the Moon's Longitude. Equation of the Centre.*

Argument IX. Moon's Correct Anomaly.

S.	— 0	— 1	— 2	— 3	— 4	— 5	S.
°	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	°
0	0 0 0	2 58 30	5 16 21	6 17 38	5 38 46	3 20 56	30
1	0 6 11	3 3 59	5 19 48	6 18 2	5 35 41	3 14 57	29
2	0 12 22	3 9 24	5 23 10	6 18 18	5 32 28	3 8 54	28
3	0 18 32	3 14 46	5 26 26	6 18 28	5 29 8	3 2 47	27
4	0 24 42	3 20 5	5 29 38	6 18 32	5 25 42	2 56 36	26
5	0 30 52	3 25 22	5 32 43	6 18 28	5 22 10	2 50 21	25
6	0 37 2	3 30 35	5 35 43	6 18 17	5 18 30	2 44 2	24
7	0 43 11	3 35 45	5 38 38	6 17 59	5 14 44	2 37 40	23
8	0 49 19	3 40 51	5 41 27	6 17 34	5 10 52	2 31 14	22
9	0 55 26	3 45 54	5 44 11	6 17 3	5 6 53	2 24 45	21
10	1 1 33	3 50 54	5 46 48	6 16 24	5 2 48	2 18 13	20
11	1 7 39	3 55 50	5 49 20	6 15 38	4 58 37	2 11 37	19
12	1 13 44	4 0 42	5 51 45	6 14 45	4 54 19	2 4 59	18
13	1 19 47	4 5 30	5 54 5	6 13 45	4 49 55	1 58 18	17
14	1 25 50	4 10 15	5 56 19	6 12 38	4 45 25	1 51 34	16
15	1 31 51	4 14 56	5 58 27	6 11 24	4 40 50	1 44 48	15
16	1 37 51	4 19 33	6 0 28	6 10 3	4 36 8	1 37 59	14
17	1 43 49	4 24 5	6 2 24	6 8 34	4 31 20	1 31 8	13
18	1 49 45	4 28 34	6 4 13	6 6 59	4 26 27	1 24 16	12
19	1 55 41	4 32 58	6 5 56	6 5 16	4 21 29	1 17 21	11
20	2 1 34	4 37 18	6 7 32	6 3 27	4 16 24	1 10 25	10
21	2 7 25	4 41 33	6 9 2	6 1 30	4 11 14	1 3 27	9
22	2 13 15	4 45 44	6 10 26	5 59 27	4 5 59	0 56 27	8
23	2 19 2	4 49 50	6 11 43	5 57 16	4 0 38	0 49 27	7
24	2 24 48	4 53 52	6 12 54	5 54 58	3 55 13	0 42 25	6
25	2 30 31	4 57 49	6 13 58	5 52 34	3 49 42	0 35 22	5
26	2 36 12	5 1 41	6 14 55	5 50 2	3 44 6	0 28 19	4
27	2 41 50	5 5 29	6 15 46	5 47 23	3 38 26	0 21 15	3
28	2 47 26	5 9 11	6 16 30	5 44 38	3 32 40	0 14 10	2
29	2 53 0	5 12 49	6 17 7	5 41 46	3 26 51	0 7 5	1
30	2 58 30	5 16 21	6 17 38	5 38 46	3 20 56	0 0 0	0
S.	+ 11	+ 10	+ 9	+ 8	+ 7	+ 6	S.

XII. *For the ☽'s Long. Variation.*

Argument XII.

☽'s equated Long. — ☉'s true Long.

S.	+ 0		+ 1		± 2		— 3		— 4		— 5		S.
°	′	″	′	″	′	″	′	″	′	″	′	″	°
0	0	0	30	9	29	6	2	2	32	28	31	55	30
1	1	14	30	43	28	26	3	16	33	3	31	15	29
2	2	27	31	15	27	44	4	30	33	36	30	33	28
3	3	40	31	44	27	0	5	43	34	6	29	48	27
4	4	53	32	11	26	13	6	57	34	35	29	2	26
5	6	6	32	36	25	25	8	10	35	1	28	13	25
6	7	18	32	58	24	35	9	22	35	24	27	22	24
7	8	30	33	18	23	43	10	34	35	45	26	29	23
8	9	41	33	35	22	49	11	46	36	3	25	35	22
9	10	51	33	49	21	53	12	56	36	19	24	38	21
10	12	0	34	1	20	56	14	6	36	32	23	40	20
11	13	9	34	11	19	57	15	15	36	43	22	40	19
12	14	16	34	18	18	57	16	23	36	51	21	39	18
13	15	23	34	22	17	55	17	29	36	57	20	35	17
14	16	28	34	23	16	52	18	35	36	59	19	30	16
15	17	31	34	22	15	47	19	39	37	0	18	24	15
16	18	34	34	19	14	41	20	42	36	57	17	17	14
17	19	35	34	13	13	34	21	44	36	53	16	8	13
18	20	35	34	4	12	26	22	44	36	45	14	58	12
19	21	33	33	53	11	17	23	43	36	35	13	47	11
20	22	30	33	39	10	8	24	40	36	22	12	36	10
21	23	24	33	23	8	57	25	35	36	6	11	23	9
22	24	17	33	4	7	45	26	29	35	48	10	9	8
23	25	8	32	43	6	34	27	21	35	28	8	54	7
24	25	57	32	19	5	21	28	10	35	5	7	39	6
25	26	45	31	53	4	8	28	59	34	39	6	24	5
26	27	30	31	24	2	54	29	45	34	11	5	7	4
27	28	13	30	53	1	41	30	29	33	41	3	51	3
28	28	54	30	19	+0	27	31	10	33	8	2	34	2
29	29	32	29	44	0	47	31	50	32	32	1	17	1
30	30	9	29	6	—2	2	32	28	31	55	0	0	0
S.	— 11		— 10		∓ 9		+ 8		+ 7		+ 6		S.

XIII. *For the ☽'s Long.*

Argument XIII.

Doub. eq. dist. ☽ from ☊. — ☽ C. An.

S.	+ 0		+ 1		+ 2		S.
S.	— 6		— 7		— 8		S.
°	′	″	′	″	′	″	°
0	0	0	0	42	1	13	30
5	0	7	0	48	1	16	25
10	0	15	0	54	1	19	20
15	0	22	0	59	1	21	15
20	0	29	1	4	1	23	10
25	0	35	1	9	1	24	5
30	0	42	1	13	1	24	0
S.	—11		—10		— 9		S.
S.	+ 5		+ 4		+ 3		S.

Equation of the Equinoctial Points.

Argument.

Mean Long. Moon's ☊.

S.	— 0	— 1	— 2	S.
S.	+ 6	+ 7	+ 8	S.
°	″	″	″	°
0	0	9	16	30
5	1	10	16	25
10	3	12	17	20
15	5	13	17	15
20	6	14	18	10
25	8	15	18	5
30	9	16	18	0
S.	+11	+10	+ 9	S.
S.	—5	—4	—3	S

XIV. *For the ☽'s Long. Red.*

Argument. XIV.

Long. ☽ in Orb. — C. Long. ☊.

S.	— 0		— 1		— 2		S.
S.	— 6		— 7		— 8		S.
°	′	″	′	″	′	″	°
0	0	0	5	53	5	53	30
1	0	14	6	0	5	46	29
2	0	28	6	6	5	38	28
3	0	43	6	12	5	30	27
4	0	57	6	18	5	21	26
5	1	11	6	23	5	12	25
6	1	25	6	28	5	3	24
7	1	39	6	32	4	53	23
8	1	52	6	36	4	43	22
9	2	6	6	39	4	33	21
10	2	19	6	41	4	22	20
11	2	33	6	44	4	11	19
12	2	46	6	45	4	0	18
13	2	59	6	47	3	48	17
14	3	11	6	47	3	36	16
15	3	24	6	48	3	24	15
16	3	36	6	47	3	11	14
17	3	48	6	47	2	59	13
18	4	0	6	45	2	46	12
19	4	11	6	44	2	33	11
20	4	22	6	41	2	19	10
21	4	33	6	39	2	6	9
22	4	43	6	36	1	52	8
23	4	53	6	32	1	39	7
24	5	3	6	28	1	25	6
25	5	12	6	23	1	11	5
26	5	21	6	18	0	57	4
27	5	30	6	12	0	43	3
28	5	38	6	6	0	28	2
29	5	46	6	0	0	14	1
30	5	53	5	53	0	0	0
S.	+11		+10		+ 9		S.
S.	+ 5		+ 4		+ 3		S.

TABLES FOR FINDING THE MOON'S LATITUDE.

I. *For the Moon's Latitude.*

Argument I.

Long. ☽ in orbit — Cor. Long. ☊

S.	+ 0			+ 1			+ 2			S.
S.	— 6			— 7			— 8			S.
°	°	′	″	°	′	″	°	′	″	°
0	0	0	0	2	34	18	4	27	23	30
1	0	5	23	2	38	56	4	30	2	29
2	0	10	46	2	43	32	4	32	37	28
3	0	16	9	2	48	5	4	35	6	27
4	0	21	31	2	52	35	4	37	31	26
5	0	26	53	2	57	1	4	39	50	25
6	0	32	15	3	1	24	4	42	4	24
7	0	37	36	3	5	44	4	44	14	23
8	0	42	56	3	10	1	4	46	17	22
9	0	48	16	3	14	14	4	48	16	21
10	0	53	35	3	18	23	4	50	9	20
11	0	58	52	3	22	29	4	51	58	19
12	1	4	9	3	26	32	4	53	40	18
13	1	9	24	3	30	30	4	55	18	17
14	1	14	39	3	34	25	4	56	50	16
15	1	19	51	3	38	16	4	58	16	15
16	1	25	3	3	42	2	4	59	38	14
17	1	30	13	3	45	45	5	0	53	13
18	1	35	21	3	49	24	5	2	3	12
19	1	40	27	3	52	58	5	3	8	11
20	1	45	32	3	56	28	5	4	7	10
21	1	50	35	3	59	54	5	5	0	9
22	1	55	35	4	3	16	5	5	48	8
23	2	0	34	4	6	33	5	6	31	7
24	2	5	30	4	9	45	5	7	7	6
25	2	10	25	4	12	53	5	7	38	5
26	2	15	16	4	15	57	5	8	4	4
27	2	20	6	4	18	55	5	8	23	3
28	2	24	52	4	21	49	5	8	38	2
29	5	29	36	4	24	38	5	8	46	1
30	2	34	18	4	27	23	5	8	49	0
S.	— 11			— 10			— 9			S.
S.	+ 5			+ 4			+ 3			S.

II. *For the Moon's Latitude.*

Argument II.

Double Dist. ☽ in Orb. from ⊙ — Arg. I.

S.	+ 0		+ 1		+ 2		S.
S.	— 6		— 7		— 8		S.
°	′	″	′	″	′	″	°
0	0	0	4	24	7	38	30
1	0	9	4	32	7	42	29
2	0	18	4	40	7	46	28
3	0	28	4	48	7	51	27
4	0	37	4	55	7	55	26
5	0	46	5	3	7	59	25
6	0	55	5	11	8	3	24
7	1	4	5	18	8	6	23
8	1	14	5	25	8	10	22
9	1	23	5	32	8	13	21
10	1	32	5	40	8	16	20
11	1	41	5	47	8	20	19
12	1	50	5	54	8	22	18
13	1	59	6	0	8	25	17
14	2	8	6	7	8	28	16
15	2	17	6	14	8	30	15
16	2	26	6	20	8	33	14
17	2	34	6	26	8	35	13
18	2	43	6	33	8	37	12
19	2	52	6	39	8	39	11
20	3	1	6	45	8	40	10
21	3	9	6	51	8	42	9
22	3	18	6	56	8	43	8
23	3	26	7	2	8	44	7
24	3	35	7	7	8	45	6
25	3	43	7	13	8	46	5
26	3	52	7	18	8	47	4
27	4	0	7	23	8	48	3
28	4	8	7	28	8	48	2
29	4	16	7	33	8	48	1
30	4	24	7	38	8	48	0
S.	—11		—10		— 9		S.
S.	+ 5		+ 4		+ 3		S.

III. *For the Moon's Latitude.*

Argument.

Arg. I. — ☽'s Mean Anom.

S.	— 0	— 1	— 2	S.
S.	+ 6	+ 7	+ 8	S.
°	"	"	"	°
0	0	9	15	30
10	3	11	16	20
20	6	13	17	10
30	9	15	18	0
S.	+11	+10	+ 9	S.
S.	— 5	— 4	— 3	S.

IV. *For the Moon's Latitude.*

Argument.

Arg. III. — ☽'s Mean Anom.

S.	— 0	— 1	— 2	S.
S.	+ 6	+ 7	+ 8	S.
°	"	"	"	°
0	0	13	22	30
10	4	16	24	20
20	9	19	25	10
30	13	22	25	0
S.	+11	+10	+ 9	S.
S.	— 5	— 4	— 3	S.

V. *For the Moon's Latitude.*

Argument.

Arg. IV. — ☽'s Mean Anom.

S.	+ 0	+ 1	+ 2	S.
S.	— 6	— 7	— 8	S.
°	"	"	"	°
0	0	8	14	30
10	3	10	15	20
20	5	12	16	10
30	8	14	16	0
S.	—11	—10	— 9	S.
S.	+ 5	+ 4	+ 3	S.

FOR THE MOON'S EQUATORIAL PARALLAX.

I. Argument V. of Longitude, viz. 2 ☽ from ⊙ — ☽'s Mean Anomaly.

S.	— 0	— 1	— 2	+ 3	+ 4	+ 5	S.
°	"	"	"	"	"	"	°
0	37	32	19	0	18	32	30
5	37	30	16	3	21	34	25
10	36	28	13	6	24	35	20
15	36	26	10	9	26	36	15
20	35	24	7	13	29	37	10
25	34	21	3	16	31	37	5
30	32	19	0	18	32	38	0
S.	—11	—10	— 9	+ 8	+ 7	+ 6	S.

II. Argument XI. of Longitude, viz. correct Anomaly of ☽.

S.	+ 0		+ 1		+ 2		+ 3		+ 4		+ 5		S.
°	'	"	'	"	'	"	'	"	'	"	'	"	°
0	54	13	54	33	55	32	57	1	58	39	59	59	30
5	54	14	54	41	55	46	57	17	58	55	60	8	25
10	54	15	54	49	55	59	57	34	59	10	60	15	20
15	54	18	54	58	56	14	57	51	59	24	60	21	15
20	54	22	55	9	56	29	58	7	59	36	60	25	10
25	54	27	55	20	56	45	58	24	59	48	60	28	5
30	54	33	55	32	57	1	58	39	59	59	60	29	0
S.	+ 11		+ 10		+ 9		+ 8		+ 7		+ 6		S.

III. Argument XII. of Longitude, viz. ☽'s equated Long. — ⊙'s true Long.

S.	+ 0	± 1	— 2	— 3	± 4	+ 5	S.
°	"	"	"	"	"	"	°
0	25	12	14	26	13	14	30
5	25	8	17	25	8	18	25
10	24	+ 3	20	24	— 4	21	20
15	22	— 1	23	22	+ 0	24	15
20	19	5	24	19	5	25	10
25	16	10	25	16	10	27	5
30	12	14	26	13	14	27	0
S.	+ 11	± 10	— 9	— 8	∓ 7	+ 6	S.

TABLE *for finding the Moon's Diameter.*

Argument. Equatorial Parallax.

Equat. Par.	☽'s Diam.	Equat. Par.	☽'s Diam.
′ ″	′ ″	′ ″	′ ″
54 0	29 26	58 0	31 37
54 10	29 31	58 10	31 42
54 20	29 37	58 20	31 47
54 30	29 42	58 30	31 53
54 40	29 48	58 40	31 58
54 50	29 53	58 50	32 4
55 0	29 58	59 0	32 9
55 10	30 4	59 10	32 15
55 20	30 9	59 20	32 20
55 30	30 15	59 30	32 26
55 40	30 20	59 40	32 31
55 50	30 26	59 50	32 36
56 0	30 31	60 0	32 42
56 10	30 37	60 10	32 47
56 20	30 42	60 20	32 53
56 30	30 47	60 30	32 58
56 40	30 53	60 40	33 4
56 50	30 58	60 50	33 9
57 0	31 4	61 0	33 15
57 10	31 9	61 10	33 20
57 20	31 15	61 20	33 26
57 30	31 20	61 30	33 31
57 40	31 26	61 40	33 36
57 50	31 31	61 50	33 42
58 0	31 37	62 0	33 47

TABLE *for the reduction of lat. and Hor. Par. for Ellipticity.* $\frac{1}{300}$

Lat.	Reduct. of Lat.	Red. Horizontal 53′	Hor. Par. Par. 57′	61′
°	"	"	"	"
0	0. 0.0	0.0	0.0	0.0
2	0.47.9	0.0	0.0	0.0
4	1.35.5	0.1	0.1	0.1
6	2.22.7	0.1	0.1	0.1
8	3. 9.2	0.2	0.2	0.2
10	3.54.3	0.3	0.3	0.4
12	4.39.3	0.5	0.5	0.5
14	5.22.4	0.6	0.7	0.7
16	6. 3.9	0.8	0.9	0.9
18	6.43.7	1.0	1.1	1.2
20	7.21.5	1.2	1.3	1.4
22	7.57.2	1.5	1.6	1.7
24	8.30.7	1.8	1.9	2.0
26	9. 1.6	2.0	2.2	2.3
28	9.29.9	2.3	2.5	2.7
30	9.55.4	2.7	2.9	3.1
32	10.18.1	3.0	3.2	3.4
34	10.37.8	3.3	3.6	3.8
36	10.54.3	3.7	3.9	4.2
38	11. 7.7	4.0	4.3	4.6
40	11.17.8	4.4	4.7	5.0
42	11.24.7	4.7	5.1	5.5
44	11.28.2	5.1	5.5	5.9
46	11.28.4	5.5	5.9	6.3
48	11.25.1	5.9	6.3	6.7
50	11.18.6	6.2	6.7	7.2
52	11. 8.8	6.6	7.1	7.6
54	10.55.6	6.9	7.5	8.0
56	10.39.3	7.3	7.8	8.4
58	10.19.9	7.6	8.2	8.8
60	9.57.4	7.9	8.5	9.1
62	9.32.0	8.3	8.9	9.5
64	9. 3.8	8.6	9.2	9.9
66	8.32.9	8.8	9.5	10.2
68	7.59.6	9.1	9.8	10.5
70	7.23.8	9.4	10.1	10.8
72	6.45.9	9.6	10.3	11.0
74	6. 6.0	9.8	10.5	11.3
76	5.24.3	10.0	10.7	11.5
78	4.41.0	10.1	10.9	11.7
80	3.56.3	10.3	11.1	11.8
82	3.10.4	10.4	11.2	12.0
84	2.23.7	10.5	11.3	12.1
86	1.36.2	10.5	11.3	12.1
88	0.48.2	10.6	11.4	12.2
90	0. 0.0	10.6	11.4	12.2

The argument of this Table is the latitude of the place in the left hand column, and the equatorial parallax on the head; in the angle of meeting is the reduction of parallax; and in the same line in the second column is the reduction of the latitude.

TABLES FOR FINDING THE MOON'S HOURLY MOTION IN LONGITUDE AND LATITUDE.

I. *For the Moon's Hourly Motion in Long.*

Arg. V. of Long. viz. 2 ☽ from ☉ — ☽'s Mean Anomaly.

S.	− 0	− 1	− 2	+ 3	+ 4	+ 5	S.
°	″	″	″	″	″	″	°
0	41	36	21	0	21	37	30
5	41	34	18	3	24	38	25
10	41	32	15	7	27	40	20
15	40	30	11	10	30	41	15
20	39	27	8	14	32	42	10
25	38	24	4	17	35	42	5
30	36	21	0	21	37	43	0
S.	− 11	− 10	− 9	+ 8	+ 7	+ 6	S.

II. *For the Moon's Hourly Motion in Long.*

Arg. XI. of Long. viz. the correct Anomaly of the ☽.

S.	+ 0		+ 1		+ 2		+ 3		+ 4		+ 5		S.
°	′	″	′	″	′	″	′	″	′	″	′	″	°
0	29	35	29	57	31	2	32	42	34	36	36	10	30
5	29	35	30	5	31	17	33	0	34	54	36	21	25
10	29	37	30	15	31	32	33	19	35	11	36	30	20
15	29	40	30	25	31	49	33	39	35	28	36	38	15
20	29	45	30	36	32	6	33	58	35	43	36	43	10
25	29	50	30	49	32	23	34	17	35	57	36	46	5
30	29	57	31	2	32	42	34	36	36	10	36	48	0
S.	+ 11		+ 10		+ 9		+ 8		+ 7		+ 6		S.

III. *For the Moon's Hourly Motion in Long.*

Arg. XII. of Long. viz. ☽'s equat. Long.—☉'s true Long.

S.	+ 0	± 1	− 2	− 3	∓ 4	+ 5	S.
°	″	″	″	″	″	″	°
0	40	19	21	40	20	21	30
5	39	13	26	39	13	27	25
10	37	+ 6	31	37	− 7	32	20
15	34	− 1	35	34	+ 0	36	15
20	30	8	38	30	7	39	10
25	25	15	39	25	14	41	5
30	19	31	40	20	21	42	0
S.	+ 11	± 10	− 9	− 8	∓ 7	+ 6	S.

I. *For the Moon's Hourly Motion in Lat.*

Argument I. of Lat.

S.	+ 0		+ 1		+ 2		S.
S.	− 6		− 7		− 8		S.
°	′	″	′	″	′	″	°
0	2	58	2	34	1	29	30
5	2	57	2	26	1	15	25
10	2	55	2	16	1	1	20
15	2	52	2	6	0	46	15
20	2	47	1	54	0	31	10
25	2	41	1	42	0	15	5
30	2	34	1	29	0	0	0
S.	+ 11		+ 10		+ 9		S.
S.	− 5		− 4		− 3		S.

II. *For the Moon's Hourly Motion in Lat.*

Argument II. of Lat.

S.	+ 0	+ 1	+ 2	S.
S.	− 6	− 7	− 8	S.
°	′	″	″	°
0	4	4	2	30
5	4	3	2	25
10	4	3	1	20
15	4	3	1	15
20	4	3	1	10
25	4	2	0	5
30	4	2	0	0
S.	+ 11	+ 10	+ 9	S.
S.	− 5	− 4	− 3	S.

Angle of the visible Path of the Moon with the Ecliptic in Eclipses.

Argument. Long. ☽ in her orbit — Long. ☊.

	0 Signs 6.										
	Horary Motion of the Moon from the Sun.										
	′ 27	′ 28	′ 29	′ 30	′ 31	′ 32	′ 33	34	′ 35	′ 36	
°	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	°
0	5 47	5 46	5 45	5 44	5 43	5 42	5 41	5 41	5 40	5 39	30
3	5 46	5 45	5 44	5 43	5 42	5 42	5 41	5 40	5 40	5 39	27
6	5 45	5 44	5 43	5 42	5 41	5 40	5 39	5 39	5 38	5 38	24
9	5 42	5 41	5 40	5 39	5 39	5 38	5 37	5 37	5 36	5 35	21
12	5 39	5 38	5 37	5 36	5 35	5 35	5 34	5 33	5 33	5 32	18
15	5 35	5 34	5 33	5 32	5 31	5 31	5 30	5 29	5 29	5 28	15
	11 Signs 5.										

Tables of the Mean Motions of the Moon from the Sun.

A. D.	S.	°	′	″
1761	9	20	47	17
1781	2	4	12	47
1791	10	4	49	47
B 1792	2	26	38	39
1793	7	6	16	4
1794	11	15	53	29
1795	3	25	30	54
B 1796	8	17	19	45
1797	0	26	57	10
1798	5	6	34	35
1799	9	16	12	0
1800	1	25	49	25
1801	6	5	26	49
1802	10	15	4	15
1803	2	24	41	39
B 1804	7	16	30	31
1805	11	26	7	56
1806	4	5	45	20
1807	8	15	22	45
B 1808	1	7	11	37
1809	5	16	49	3
1810	9	26	26	28
1811	2	6	3	54
B 1812	6	27	52	47
1813	11	7	30	12
1814	3	17	7	38
1815	7	26	45	3
B 1816	0	18	33	56
1817	4	28	11	22
1818	9	7	48	47
1819	1	17	26	13
B 1820	6	9	15	6
1821	10	18	52	32

Years Complete.	S.	°	′	″
1	4	9	37	24
2	8	19	14	49
3	0	28	52	14
B 4	5	20	41	6
5	10	0	18	30
6	2	9	55	54
7	6	19	33	19
B 8	11	11	22	10
9	3	20	59	36
10	8	0	37	0
11	0	10	14	24
B 12	5	2	3	16
13	9	11	40	40
14	1	21	18	5
15	6	0	55	30
B 16	10	22	44	21
17	3	2	21	46
18	7	11	59	10
19	11	21	36	35
B 20	4	13	25	26

Months.	S.	°	′	″
January	0	0	0	0
February	0	17	54	48
March	11	29	15	16
April	0	17	10	3
May	0	22	53	23
June	1	10	48	11
July	1	16	31	22
August	2	4	26	20
Septem.	2	22	21	8
October	2	28	4	29
Novem.	3	15	59	16
Decem.	3	21	42	37

In months after February of bissextile years subtract one day from the time found by the Tables.

TABLES *of the Mean Motions of the Moon from the Sun.*

Days.	S.	°	′	″
1	0	12	11	27
2	0	24	22	53
3	1	6	34	20
4	1	18	45	47
5	2	0	57	13
6	2	13	8	40
7	2	25	20	7
8	3	7	31	33
9	3	19	43	0
10	4	1	54	27
11	4	14	5	53
12	4	26	17	20
13	5	8	28	47
14	5	20	40	13
15	6	2	15	40
16	6	15	3	7
17	6	27	14	33
18	7	9	26	0
19	7	21	37	27
20	8	3	48	53
21	8	16	0	20
22	8	28	11	47
23	9	10	23	13
24	9	22	34	40
25	10	4	46	7
26	10	16	57	33
27	10	29	9	0
28	11	11	20	27
29	11	23	31	53
30	0	5	43	20
31	0	17	54	47

Hours and Minutes.							
h. ′ ″	° ′ ″	′ ″ ‴	″ ‴ ⁗	h. ′ ″	° ′ ″	′ ″ ‴	″ ‴ ⁗
1	0	30	29	31	15	44	48
2	1	0	57	32	16	15	17
3	1	31	26	33	16	45	45
4	2	1	54	34	17	16	13
5	2	32	23	35	17	46	42
6	3	2	52	36	18	17	10
7	3	33	20	37	18	47	39
8	4	3	49	38	19	18	7
9	4	34	18	39	19	48	36
10	5	4	46	40	20	19	5
11	5	35	15	41	20	49	33
12	6	5	43	42	21	20	2
13	6	36	12	43	21	50	31
14	7	6	41	44	22	20	59
15	7	37	9	45	22	51	28
16	8	7	38	46	23	21	56
17	8	38	6	47	23	52	25
18	9	8	35	48	24	22	54
19	9	39	4	49	24	53	22
20	10	9	32	50	25	23	51
21	10	40	1	51	25	54	19
22	11	10	30	52	26	24	48
23	11	40	59	53	26	55	17
24	12	11	28	54	27	25	45
25	12	41	56	55	27	56	14
26	13	12	25	56	28	26	43
27	13	42	54	57	28	57	12
28	14	13	22	58	29	27	40
29	14	43	51	59	29	58	9
30	15	14	19	60	30	28	37

	d.	h.	m.	h.
A Lunation =	29	12	44	3
Half a Lunation =	14	18	22	2

TABLE I. *Of Mean New Moons, &c. in March, from* 1791 *to* 1821.

	Mean New ☽ in March.				Sun's Mean Anomaly.				Moon's Mean Anomaly.				☉'s Mean Dist. from the ☽'s node.			
A.D.	d.	h.	m.	s.	S.	°	′	″	S.	°	′	″	S.	°	′	″
1791	4	14	8	35	8	3	31	47	9	26	23	45	4	18	52	22
1792	22	11	41	15	8	21	53	59	9	2	0	52	5	27	35	24
1793	11	20	29	51	8	11	9	51	7	11	48	57	6	5	38	11
1794	30	18	2	32	8	29	32	3	6	17	26	4	7	14	21	13
1795	20	2	51	8	8	18	47	55	4	27	14	9	7	22	24	0
1796	8	11	39	44	8	8	3	47	3	7	2	14	8	0	26	47
1797	27	9	12	24	8	26	25	59	2	12	39	19	9	9	9	48
1798	16	18	1	1	8	15	41	51	0	22	27	25	9	17	12	35
1799	6	2	49	37	8	4	57	43	11	2	15	30	9	25	15	22
1800	25	0	22	17	8	23	19	55	10	7	52	36	11	3	58	23
1801	14	9	10	53	8	12	35	47	8	17	40	41	11	12	1	10
1802	3	17	59	29	8	1	51	39	6	27	28	46	11	20	3	57
1803	22	15	32	9	8	20	13	51	6	3	5	52	0	28	46	58
1804	11	0	20	45	8	9	29	43	4	12	53	57	1	6	49	45
1805	0	9	9	21	7	28	45	31	2	22	42	2	1	14	52	35
1806	19	6	42	1	8	17	7	43	1	26	19	8	2	23	35	36
1807	8	15	30	37	8	6	23	35	0	8	7	13	3	1	38	23
1808	26	13	3	17	8	24	45	47	11	13	44	19	4	10	21	24
1809	15	21	51	53	8	14	1	39	9	23	32	24	4	18	24	11
1810	5	6	40	29	8	3	17	31	8	3	20	29	4	26	26	58
1811	24	4	13	9	8	21	39	43	7	8	57	35	6	5	9	59
1812	12	13	1	45	8	10	55	35	5	18	45	40	6	13	12	46
1813	1	21	50	21	8	0	11	27	3	28	33	45	6	21	15	23
1814	20	19	23	1	8	18	33	39	3	4	10	51	7	29	58	24
1815	10	4	11	37	8	7	53	31	1	13	58	56	8	8	1	11
1816	28	1	44	17	8	26	15	43	0	19	36	2	9	16	44	12
1817	17	10	32	53	8	15	31	35	10	29	24	7	9	24	46	59
1818	6	19	21	29	8	4	47	27	9	9	12	12	10	2	49	46
1819	25	16	54	9	8	23	9	39	8	14	49	18	11	11	32	47
1820	14	1	42	45	8	12	25	31	6	24	37	23	11	19	35	34
1821	3	10	31	21	8	1	41	23	5	4	25	28	11	27	38	21

II. *Mean Anomalies, and Sun's Mean Distance from the Node, for $13\frac{1}{2}$ Mean Lunations.*

	Mean Lunations.				Sun's Mean Anomaly.				Moon's Mean Anomaly.				Sun's Mean dis. from the Node.			
No	d.	h.	m.	s.	S.	°	′	″	S.	°	′	″	S.	°	′	″
1	29	12	44	3	0	29	6	19	0	25	49	0	1	0	40	14
2	59	1	28	6	1	28	12	39	1	21	38	1	2	1	20	28
3	88	14	12	9	2	27	18	58	2	17	27	1	3	2	0	42
4	118	2	56	12	3	26	25	17	3	13	16	2	4	2	40	56
5	147	15	40	15	4	25	31	37	4	9	5	2	5	3	21	10
6	177	4	24	18	5	24	37	56	5	4	54	3	6	4	1	24
7	206	17	8	21	6	23	44	15	6	0	43	3	7	4	41	38
8	236	5	52	24	7	22	50	35	6	26	32	3	8	5	21	53
9	265	18	36	27	8	21	56	54	7	22	21	4	9	6	2	6
10	295	7	20	30	9	21	3	14	8	18	10	4	10	6	42	20
11	324	20	4	33	10	20	9	33	9	13	59	5	11	7	22	34
12	354	8	48	36	11	19	15	52	10	9	48	5	0	8	2	47
13	385	21	32	40	0	18	22	12	11	5	37	6	1	8	43	1
$\frac{1}{2}$	14	18	22	2	0	14	33	10	6	12	54	30	0	15	20	7

III *The Days of the Year reckoned from the beginning of March.*

Days.	Mar.	Apr.	May.	June.	July.	Aug.	Sep.	Oct.	Nov.	Dec.	Jan.	Feb.
1	1	32	62	93	123	154	185	215	246	276	307	338
2	2	33	63	94	124	155	186	216	247	277	308	339
3	3	34	64	95	125	156	187	217	248	278	309	340
4	4	35	65	96	126	157	188	218	249	279	310	341
5	5	36	66	97	127	158	189	219	250	280	311	342
6	6	37	67	98	128	159	190	220	251	281	312	343
7	7	38	68	99	129	160	191	221	252	282	313	344
8	8	39	69	100	130	161	192	222	253	283	314	345
9	9	40	70	101	131	162	193	223	254	284	315	346
10	10	41	71	102	132	163	194	224	255	285	316	347
11	11	42	72	103	133	164	195	225	256	286	317	348
12	12	43	73	104	134	165	196	226	257	287	318	349
13	13	44	74	105	135	166	197	227	258	288	319	350
14	14	45	75	106	136	167	198	228	259	289	320	351
15	15	46	76	107	137	168	199	229	260	290	321	352
16	16	47	77	108	138	169	200	230	261	291	322	353
17	17	48	78	109	139	170	201	231	262	292	323	354
18	18	49	79	110	140	171	202	232	263	293	324	355
19	19	50	80	111	141	172	203	233	264	294	325	356
20	20	51	81	112	142	173	204	234	265	295	326	357
21	21	52	82	113	143	174	205	235	266	296	327	358
22	22	53	83	114	144	175	206	236	267	297	328	359
23	23	54	84	115	145	176	207	237	268	298	329	360
24	24	55	85	116	146	177	208	238	269	299	330	361
25	25	56	86	117	147	178	209	239	270	300	331	362
26	26	57	87	118	148	179	210	240	271	301	332	363
27	27	58	88	119	149	180	211	241	272	302	333	364
28	28	59	89	120	150	181	212	242	273	303	334	365
29	29	60	90	121	151	182	213	243	274	304	335	366
30	30	61	91	122	152	183	214	244	275	305	336	
31	31		92		153	184		245		306	337	

IV. *The Annual or First Equation of the Mean to the true Syzygy.*

Argument. Sun's Mean Anomaly.

°	S. 0 h.	m.	s.	1 h.	m.	s.	2 h.	m.	s.	3 h.	m.	s.	4 h.	m.	s.	5 h.	m.	s.	°
0	0	0	0	2	3	12	3	35	0	4	10	53	3	39	30	2	7	45	30
1	0	4	18	2	6	55	3	37	10	4	10	57	3	37	19	2	3	55	29
2	0	8	35	2	10	36	3	39	18	4	10	55	3	35	6	2	0	1	28
3	0	12	51	2	14	14	3	41	23	4	10	49	3	32	50	1	56	5	27
4	0	17	8	2	17	52	3	43	26	4	10	39	3	30	30	1	52	6	26
5	0	21	24	2	21	27	3	45	25	4	10	24	3	28	5	1	48	4	25
6	0	25	39	2	25	9	3	47	19	4	10	4	3	25	35	1	41	1	24
7	0	28	55	2	28	29	3	49	7	4	9	39	3	23	0	1	39	56	23
8	0	34	11	2	31	57	3	50	50	4	9	10	3	20	20	1	35	49	22
9	0	38	26	2	35	22	3	52	29	4	8	37	3	17	35	1	31	41	21
10	0	42	39	2	38	44	3	54	4	4	7	59	3	14	49	1	27	31	20
11	0	46	52	2	42	3	3	55	35	4	7	16	3	11	59	1	23	19	19
12	0	51	4	2	45	18	3	57	2	4	6	29	3	9	6	1	19	5	18
13	0	55	17	2	48	30	3	58	27	4	5	37	3	6	10	1	14	49	17
14	0	59	27	2	51	40	3	59	49	4	4	41	3	3	10	1	10	32	16
15	1	3	36	2	54	48	4	1	7	4	3	40	3	0	7	1	6	15	15
16	1	7	45	2	57	53	4	2	18	4	2	35	2	57	0	1	1	55	14
17	1	11	53	3	0	54	4	3	23	4	1	26	2	53	47	0	57	36	13
18	1	16	0	3	3	51	4	4	22	4	0	12	2	50	36	0	53	15	12
19	1	20	6	3	6	45	4	5	18	3	58	52	2	47	18	0	48	52	11
20	1	24	10	3	9	36	4	6	10	3	57	27	2	43	57	0	44	28	10
21	1	28	12	3	12	24	4	6	58	3	55	59	2	40	33	0	40	2	9
22	1	32	12	3	15	9	4	7	41	3	54	26	2	37	6	0	35	36	8
23	1	36	10	3	17	51	4	8	21	3	52	49	2	33	35	0	31	10	7
24	1	40	6	3	20	30	4	8	57	3	51	9	2	30	2	0	26	44	6
25	1	44	1	3	23	5	4	9	29	3	49	26	2	26	26	0	22	17	5
26	1	47	54	3	25	36	4	9	57	3	47	38	2	22	47	0	17	50	4
27	1	51	46	3	28	3	4	10	16	3	45	44	2	19	5	0	13	23	3
28	1	55	37	3	30	26	4	10	33	3	43	45	2	15	20	0	8	56	2
29	1	59	26	3	32	45	4	10	45	3	41	40	2	11	35	0	4	29	1
30	2	3	12	3	35	0	4	10	53	3	39	30	2	7	45	0	0	0	0
	S. 11			10			9			8			7			6			

40½

+

V. *Equation of the Moon's Mean Anomaly.*

Argument. Sun's Mean Anomaly.

°	S. 0 °	′	″	1 °	′	″	2 °	′	″	3 °	′	″	4 °	′	″	5 °	′	″	°
0	0	0	0	0	46	45	1	21	32	1	35	1	1	23	4	0	48	19	30
1	0	1	37	0	48	10	1	22	21	1	35	2	1	22	14	0	46	51	29
2	0	3	13	0	49	34	1	23	10	1	35	1	1	21	24	0	45	23	28
3	0	4	52	0	50	53	1	23	57	1	35	0	1	20	32	0	43	54	27
4	0	6	28	0	52	19	1	24	41	1	34	57	1	19	38	0	42	24	26
5	0	8	6	0	53	40	1	25	24	1	34	50	1	18	42	0	40	53	25
6	0	9	42	0	55	0	1	26	6	1	34	43	1	17	45	0	39	21	24
7	0	11	20	0	56	21	1	26	48	1	34	33	1	16	48	0	37	49	23
8	0	12	56	0	57	38	1	27	28	1	34	22	1	15	47	0	30	15	22
9	0	14	33	0	58	56	1	28	6	1	34	9	1	14	44	0	34	40	21
10	0	16	10	1	0	13	1	28	43	1	33	53	1	13	41	0	33	5	20
11	0	17	47	1	1	29	1	29	17	1	33	37	1	12	37	0	31	31	19
12	0	19	23	1	2	43	1	29	51	1	33	20	1	11	33	0	29	51	18
13	0	20	59	1	3	56	1	30	22	1	33	0	1	10	26	0	28	18	17
14	0	22	35	1	5	8	1	30	50	1	32	38	1	9	17	0	26	40	16
15	0	24	10	1	6	18	1	31	19	1	32	14	1	8	8	0	25	3	15
16	0	25	45	1	7	27	1	31	45	1	31	50	1	6	58	0	23	23	14
17	0	27	19	1	8	36	1	32	12	1	31	23	1	5	46	0	21	45	13
18	0	28	52	1	9	42	1	32	34	1	30	55	1	4	32	0	20	7	12
19	0	30	25	1	10	49	1	32	57	1	30	25	1	3	19	0	18	28	11
20	0	31	57	1	11	54	1	33	17	1	29	54	1	2	1	0	16	48	10
21	0	33	29	1	12	58	1	33	36	1	29	20	1	0	45	0	15	8	9
22	0	35	2	1	14	1	1	33	52	1	28	45	0	59	26	0	13	28	8
23	0	36	32	1	15	1	1	34	6	1	28	9	0	58	7	0	11	48	7
24	0	38	1	1	16	0	1	34	18	1	27	30	0	56	45	0	10	7	6
25	0	39	29	1	16	59	1	34	30	1	26	50	0	55	23	0	8	20	5
26	0	40	59	1	17	57	1	34	40	1	26	27	0	54	1	0	6	44	4
27	0	42	26	1	18	52	1	34	48	1	25	5	0	52	37	0	5	3	3
28	0	43	54	1	19	47	1	34	54	1	24	39	0	51	12	0	3	21	2
29	0	45	19	1	20	40	1	34	58	1	23	57	0	49	45	0	1	40	1
30	0	46	45	1	21	32	1	35	1	1	23	4	0	48	19	0	0	0	0
	S. 11			10			9			8			7			6			

+

VI. *The Second Equation of the Mean to the True Syzygy.*

Argument. Moon's equated Anomaly.

	S. 0			1			2			3			4			5			
°	h.	m.	s.	h.	m.	s.	h.	m.	s.	h.	m.	s.	h.	m.	s.	h.	m.	s.	°
0	0	0	0	5	12	48	8	47	8	9	46	44	8	8	59	4	34	33	30
1	0	10	58	5	21	56	8	51	45	9	45	3	8	3	12	4	26	1	29
2	0	21	56	5	30	57	8	56	10	9	45	12	7	57	23	4	17	25	28
3	0	32	51	5	39	51	9	0	25	9	44	11	7	51	33	4	8	47	27
4	0	43	52	5	48	37	9	4	31	9	42	59	7	45	46	4	0	7	26
5	0	54	50	5	57	17	9	8	25	9	41	36	7	39	46	3	51	23	25
6	1	5	48	6	5	51	9	12	9	9	40	3	7	33	36	3	42	32	24
7	1	16	46	6	14	19	9	15	43	9	38	19	7	27	22	3	33	38	23
8	1	27	44	6	22	41	9	19	5	9	36	24	7	21	2	3	24	42	22
9	1	38	40	6	30	57	9	22	14	9	34	18	7	14	30	3	15	44	21
10	1	49	33	6	39	4	9	25	12	9	32	1	7	7	50	3	6	45	20
11	2	0	23	6	47	0	9	27	54	9	29	33	7	1	2	2	57	43	19
12	2	11	10	6	54	46	9	30	32	9	26	54	6	54	8	2	48	39	18
13	2	21	54	7	2	21	9	32	58	9	24	4	6	47	9	2	39	34	17
14	2	32	31	7	9	52	9	35	12	9	21	3	6	40	6	2	30	28	16
15	2	43	9	7	17	9	9	37	14	9	17	51	6	32	56	2	21	19	15
16	2	53	38	7	24	19	9	39	8	9	14	28	6	25	40	2	12	8	14
17	3	4	3	7	31	18	9	40	51	9	10	54	6	18	13	2	2	53	13
18	3	14	21	7	38	9	9	42	21	9	7	9	6	10	49	1	53	36	12
19	3	24	42	7	44	51	9	43	42	9	3	13	6	3	16	1	44	16	11
20	3	34	58	7	51	24	9	44	53	8	59	6	5	55	38	1	34	54	10
21	3	45	11	7	57	45	9	45	52	8	54	50	5	47	54	1	25	31	9
22	3	55	21	8	3	56	9	46	38	8	50	24	5	40	4	1	16	7	8
23	4	5	26	8	9	57	9	47	13	8	45	48	5	32	9	1	6	41	7
24	4	15	26	8	15	46	9	47	36	8	41	2	5	24	9	0	57	13	6
25	4	25	20	8	21	24	9	47	49	8	36	6	5	16	5	0	47	44	5
26	4	35	6	8	26	53	9	47	54	8	31	0	5	7	56	0	38	13	4
27	4	44	42	8	32	11	9	47	46	8	25	44	4	59	42	0	28	41	3
28	4	54	11	8	37	19	9	47	33	8	20	18	4	51	15	0	19	8	2
29	5	3	33	8	42	18	9	47	14	8	14	33	4	43	2	0	9	34	1
30	5	12	48	8	47	8	9	46	44	8	8	59	4	34	33	0	0	0	0
	S. 11			10			9			8			7			6			

VII. *The third Equation of the Mean to the True Syzygy.*

Arg. ☉'s Mean Anom. — ☽'s Mean Anom.

	S.—0		— 1		— 2		
	S.+ 6		+ 7		+ 8		
°	′	″	′	″	′	″	°
0	0	0	2	22	4	12	30
1	0	5	2	26	4	15	29
2	0	10	2	30	4	18	28
3	0	15	2	34	4	21	27
4	0	20	2	38	4	24	26
5	0	25	2	42	4	27	25
6	0	30	2	46	4	30	24
7	0	35	2	50	4	32	23
8	0	40	2	54	4	34	22
9	0	45	2	58	4	36	21
10	0	50	3	2	4	38	20
11	0	55	3	6	4	40	19
12	1	0	3	10	4	42	18
13	1	5	3	14	4	44	17
14	1	10	3	18	4	46	16
15	1	15	3	22	4	48	15
16	1	20	3	6	4	50	14
17	1	25	3	30	4	51	13
18	1	30	3	34	4	52	12
19	1	35	3	38	4	53	11
20	1	40	3	42	4	54	10
21	1	45	3	45	4	55	9
22	1	49	3	48	4	56	8
23	1	52	3	51	4	57	7
24	1	56	3	54	4	57	6
25	2	0	3	57	4	57	5
26	2	4	4	0	4	58	4
27	2	9	4	3	4	58	3
28	2	13	4	6	4	58	2
29	2	18	4	9	4	58	1
30	2	22	4	12	4	58	0
	S.— 5		— 4		—3 S.		
	S. +11		+ 10		+9 S.		

VIII. *The fourth Equation of the Mean to the True Syzygy.*

Argument. ☉'s Mean Dist. from ☽'s ☊.

	+ 0		+ 1		+ 2		
	+ 6		+ 7		+ 8		
°	′	″	′	″	′	″	°
0	0	0	1	22	1	22	30
1	0	4	1	23	1	21	29
2	0	7	1	24	1	20	28
3	0	10	1	25	1	18	27
4	0	13	1	26	1	16	26
5	0	16	1	7	1	14	25
6	0	20	1	28	1	12	24
7	0	23	1	29	1	10	23
8	0	26	1	30	1	8	22
9	0	29	1	31	1	6	21
10	0	32	1	32	1	3	20
11	0	35	1	33	1	0	19
12	0	38	1	33	0	57	18
13	0	41	1	34	0	54	17
14	0	44	1	34	0	51	16
15	0	47	1	34	0	49	15
16	0	50	1	34	0	45	14
17	0	52	1	34	0	41	13
18	0	54	1	31	0	37	12
19	0	57	1	33	0	34	11
20	1	0	1	33	0	31	10
21	1	2	1	32	0	28	9
22	1	5	1	31	0	25	8
23	1	8	1	30	0	22	7
24	1	10	1	29	0	19	6
25	1	12	1	28	0	16	5
26	1	14	1	27	0	13	4
27	1	16	1	26	0	10	3
28	1	18	1	25	0	6	2
29	1	20	1	24	0	3	1
30	1	22	1	22	0	0	0
	S.— 5		— 4		—3 S.		
	S.—11		—10		—9 S.		

TABLE *of Logistical Logarithms.*

	0	1	2	3	4	5	6	7		0	1	2	3	4	5	6	7
	0	60	120	180	240	300	360	420		0	60	120	180	240	300	360	420
0	00000	17782	14771	13010	11761	10792	10000	9331	30	20792	16021	13802	12341	11249	10378	9652	9031
1	35563	17710	14735	12986	11743	10777	9988	9320	31	20649	15973	13773	12320	11233	10365	9641	9021
2	32553	17639	14699	12962	11725	10763	9976	9310	32	20512	15925	13745	12300	11217	10352	9630	9012
3	30792	17570	14664	12939	11707	10749	9964	9300	33	20378	15878	13716	12279	11201	10339	9619	9003
4	29542	17501	14629	12915	11689	10734	9952	9289	34	20248	15832	13688	12259	11186	10326	9608	8992
5	28573	17434	14594	12891	11671	10720	9940	9279	35	20122	15786	13660	12239	11170	10313	9597	8983
6	27782	17368	14559	12868	11654	10706	9928	9269	36	20000	15740	13632	12218	11154	10300	9586	8973
7	27112	17302	14525	12845	11636	10692	9916	9259	37	19881	15695	13604	12198	11138	10287	9575	8964
8	26532	17238	14491	12821	11619	10678	9905	9249	38	19765	15651	13576	12178	11123	10274	9564	8954
9	26021	17175	14457	12798	11601	10663	9893	9238	39	19652	15607	13549	12159	11107	10261	9553	8945
10	25563	17112	14424	12775	11584	10649	9881	9228	40	19542	15563	13522	12139	11091	10248	9542	8935
11	25149	17050	14390	12753	11566	10635	9869	9218	41	19435	15520	13495	12119	11076	10235	9532	8926
12	24771	16990	14357	12730	11549	10621	9858	9208	42	19331	15477	13468	12099	11061	10223	9521	8917
13	24424	16930	14325	12707	11532	10608	9846	9198	43	19228	15435	13441	12080	11045	10210	9510	8907
14	24102	16871	14292	12685	11515	10594	9834	9188	44	19128	15393	13415	12061	11030	10197	9499	8898
15	23802	16812	14260	12663	11498	10580	9823	9178	45	19031	15351	13388	12041	11015	10185	9488	8888
16	23522	16755	14228	12640	11481	10566	9811	9168	46	18935	15310	13362	12022	10999	10172	9478	8879
17	23259	16698	14196	12618	11464	10552	9800	9158	47	18842	15269	13336	12003	10984	10160	9467	8870
18	23010	16642	14165	12596	11447	10539	9788	9148	48	18751	15229	13310	11984	10969	10147	9456	8861
19	22775	16587	14133	12574	11430	10525	9777	9138	49	18661	15189	13284	11965	10954	10135	9446	8851
20	22553	16532	14102	12553	11413	10512	9765	9128	50	18573	15149	13259	11946	10939	10122	9435	8842
21	22341	16478	14071	12531	11397	10498	9754	9119	51	18487	15110	13233	11927	10924	10110	9425	8833
22	22139	16425	14040	12510	11380	10484	9742	9109	52	18403	15071	13208	11908	10909	10098	9414	8824
23	21946	16372	14010	12488	11363	10471	9731	9099	53	18320	15032	13183	11889	10894	10085	9404	8814
24	21761	16320	13979	12467	11347	10458	9720	9089	54	18239	14994	13158	11871	10880	10073	9393	8805
25	21584	16269	13949	12445	11331	10444	9708	9079	55	18159	14956	13133	11852	10865	10061	9383	8796
26	21413	16218	13919	12424	11314	10431	9697	9070	56	18081	14918	13108	11834	10850	10049	9372	8787
27	21249	16168	13890	12403	11298	10418	9686	9060	57	18004	14881	13083	11816	10835	10036	9362	8778
28	21091	16118	13860	12382	11282	10404	9675	9050	58	17929	14844	13059	11797	10821	10024	9351	8769
29	20939	16069	13831	12362	11266	10391	9664	9041	59	17855	14808	13034	11779	10806	10012	9341	8760
30	20792	16021	13802	12341	11249	10378	9652	9031	60	17782	14771	13010	11761	10792	10000	9331	8752

TABLE *of Logistical Logarithms.*

	8	9	10	11	12	13	14	15	16		8	9	10	11	12	13	14	15	16
	480	540	600	660	720	780	840	900	960		480	540	600	660	720	780	840	900	960
0	8751	8239	7782	7368	6990	6642	6320	6021	5740	30	8487	8004	7575	7171	6812	6478	6168	5878	5607
1	8742	8231	7774	7361	6984	6637	6315	6016	5736	31	8479	7997	7563	7168	6807	6473	6163	5874	5602
2	8733	8223	7767	7354	6978	6631	6310	6011	5731	32	8470	7989	7556	7162	6801	6466	6158	5879	5598
3	8724	8215	7760	7348	6972	6625	6305	6006	5727	33	8462	7981	7549	7156	6795	6462	6153	5864	5594
4	8715	8207	7753	7341	6966	6620	6300	6001	5722	34	8453	7974	7542	7149	6789	6457	6148	5860	5589
5	8706	8199	7745	7335	6960	6614	6294	6997	5718	35	8445	7966	7535	7143	6784	6451	6143	5855	5585
6	8697	8191	7738	7328	6954	6609	6289	5992	5713	36	8437	7959	7528	7137	6778	6446	6138	5850	5580
7	8688	8183	7731	7322	6948	6603	6284	5987	5709	37	8428	7951	7522	7131	6772	6441	6133	5846	5576
8	8679	8175	7724	7315	6942	6598	6279	5982	5704	38	8420	7944	7515	7124	6766	6435	6128	5841	5572
9	8670	8167	7717	7309	6936	6592	6274	5977	5700	39	8411	7936	7508	7118	6761	6430	6123	5836	5567
10	8661	8159	7710	7302	6930	6587	6269	5973	5695	40	8403	7929	7501	7112	6755	6425	6118	5832	5563
11	8652	8152	7703	7296	6924	6581	6264	5968	5691	41	8395	7921	7494	7106	6749	6420	6113	5827	5559
12	8643	8144	7696	7289	6918	6576	6259	5963	5686	42	8386	7914	7488	7100	6743	6414	6108	5823	5554
13	8635	8136	7688	7283	6912	6570	6254	5958	5682	43	8378	7906	7481	7093	6738	6409	6103	5818	5550
14	8626	8128	768	7276	6906	6565	6248	5954	5677	44	8370	7899	7474	7087	6732	6404	6099	5813	5546
15	8617	8120	7674	7270	6900	6559	6243	5949	5673	45	8361	7891	7467	7081	6726	6398	6094	5809	5541
16	8608	8112	7667	7264	6894	6554	6238	5944	5669	46	8353	7884	7461	7075	6721	6393	6089	5804	5537
17	8599	8104	7660	7257	6888	6548	6233	5939	5664	47	8345	7877	7454	7069	6715	6388	6084	5800	5533
18	8591	8097	7653	7251	6882	6543	6228	5935	5660	48	8337	7869	7447	7063	6709	6383	6079	5795	5528
19	8582	8089	7646	7244	6877	6538	6223	5930	5655	49	8328	7862	7441	7057	6704	6377	6074	5790	5524
20	8573	8081	7639	7238	6871	6532	6218	5925	5651	50	8320	7855	7434	7050	6698	6372	6069	5786	5520
21	8565	8073	7632	7232	6865	6527	6213	5920	5640	51	8312	7847	7427	7044	669	6367	6064	781	5516
22	8556	8066	7625	7225	6859	6521	6208	5916	5642	52	8304	7840	7421	7038	6687	6362	6059	5777	5511
23	8547	8058	7618	7219	6853	6516	6203	5911	5637	53	8296	7832	7414	7032	6681	6357	6058	772	5507
24	8539	8050	7611	7212	6847	6510	6198	5906	5633	54	8288	7825	7407	7026	6676	6351	6050	5768	5503
25	8530	8043	7604	7206	6841	6505	6193	5902	5629	55	8279	7818	7401	7020	6670	6346	6045	5763	5498
26	8522	8035	7597	7200	6836	6500	6188	5897	5624	56	8271	7811	7394	7014	6664	6341	6040	5758	5494
27	8513	8027	7590	7193	6830	6494	6183	5892	5620	57	8263	7805	7387	7008	6659	6336	6035	5754	5490
28	8504	8020	7583	7187	6824	6489	6178	5888	5615	58	8255	7796	7381	7002	6653	6331	6030	5749	5486
29	8496	8012	7577	7181	6818	6484	6173	5883	5611	59	8247	7789	7374	6996	6648	6325	6025	5745	5481
30	8487	8004	7570	7175	6812	6478	6168	5878	5607	60	8239	7782	7368	6990	6642	6320	6021	5740	5477

TABLE *of Logistical Logarithms.*

	17	18	19	20	21	22	23	24	25		17	18	19	20	21	22	23	24	25
	1020	1080	1140	1200	1260	1320	1380	1440	1500		1020	1080	1140	1200	1260	1320	1380	1440	1500
0	5477	5229	4994	4771	4559	4357	4164	3979	3802	30	5351	5110	4881	4664	4457	4260	4071	3890	3716
1	5473	5225	4990	4768	4556	4354	4161	3976	3799	31	5347	5106	4877	4660	4454	4256	4068	3887	3713
2	5469	5221	4986	4764	4552	4351	4158	3973	3796	32	5343	5102	4874	4657	4450	4253	4065	3884	3710
3	5464	5217	4983	4760	4549	4347	4155	3970	3793	33	5339	5098	4870	4653	4447	4250	4062	3881	3708
4	5460	5213	4979	4757	4546	4344	4152	3967	3791	34	5335	5094	4866	4650	4444	4247	4059	3878	3705
5	5456	5209	4975	4753	4542	4341	4149	3964	3788	35	5331	5090	4863	4646	4440	4244	4055	3875	3702
6	5452	5205	4971	4750	4539	4338	4145	3961	3785	36	5326	5086	4859	4643	4437	4240	4052	3872	3699
7	5447	5201	4967	4746	4535	4334	4142	3958	3782	37	5322	5082	4855	4639	4434	4237	4049	3869	3696
8	5443	5197	4964	4742	4532	4331	4139	3955	3779	38	5318	5079	4852	4636	4430	4234	4046	3866	3693
9	5439	5193	4960	4739	4528	4328	4136	3952	3776	39	5314	5075	4848	4632	4427	4231	4043	3863	3691
10	5435	5189	4956	4735	4525	4325	4133	3949	3773	40	5310	5071	4844	4629	4424	4228	4040	3860	3688
11	5430	5185	4952	4732	4522	4321	4130	3946	3770	41	5306	5067	4841	4625	4420	4224	4037	3857	3685
12	5426	5181	4949	4728	4518	4318	4127	3943	3768	42	5302	5063	4837	4622	4417	4221	4034	3855	3682
13	5422	5177	4945	4724	4515	4315	4124	3940	3765	43	5298	5059	4833	4618	4414	4218	4031	3852	3679
14	5418	5173	4941	4721	4511	4311	4120	3937	3762	44	5294	5055	4830	4615	4410	4215	4028	3849	3677
15	5414	5169	4937	4717	4508	4308	4117	3934	3759	45	5290	5051	4826	4611	4407	4212	4025	3846	3674
16	5409	5165	4933	4714	4505	4305	4114	3931	3756	46	5285	5048	4822	4608	4404	4209	4022	3843	3671
17	5405	5161	4930	4710	4501	4302	4111	3928	3753	47	5281	5044	4819	4604	4400	4205	4019	3840	3668
18	5401	5157	4926	4707	4498	4298	4108	3925	3750	48	5277	5040	4815	4601	4397	4202	4016	3837	3665
19	5397	5153	4922	4703	4494	4295	4105	3922	3747	49	5273	5036	4811	4597	4394	4199	4013	3834	3663
20	5393	5149	4918	4699	4491	4292	4102	3919	3745	50	5269	5032	4808	4594	4390	4196	4010	3831	3660
21	5389	5145	4915	4696	4488	4289	4099	3917	3742	51	5265	5028	4804	4590	4387	4193	4007	3828	3657
22	5384	5141	4911	4692	4484	4285	4096	3914	3739	52	5261	5025	4800	4587	4384	4189	4004	3825	3654
23	5380	5137	4907	4689	4481	4282	4092	3911	3736	53	5257	5021	4797	4584	4380	4186	4001	3822	3651
24	5376	5133	4903	4685	4477	4279	4089	3908	3733	54	5253	5017	4793	4580	4377	4183	3998	3820	3649
25	5372	5129	4900	4682	4474	4276	4086	3905	3730	55	5249	5013	4789	4577	4374	4180	3995	3817	3646
26	5368	5125	4896	4678	4471	4273	4083	3902	3727	56	5245	5009	4786	4573	4370	4177	3991	3814	3643
27	5361	5122	4892	4675	4467	4269	4080	3899	3725	57	5241	5005	4782	4570	4367	4174	3988	3811	3640
28	5354	5118	4889	4671	4464	4266	4077	3896	3722	58	5237	5002	4778	4566	4364	4171	3985	3808	3637
29	5355	5114	4885	4668	4460	4263	4074	3893	3719	59	5233	4998	4775	4563	4361	4167	3982	3805	3635
30	5351	5110	4881	4664	4457	4260	4071	3890	3716	60	5229	4994	4771	4559	4357	4164	3979	3802	3632

TABLE *of Logistical Logarithms.*

	26	27	28	29	30	31	32	33	34		26	27	38	29	30	31	32	33	34
	1560	1620	1680	1740	1800	1860	1920	1980	2040		1560	1620	1680	1740	1800	1860	1920	1980	2040
0	3632	3468	3310	3158	3010	2868	2730	2596	2467	30	3549	3388	3233	3083	2939	2798	2663	2531	2403
1	3629	3465	3307	3155	3008	2866	2728	2594	2465	31	3546	3386	3231	3081	2936	2796	2660	2529	2401
2	3626	3463	3305	3153	3005	2863	2725	2592	2462	32	3544	3383	3228	3078	2934	2794	2658	2527	2399
3	3623	3460	3302	3150	3003	2861	2723	2590	2460	33	3541	3380	3225	3076	2931	2792	2656	2525	2397
4	3621	3457	3300	3148	3001	2859	2721	2588	2458	34	3538	3378	3223	3073	2929	2789	2654	2522	2395
5	3618	3454	3297	3145	2998	2856	2719	2585	2456	35	3535	3375	3220	3071	2927	2787	2652	2520	2393
6	3615	3452	3294	3143	2996	2854	2716	2583	2454	36	3533	3372	3218	3069	2924	2785	2649	2518	2391
7	3612	3449	3292	3140	2993	2852	2714	2581	2452	37	3530	3370	3215	3066	2922	2782	2647	2516	2389
8	3610	3446	3289	3138	2991	2849	2712	2579	2450	38	3527	3367	3213	3064	2920	2780	2645	2514	2387
9	3607	3444	3287	3135	2989	2847	2710	2577	2448	39	3525	3365	3210	3061	2917	2778	2643	2512	2384
10	3604	3441	3284	3133	2986	2845	2707	2574	2445	40	3522	3362	3208	3059	2915	2775	2640	2510	2382
11	3601	3438	3282	3130	2984	2842	2705	2572	2443	41	3519	3359	3205	3056	2912	2773	2638	2507	2380
12	3598	3436	3279	3128	2981	2840	2703	2570	2441	42	3516	3357	3203	3054	2910	2771	2636	2505	2378
13	3596	3433	3276	3125	2979	2838	2701	2568	2439	43	3514	3354	3200	3052	2908	2769	2634	2503	2376
14	3593	3431	3274	3123	2977	2835	2698	2566	2437	44	3511	3351	3198	3049	2905	2796	2632	2501	2374
15	3590	3428	3271	3120	2974	2833	2696	2564	2435	45	3508	3349	3195	3047	2993	2764	2629	2499	2372
16	3587	3425	3269	3118	2972	2831	2694	2561	2433	46	3506	3356	3193	3044	2901	2762	2627	2497	2370
17	3585	3423	3266	3115	2969	2828	2692	2559	2431	47	3503	3344	3190	3042	2898	2760	2625	2494	2368
18	3582	3420	3264	3113	2967	2826	2689	2557	2429	48	3500	3341	3188	3039	2896	2757	2623	2492	2366
19	3579	3417	3261	3110	2965	2824	2687	2555	2426	49	3497	3338	3185	3037	2894	2755	2621	2490	2364
20	3576	3415	3299	3108	2962	2821	2685	2553	2424	50	3495	3336	3183	3034	2891	2753	2618	2488	2362
21	3574	3412	3256	3105	2960	2819	2683	2551	2422	51	3492	3333	3180	3032	2889	2750	2616	2486	2359
22	3571	3409	3253	3103	2958	2817	2681	2548	2420	52	3489	3331	3178	3030	2887	2748	2614	2484	2357
23	3568	3407	3251	3101	2955	2815	2678	2546	2418	53	3487	3328	3175	3027	2884	2746	2612	2482	2355
24	3565	3404	3248	3098	2953	2812	2676	2544	2416	54	3484	3325	3173	3025	2882	2744	2610	2480	2353
25	3563	3401	3246	3096	2950	2810	2674	2542	2414	55	3481	3323	3170	3022	2880	2741	2607	2477	2351
26	3560	3399	3243	3093	2948	2808	2672	2540	2412	56	3479	3320	3168	3020	2877	2739	2605	2475	2349
27	3557	3396	3241	3091	2946	2805	2669	2538	2410	57	3476	3318	3165	3018	2875	2737	2603	2473	2347
28	3555	3393	3238	3088	2943	2803	2667	2535	2408	58	3473	3315	3163	3015	2873	2735	2601	2471	2345
29	3552	3391	3236	3086	2941	2801	2665	2533	2405	59	3471	3313	3160	3013	2870	2732	2599	2469	2343
30	3549	3388	3233	3083	2939	2798	2663	2531	2403	60	3468	3310	3158	3010	2868	2730	2596	2467	2341

TABLE *of Logistical Logarithms.*

·	35	36	37	38	39	40	41	42	43		35	36	37	38	39	40	41	42	34
	2100	2160	2220	2280	2340	2400	2460	2520	2580		2100	2160	2220	2280	2340	2400	2460	2520	2580
0	2341	2218	2099	1984	1871	1761	1651	1549	1447	30	2279	2159	2041	1927	1816	1707	1601	1498	1397
1	2339	2216	2098	1982	1869	1759	1652	1547	1445	31	2277	2157	2039	1925	1814	1705	1599	1496	1395
2	2337	2214	2096	1980	1867	1757	1650	1546	1443	32	2275	2155	2037	1923	1812	1703	1598	1494	1393
3	2335	2212	2094	1978	1865	1755	1648	1544	1442	33	2273	2153	2035	1921	1810	1702	1596	1493	1392
4	2333	2210	2092	1976	1863	1751	1647	1542	1440	34	2271	2151	2033	1919	1808	1700	1594	1491	1390
5	2331	2208	2090	1974	1862	1752	1645	1540	1438	35	2269	2149	2032	1918	1806	1698	1592	1489	1388
6	2328	2206	2088	1972	1860	1750	1643	1539	1437	36	2267	2147	2030	1916	1805	1696	1591	1487	1387
7	2326	2204	2086	1970	1858	1748	1641	1537	1435	37	2265	2145	2028	1914	1803	1694	1589	1486	1385
8	2324	2202	2084	1968	1856	1746	1640	1535	1433	38	2263	2143	2026	1912	1801	1693	1587	1484	1383
9	2322	2200	2082	1967	1854	1745	1638	1534	1432	39	2261	2141	2024	1910	1799	1691	1585	1482	1382
10	2320	2198	2080	1965	1852	1743	1636	1532	1430	40	2259	2139	2022	1908	1797	1689	1584	1481	1380
11	2318	2196	2078	1963	1850	1741	1634	1530	1428	41	2257	2137	2020	1906	1795	1687	1582	1479	1378
12	2316	2194	2076	1961	1849	1739	1633	1528	1427	42	2255	2135	2018	1904	1794	1686	1580	1477	1377
13	2314	2192	2074	1959	1847	1737	1631	1527	1425	43	2253	2133	2016	1903	1792	1684	1578	1476	1375
14	2312	2190	2072	1957	1845	1736	1629	1525	1423	44	2251	2131	2014	1901	1790	1682	1577	1471	1373
15	2310	2188	2070	1955	1843	1734	1627	1523	1422	45	2249	2129	2012	1899	1788	1680	1575	1472	1372
16	2308	2186	2068	1953	1841	1732	1626	1522	1420	46	2247	2127	2010	1897	1786	1678	1573	1470	1370
17	2306	2184	2066	1951	1839	1730	1624	1520	1418	47	2245	2125	2009	1895	1785	1677	1571	1469	1368
18	2304	2182	2064	1950	1838	1728	1622	1518	1417	48	2243	2123	2007	1893	1783	1675	1570	1467	1367
19	2302	2180	2062	1948	1836	1727	1620	1516	1415	49	2241	2121	2005	1891	1781	1673	1568	1465	1365
20	2300	2178	2061	1946	1834	1725	1619	1515	1413	50	2239	2119	2003	1889	1779	1671	1566	1464	1363
21	2298	2176	2059	1944	1832	1723	1617	1513	1412	51	2237	2117	2001	1888	1777	1670	1565	1462	1362
22	2296	2174	2057	1942	1830	1721	1615	1511	1410	52	2235	2115	1999	1886	1775	1668	1563	1460	1360
23	2294	2172	2055	1940	1828	1719	1613	1510	1408	53	2233	2113	1997	1884	1774	1666	1561	1459	1359
24	2291	2170	2053	1938	1827	1718	1612	1508	1407	54	2231	2111	1995	1882	1772	1664	1559	1457	1357
25	2289	2169	2051	1936	1825	1716	1610	1506	1405	55	2229	2109	1993	1880	1770	1663	1558	1455	1355
26	2287	2167	2049	1934	1823	1714	1608	1504	1403	56	2227	2107	1991	1878	1768	1661	1556	1451	1354
27	2285	2165	2047	1933	1821	1712	1606	1503	1402	57	2225	2105	1989	1876	1766	1659	1554	1452	1352
28	2283	2163	2045	1931	1819	1711	1605	1501	1400	58	2223	2103	1987	1875	1765	1657	1552	1450	1350
29	2281	2161	2043	1929	1817	1709	1603	1499	1398	59	2220	2101	1986	1873	1763	1655	1551	1449	1349
30	2279	2159	2041	1927	1816	1707	1601	1498	1397	60	2218	2099	1984	1871	1761	1654	1549	1447	1347

TABLE *of Logistical Logarithms.*

	44	45	46	47	48	49	50	51	52		44	45	46	47	48	49	50	51	52
	2640	2700	2760	2820	2880	2940	3000	3060	3120		2640	2700	2760	2820	2880	2940	3000	3060	3120
0	1347	1249	1154	1061	969	880	792	706	621	30	1298	1201	1107	1015	924	835	749	663	580
1	1345	1248	1152	1059	968	878	790	704	620	31	1296	1200	1105	1013	923	834	747	662	579
2	1344	1246	1151	1057	966	877	789	703	619	32	1295	1198	1104	1012	921	833	746	661	577
3	1342	1245	1149	1056	963	875	787	702	617	33	1293	1197	1102	1010	920	831	744	659	576
4	1340	1243	1148	1054	963	874	786	700	616	34	1291	1195	1101	1008	918	830	743	658	574
5	1339	1241	1146	1053	96[illegible]	872	785	699	615	35	1290	1193	1099	1007	917	828	741	656	573
6	1337	1240	1145	1051	960	871	783	697	613	36	1288	1192	1098	1005	915	827	740	655	572
7	1335	1238	1143	1050	959	869	782	696	612	37	1287	1190	1096	1004	914	825	739	654	570
8	1334	1237	1141	1048	957	868	780	694	610	38	1285	1189	1095	1002	912	824	737	652	569
9	1332	1235	1140	1047	956	866	779	693	609	39	1283	1187	1093	1001	911	822	736	651	568
10	1331	1233	1138	1045	954	865	777	692	608	40	1232	1186	1091	999	909	821	734	649	566
11	1329	1232	1137	1044	953	863	776	690	606	41	1280	1184	1090	998	908	819	733	648	565
12	1327	1230	1135	1042	951	862	774	689	605	42	1278	1182	1088	996	906	818	731	647	563
13	1326	1229	1134	1041	950	860	773	987	603	43	1277	1181	1087	995	905	816	730	645	562
14	1324	1227	1132	1039	948	859	772	686	602	44	1275	1179	1085	993	903	815	729	644	561
15	1322	1225	1130	1037	947	857	770	685	601	45	1274	1178	1084	992	902	814	727	642	559
16	1321	1224	1129	1036	945	856	769	683	599	46	1272	1176	1082	990	900	812	726	641	558
17	1319	1222	1127	1034	944	855	767	682	598	47	1270	1174	1081	989	899	811	724	640	557
18	1317	1221	1126	1033	942	853	766	680	596	48	1269	1173	1079	987	897	809	723	638	555
19	1316	1219	1124	1031	941	852	764	679	595	49	1267	1171	1078	986	896	808	721	637	554
20	1314	1217	1123	1030	939	850	763	678	594	50	1266	1170	1076	984	894	806	720	635	552
21	1313	1216	1121	1028	938	849	762	676	592	51	1264	1168	1074	983	893	805	719	634	551
22	1311	1214	1119	1027	936	847	760	675	591	52	1262	1167	1073	981	891	803	717	633	550
23	1309	1213	1118	1025	935	846	759	673	590	53	1261	1165	1071	980	890	802	716	631	548
24	1308	1211	1116	1024	933	844	757	672	588	54	1259	1163	1070	978	888	801	714	630	547
25	1306	1209	1115	1022	932	843	756	670	587	55	1257	1162	1068	977	887	799	713	628	546
26	1304	1208	1113	1021	930	841	754	669	585	56	1256	1160	1067	975	885	798	711	627	544
27	1303	1206	1112	1019	929	840	753	668	584	57	1254	1159	1065	974	884	796	710	626	543
28	1301	1205	1110	1018	927	838	751	666	583	58	1253	1157	1064	972	883	795	709	624	541
29	1300	1203	1109	1016	926	837	750	665	581	59	1251	1156	1062	971	831	793	707	623	540
30	1298	1201	1107	1015	924	835	749	663	580	60	1249	1154	1061	969	880	792	706	621	539

TABLE *of Logistical Logarithms.*

	53	54	55	56	57	58	59		53	54	55	56	57	58	59
	3180	3240	3300	3360	3420	3480	3540		3180	3240	3300	3360	3420	3480	3540
0	539	458	378	300	223	147	73	30	498	418	339	261	185	110	36
1	537	456	377	298	221	146	72	31	497	416	337	260	184	109	35
2	536	455	375	297	220	145	71	32	495	415	336	258	182	107	34
3	535	454	374	296	219	143	69	33	494	414	335	257	181	106	33
4	533	452	373	294	218	142	68	34	493	412	333	256	180	105	31
5	532	451	371	293	216	141	67	35	491	411	332	255	179	104	30
6	531	450	370	292	215	140	66	36	490	410	331	253	177	103	29
7	529	448	369	291	214	139	64	37	489	408	329	252	176	101	28
8	528	447	367	289	213	137	63	38	487	407	328	251	175	100	27
9	526	446	366	288	211	136	62	39	486	406	327	250	174	99	25
10	525	444	365	287	210	135	61	40	484	404	326	248	172	98	24
11	524	443	363	285	209	134	60	41	483	403	324	247	171	96	23
12	522	442	362	284	208	132	58	42	482	402	323	246	170	95	22
13	521	440	361	283	206	131	57	43	480	400	322	244	169	94	21
14	520	439	359	282	205	130	56	44	479	399	320	243	167	93	19
15	518	438	358	280	204	129	55	45	478	398	319	242	166	91	18
16	517	436	357	279	202	127	53	46	476	396	318	241	165	90	17
17	516	435	356	278	201	126	52	47	475	395	316	239	163	89	16
18	514	434	354	276	200	125	51	48	474	394	315	238	162	88	15
19	513	432	353	275	199	124	50	49	472	392	314	237	161	87	13
20	512	431	352	274	197	122	49	50	471	391	313	235	160	85	12
21	510	430	350	273	196	121	47	51	470	390	311	234	158	84	11
22	509	428	349	271	195	120	46	52	468	388	310	233	157	83	10
23	507	427	348	270	194	119	45	53	467	387	309	232	156	82	8
24	506	426	346	269	192	117	44	54	466	386	307	230	155	80	7
25	505	424	345	267	191	116	42	55	464	384	306	229	153	79	6
26	503	423	344	266	190	115	41	56	463	383	305	228	152	78	5
27	502	422	342	265	189	114	40	57	462	382	304	227	151	77	4
28	501	420	341	264	187	112	39	58	460	381	302	225	150	75	2
29	499	419	340	262	186	111	38	59	459	379	301	224	148	74	1
30	498	418	339	261	185	110	36	60	458	378	300	223	147	73	0

Home is a charm

Pl. I.

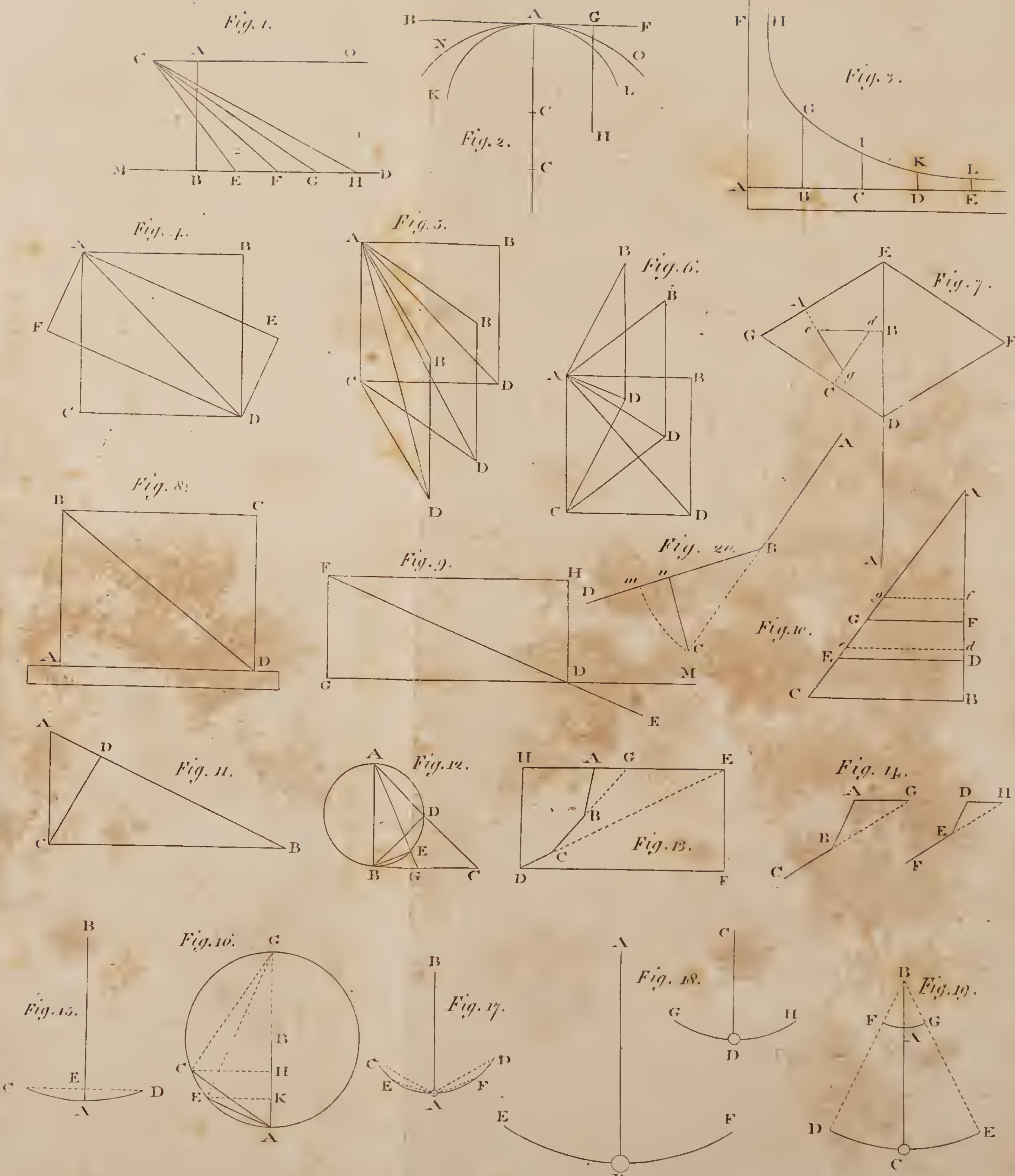

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Fig. 20.

1.

Farewell to the spot to memory dear
And hallow'd by former scenes
I ne'er shall forget in the shadow of years
The sunshine of former days

2.

No sorrow nor care ~~then~~ entered this breast
It was the seat of hope and of pleasure
It lay like the ocean when enjoying its rest
And little it thought of the future

3.

But now though care should make his abode
And fill my bosom with sorrow
Yet still should it swell though oppressed with the load
At ~~a~~ thought of former delight.

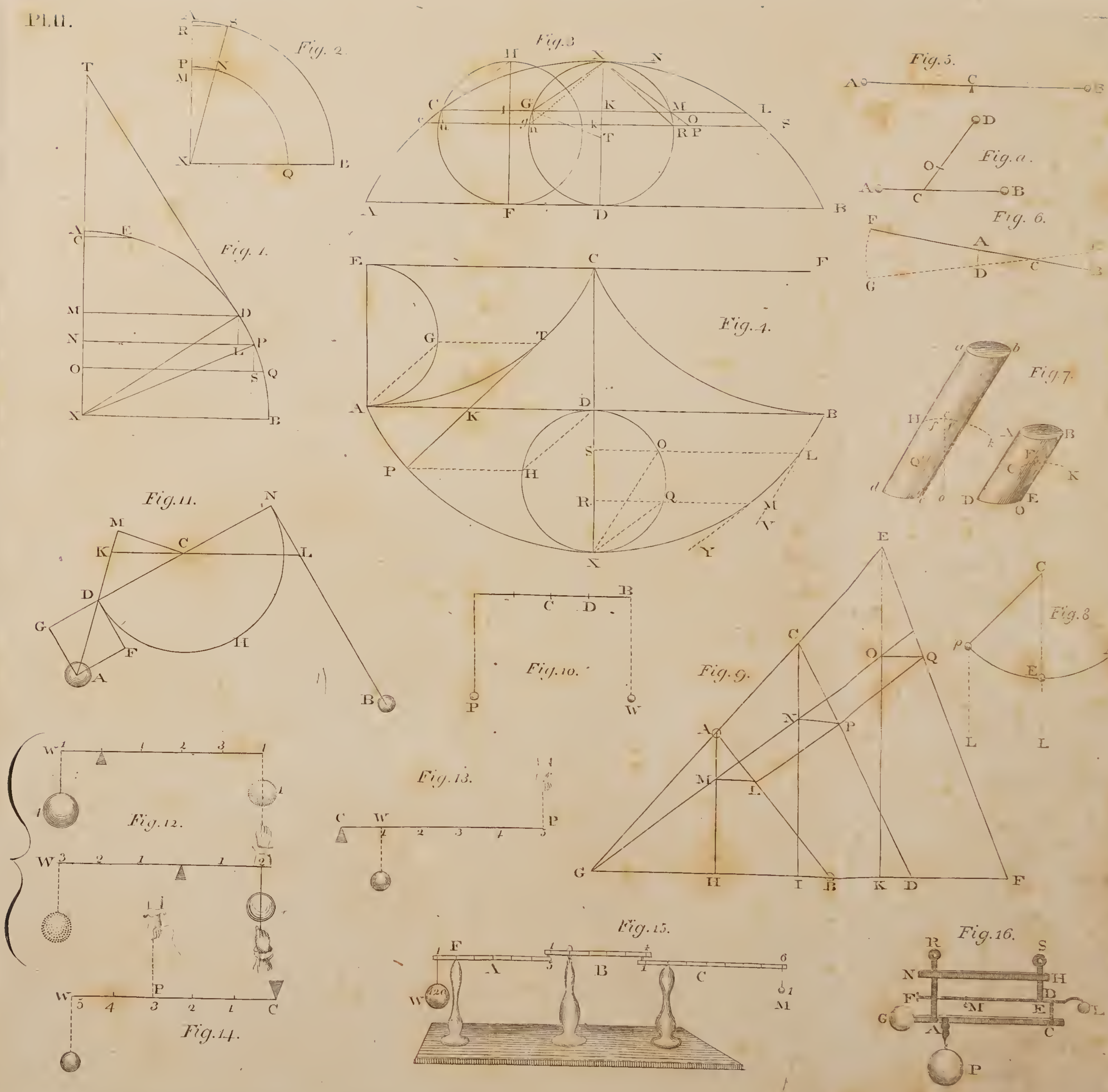
4

Farewell then farewell I ne'er shall forget thee
Though time shall sweep me away,
When the last thought in death shall by me be spent,
On my youth and my yesterday.

Written on Leaving St John's College   R J Watkins.

Dated Hall of St John's Octbr 1st 1830

Pl. II.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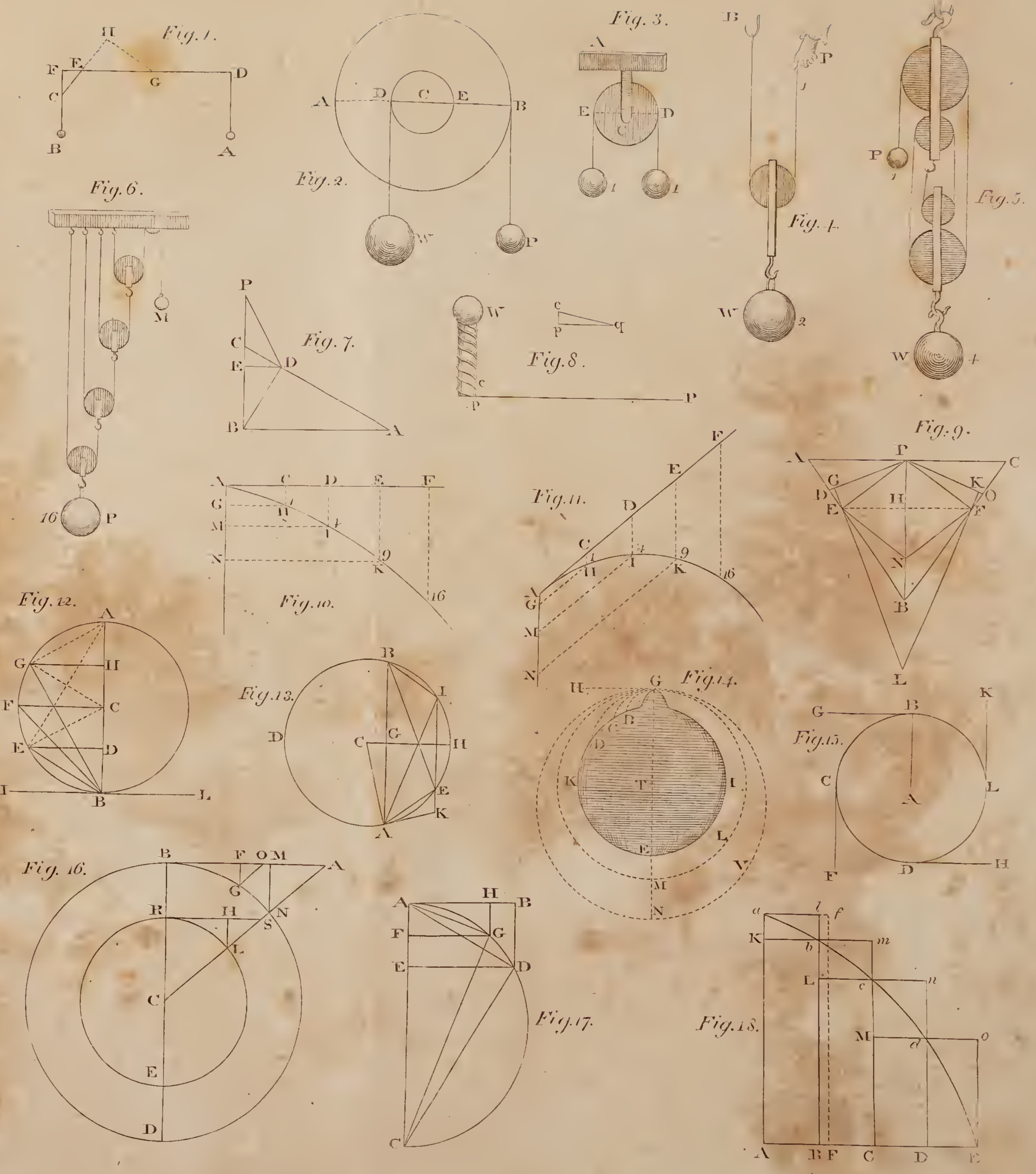
Pl. III.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Pl.IV.

Fig.1.

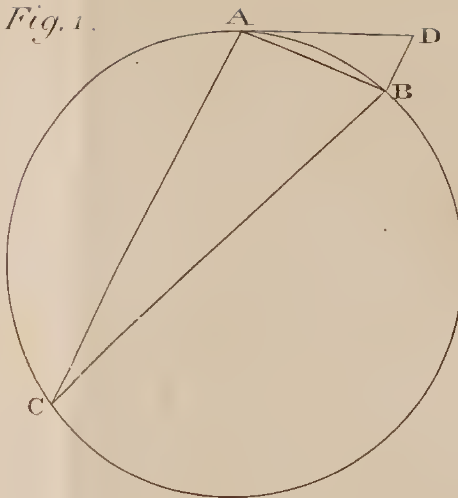
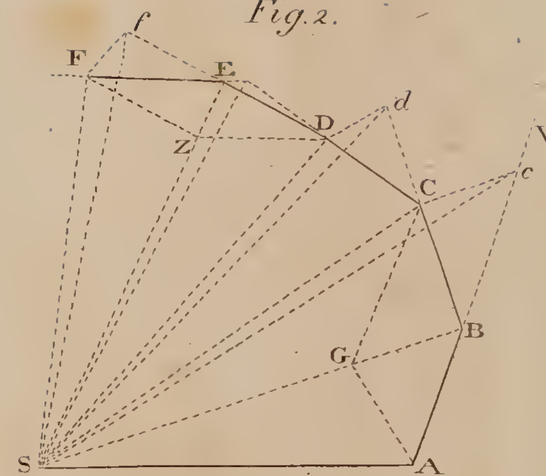


Fig.2.



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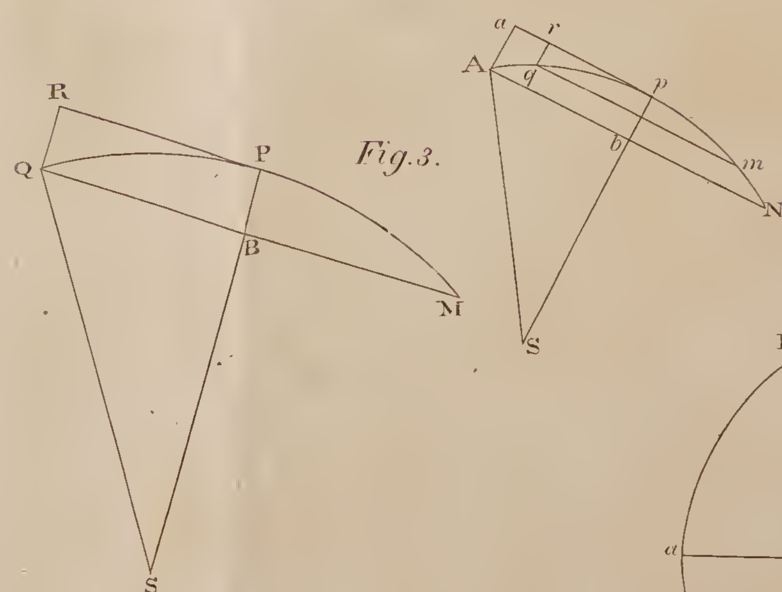

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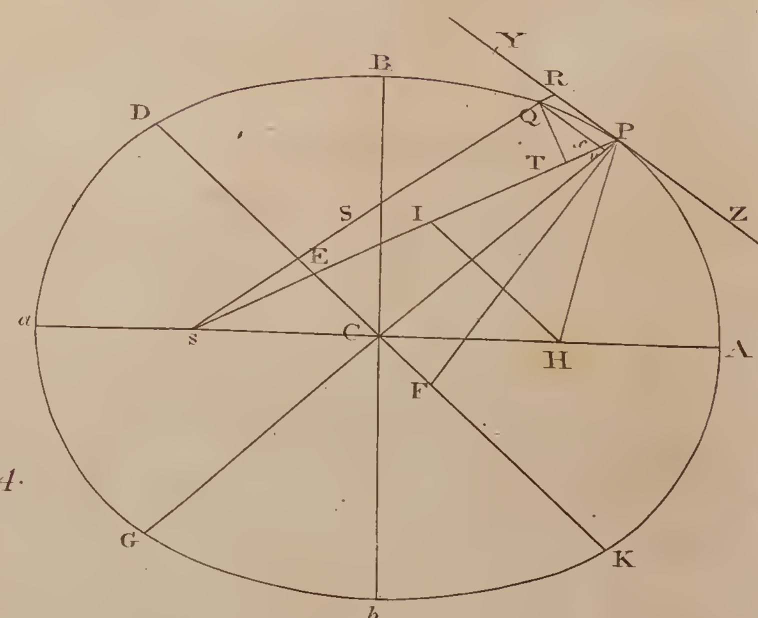
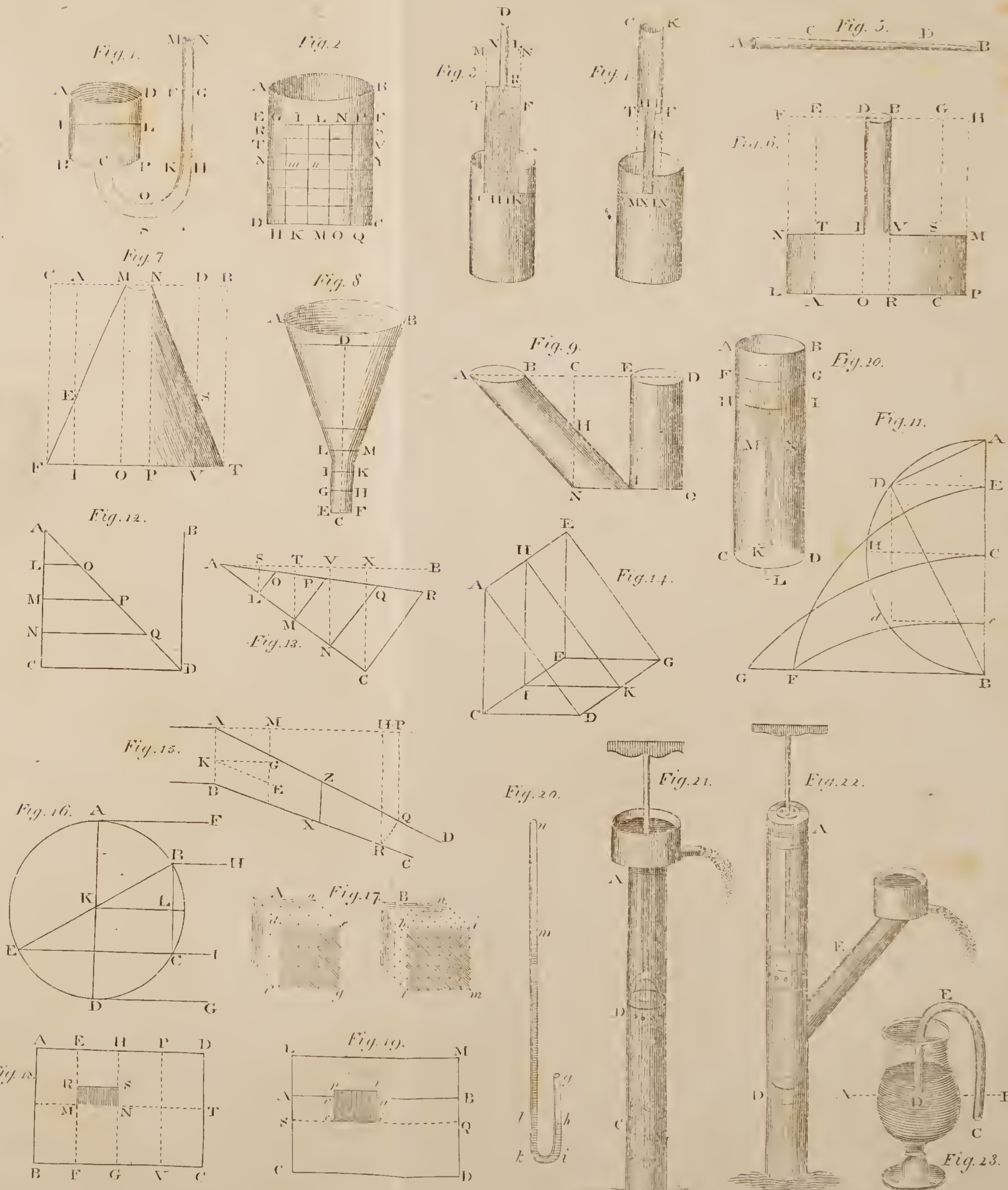


Pl. V.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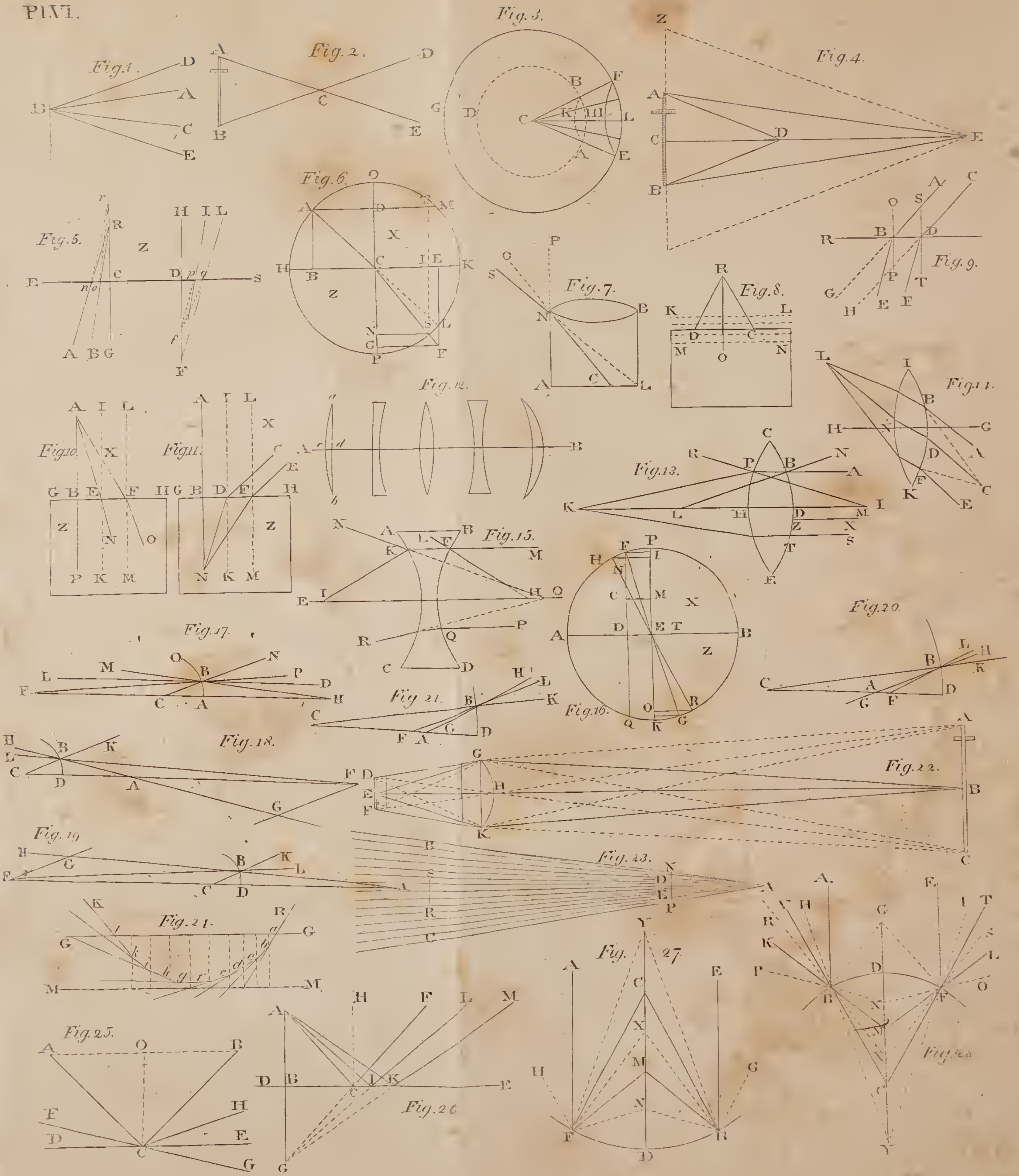
Pl. VI.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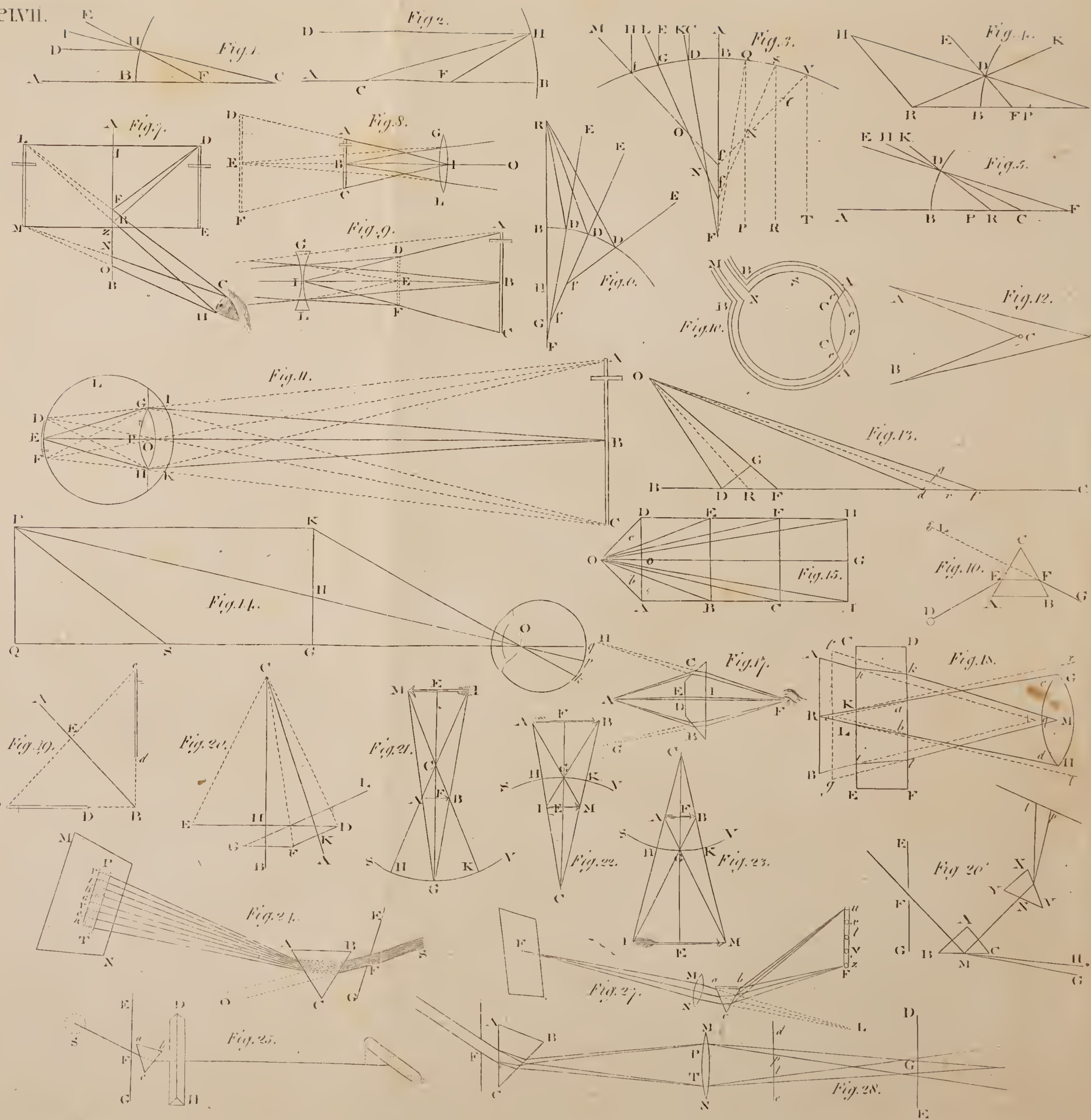
Pl. VII.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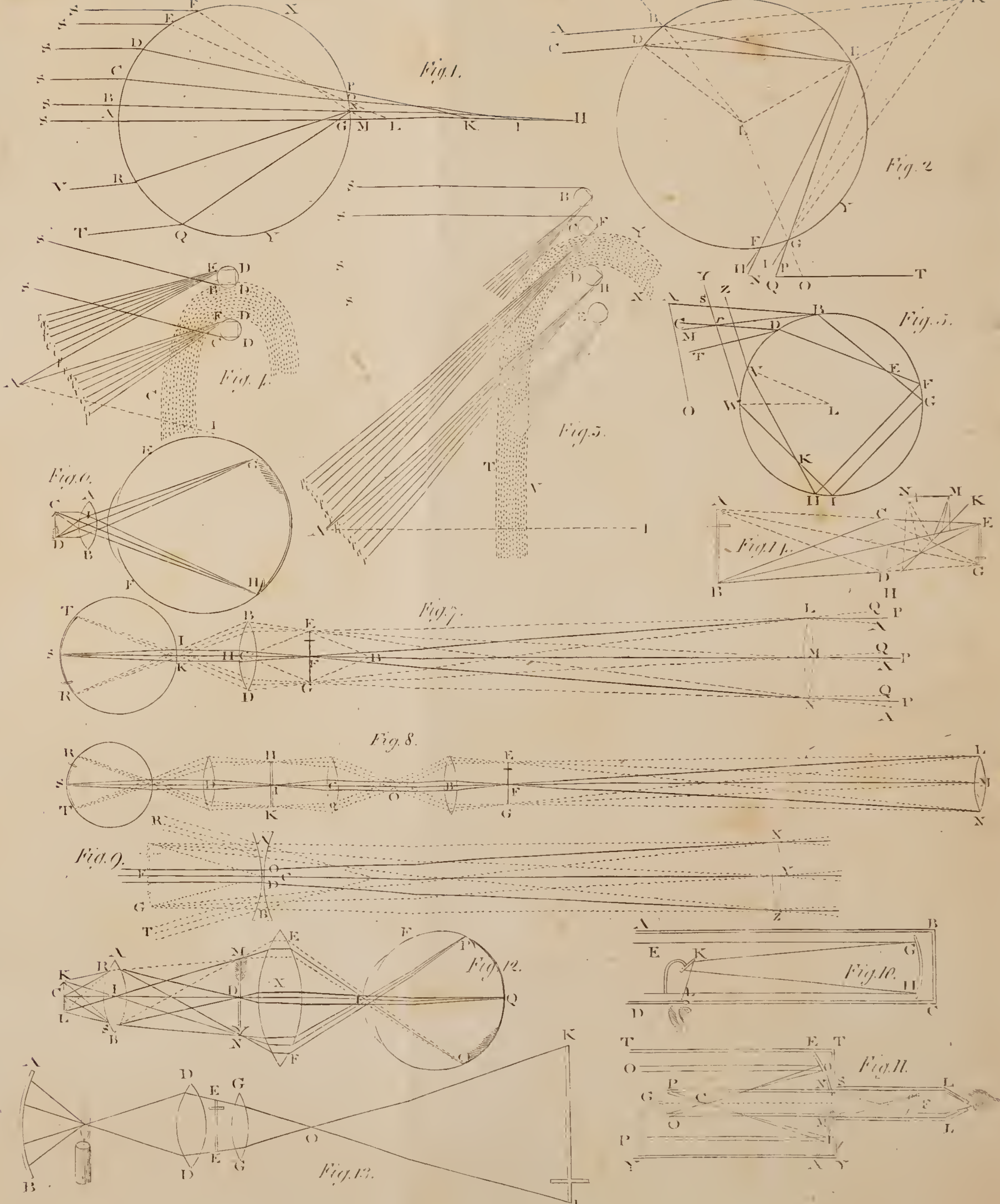
Pl. VIII.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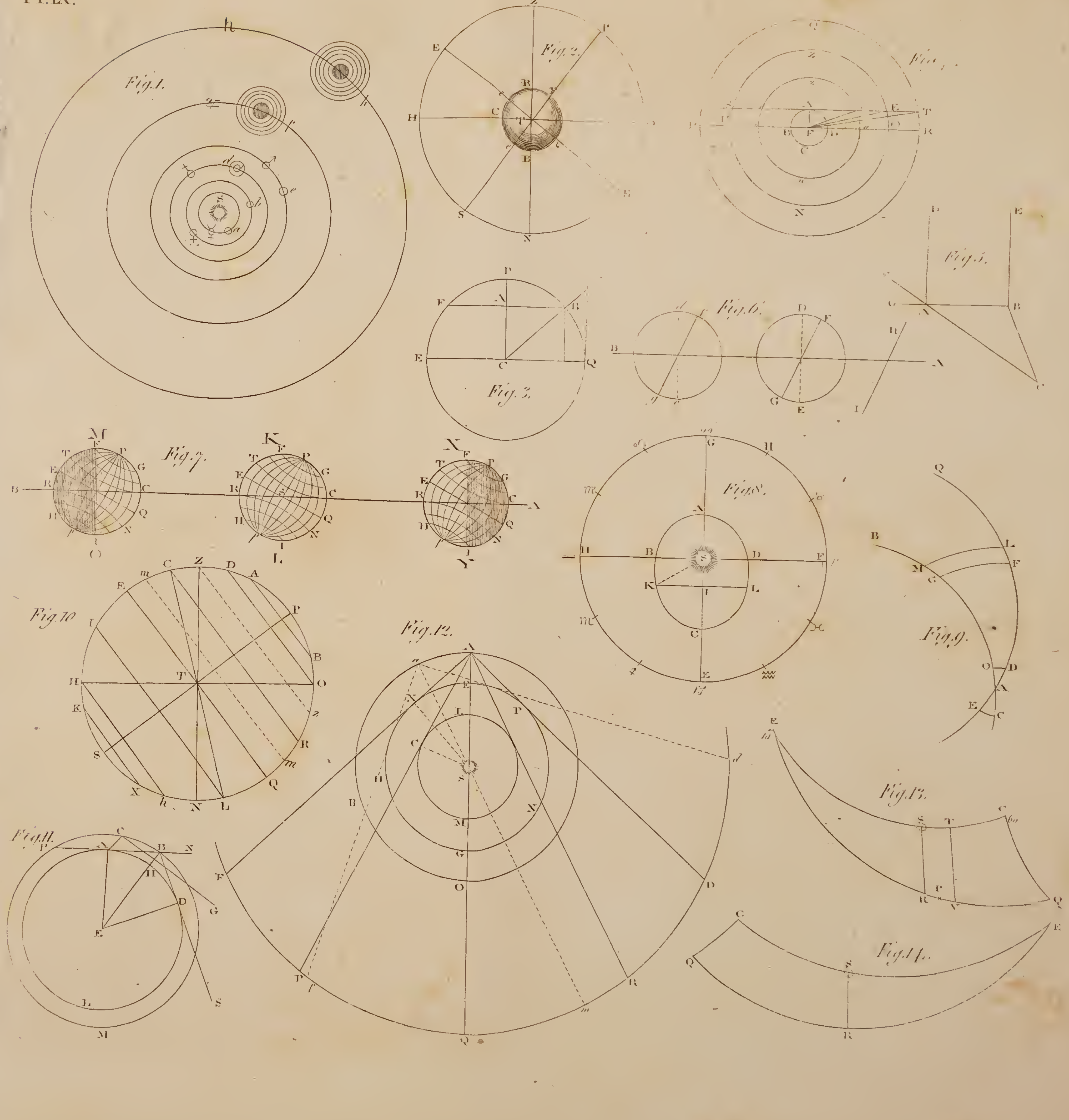
Pl. IX.



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Pl. X.

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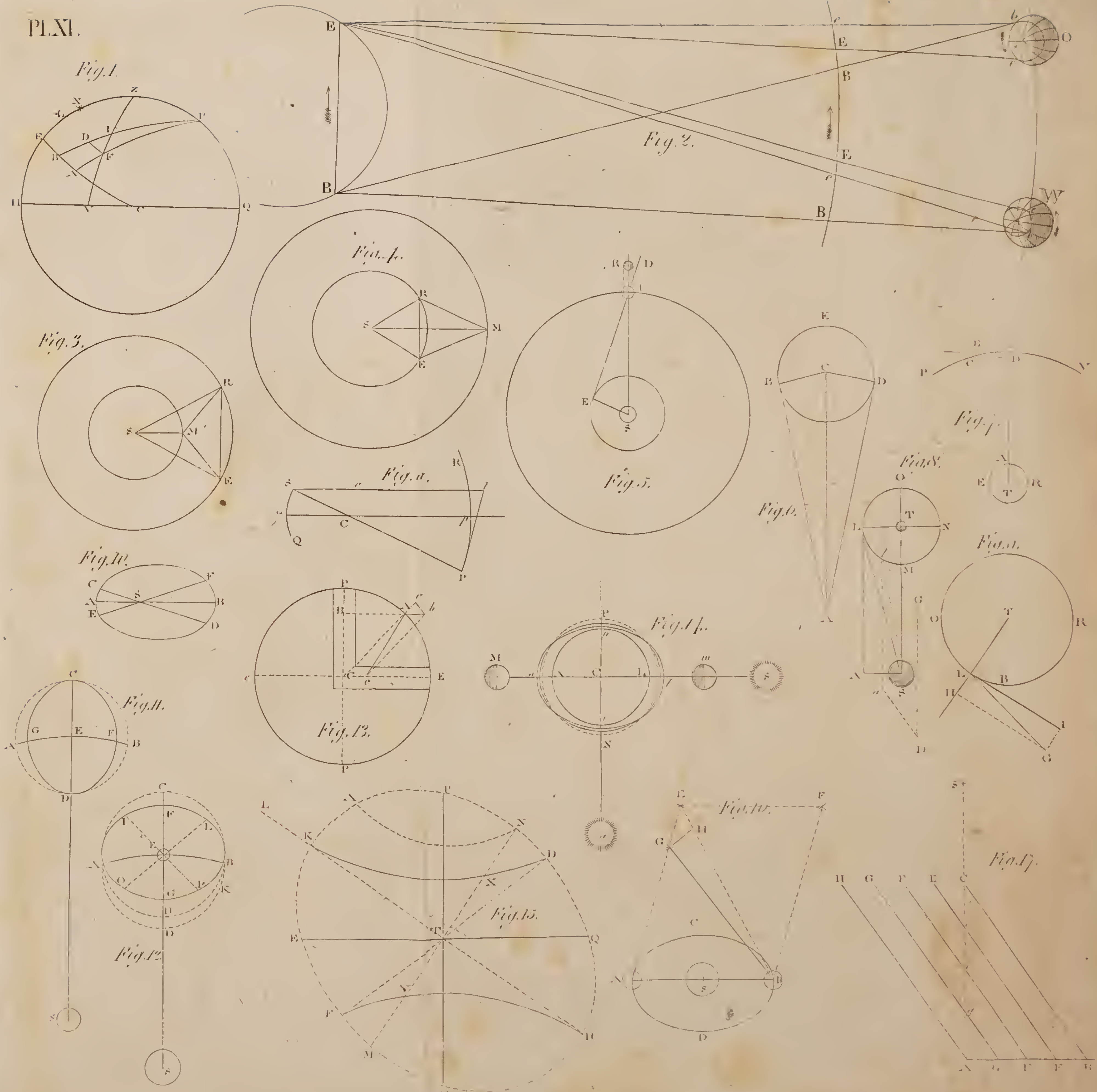
Pl. XI.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Pl. XII.

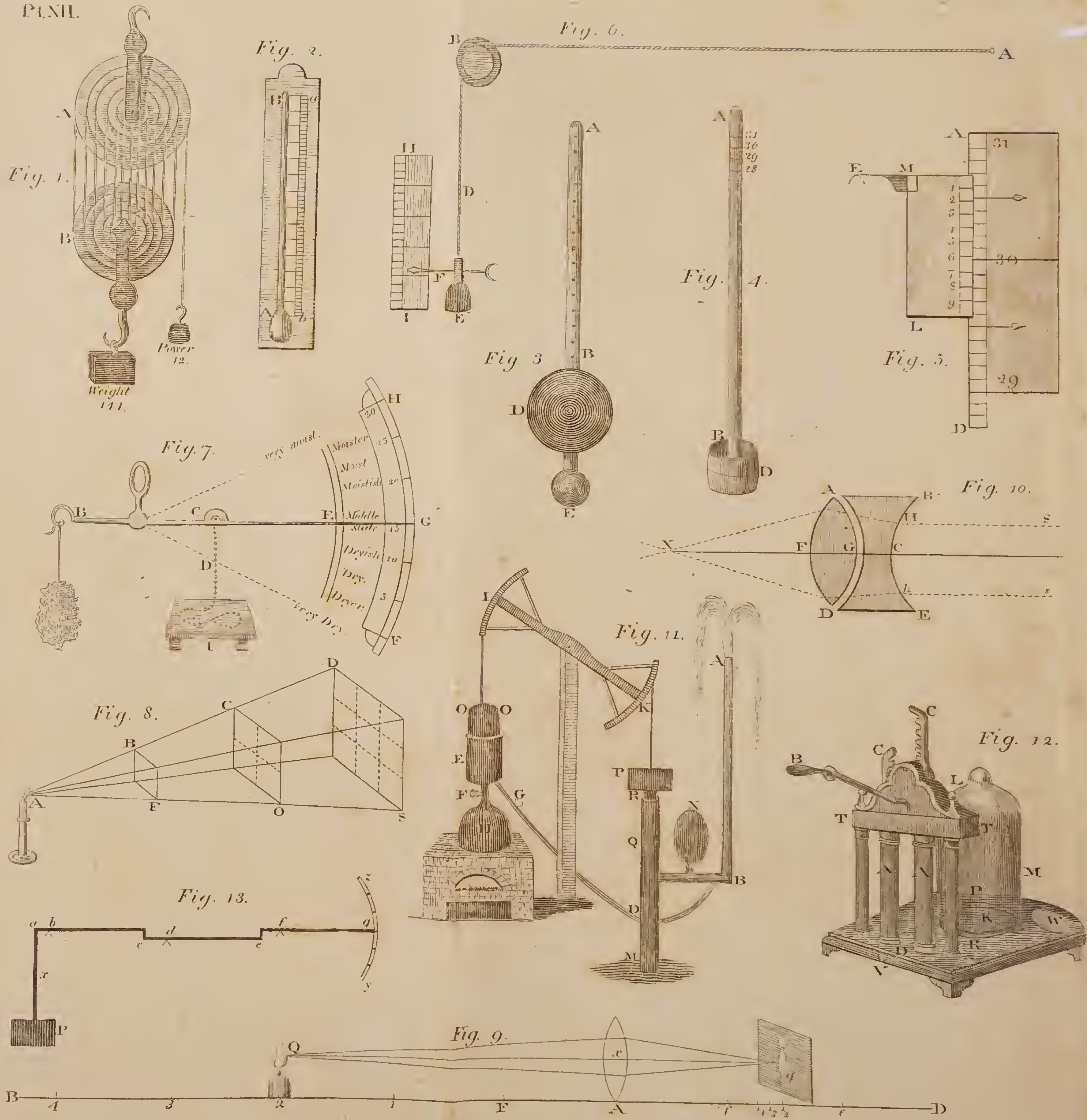

Fig. 1.
Power 12.
Weight 144.
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Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
very moist.
Moister
Moist
Moistish
Middle State.
Dryish
Dry.
Dryer.
very Dry.
Fig. 8.
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Fig. 13.

Pl.XIII.

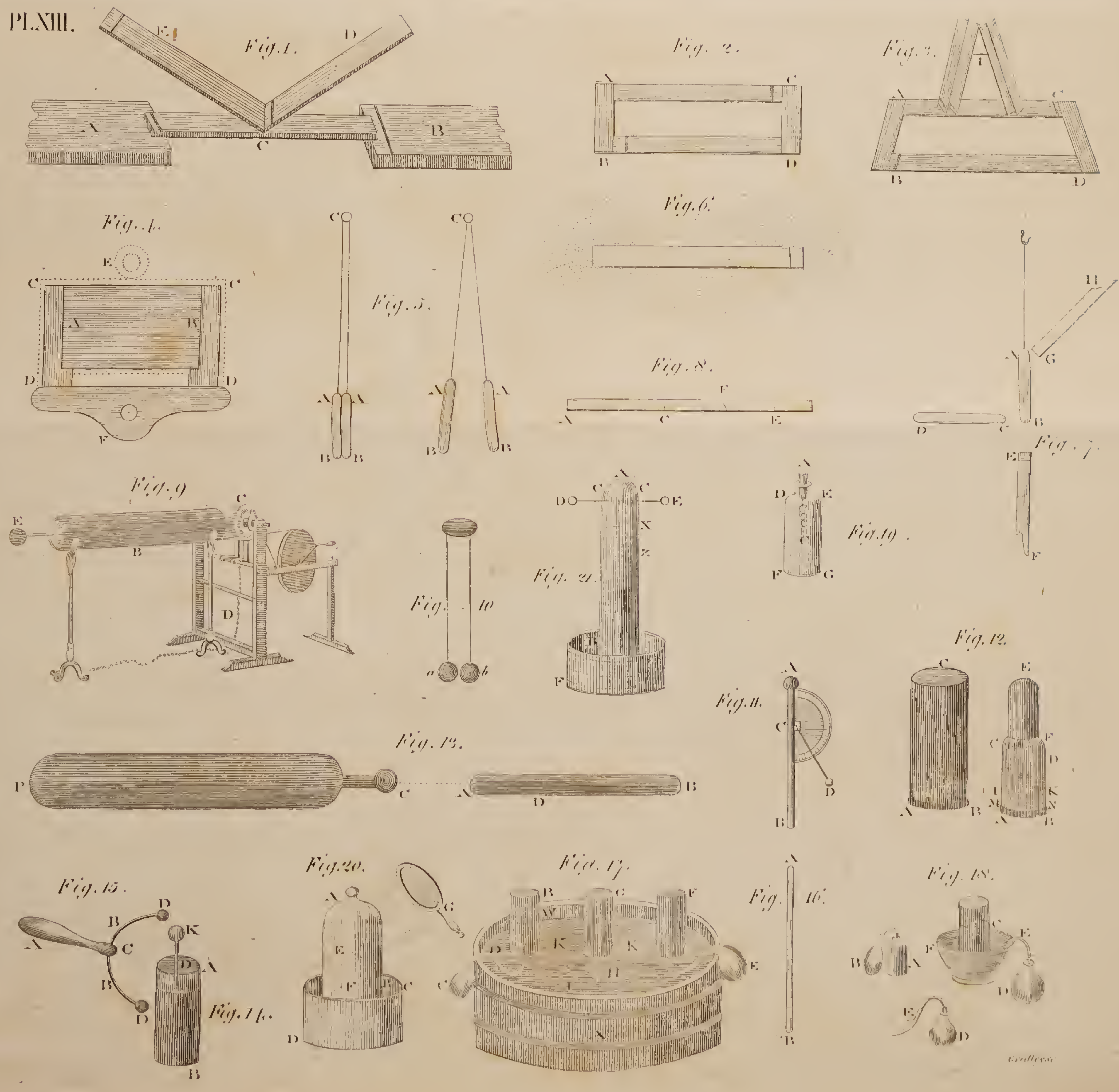

Fig. 1.
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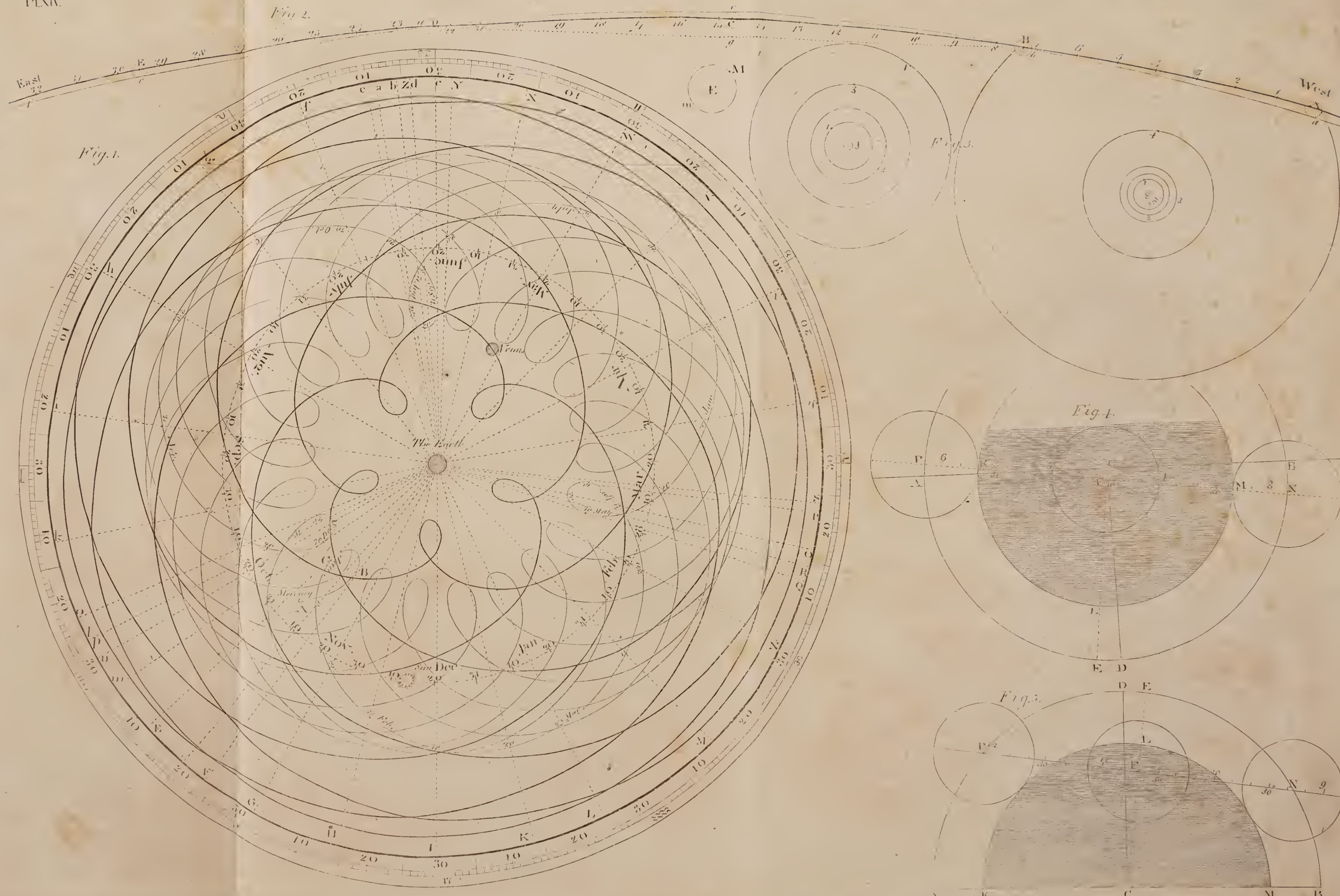
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
East
West
The Earth
Venus
Mercury
June
May
Apr.
Mar.
Feb.
Jan.
Dec.
Nov.
Oct.
Sept.
Aug.
July

Graduated 22nd Jan 1790.

Pl.XV.

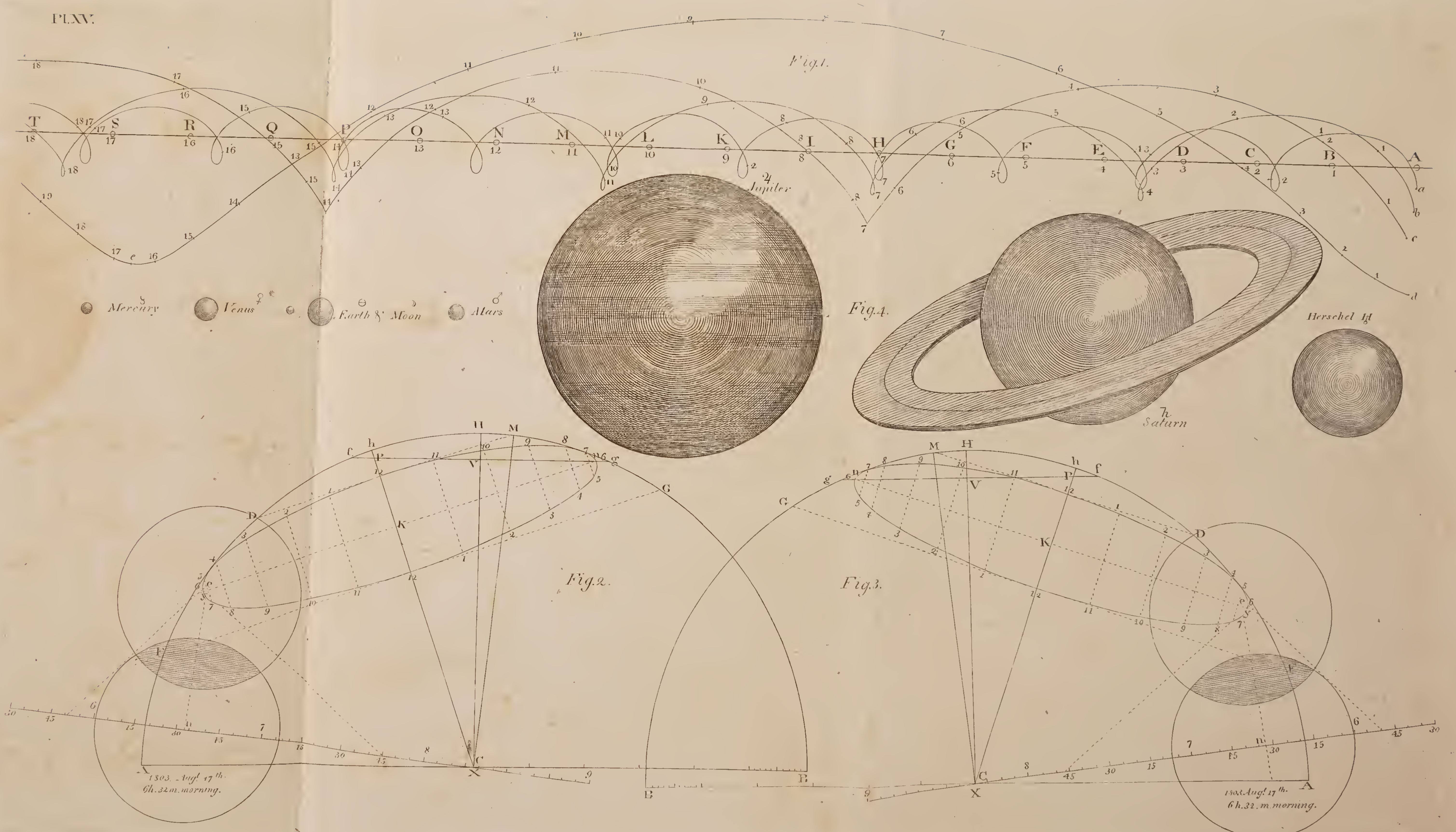

Fig.1.
Mercury
Venus
Earth & Moon
Mars
Jupiter
Fig.4.
Saturn
Herschel
Fig.2.
Fig.3.
1803. Aug.t 17th
6h.32m.morning.
1803. Aug.t 17th
6h.32.m.morning.

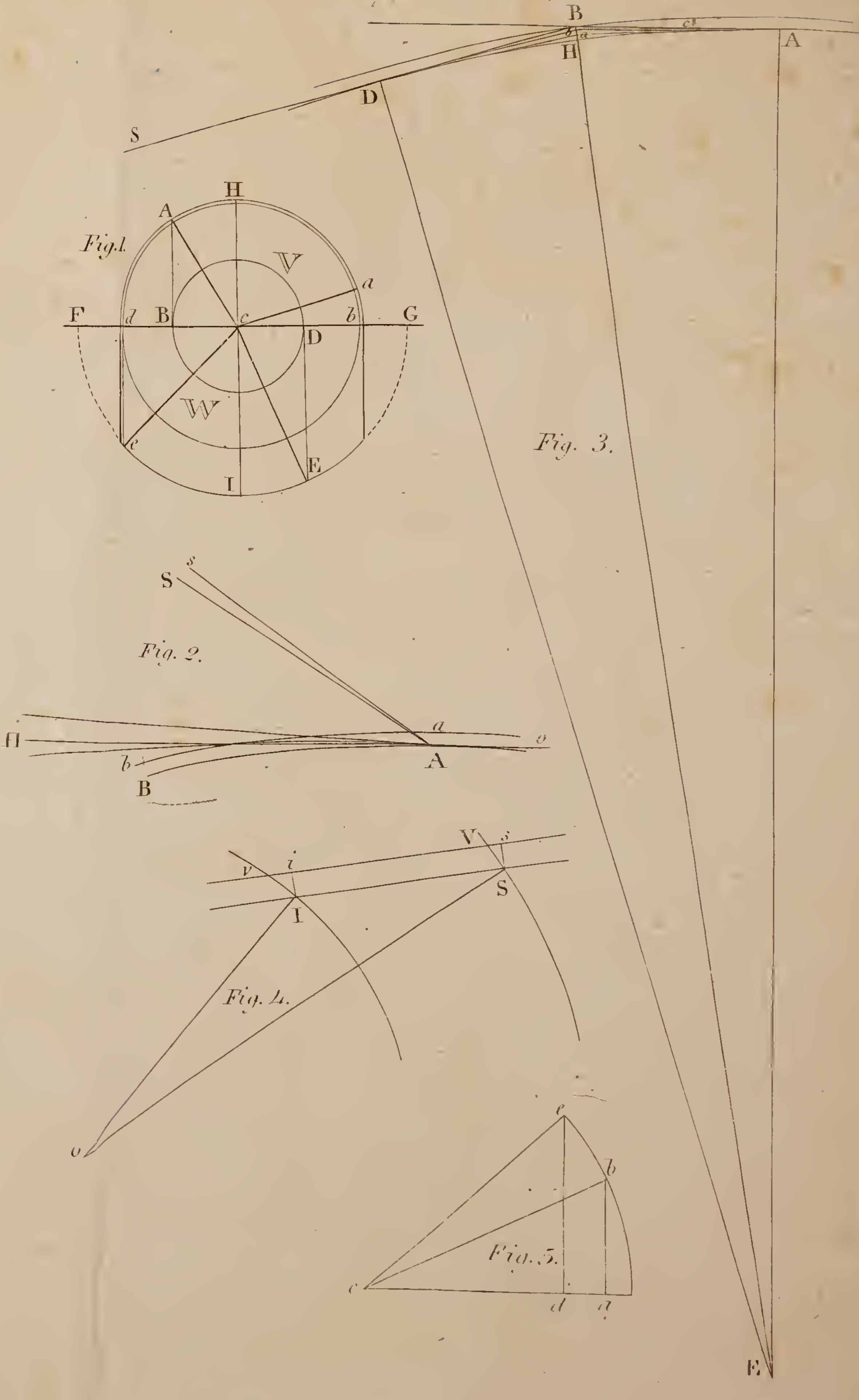
Fig. 1.
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Fig. 2.
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Fig. 3.
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b
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Fig. 4.
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Fig. 5.
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